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EXTENSION OF MULTIRESOLUTION ANALYSIS AND THE CONSTRUCTION OF ORTHOGONAL MULTIWAVELETS

SHOUZHI YANG

Abstract. A procedure of the extension of orthogonal multiresolution analysis is introduced by assuming that $\phi_1(x)$ ia an orthogonal uniscaling function and constructing a new orthogonal multiscaling function $\Phi(x) = [\phi_1(x), \phi_2(x)]^T$. Moreover, an explicit formula of the orthogonal multiwavelets associated with $\Phi(x)$ is obtained. Finally, a construction example is given.

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1. Introduction

Multiresolution analysis is one of the most important methods used in constructing wavelets. A variety of uniwavelets such as the well-known Daubechies' orthogonal wavelets ([1], [2]) and semi-orthogonal spline wavelets by Chui and Wang et al. [3] are constructed on the basis of multiresolution analysis. Multiwavelets possess some nice features that uniwavelets do not and thus multiwavelets have interesting applications in signal processing and some other areas ([4], [5]). In recent years, multiscaling functions and multiwavelets have been studied extensively ([6]–[13]). Similar to the construction of a uniscaling function, one can construct a multiscaling function of "multiple" multiresolution analysis. The main difficulty in constructing of multiwavelets is the verification of the convergence of the infinite product of two-scale matrix symbols (see [14], [15]). Is there any easier method of constructing orthogonal multiwavelets? Can multiwavelets be constructed on the basis of uniwavelets? The objective of this paper is to describe a procedure of the extension of orthogonal multiresolution analysis and to construct the multiwavelets.

The paper is organized as follows: In Section 2, we briefly recall the concept of multiresolution analysis. In Section 3, a procedure of the extension of orthogonal multiresolution analysis and the construction of orthogonal multiwavelets is introduced. In Section 4, explicit formula of the orthogonal multiwavelets is established. In Section 5, we give a construction example.

2. Basic Concept and the Lemma

2.1. **Orthogonal uniwavelet.** Let $\phi(x)$ be a scaling function satisfying the following two-scale equation:

$$\phi(x) = \sum_{n\in Z} p_n \phi(2x - n), \tag{1}$$

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where $\{p_n\}_{n\in Z}$ is a two-scale sequence.

(1) can be rewritten as

$$\hat{\phi}(w) = p\Big(\frac{w}{2}\Big)\,\hat{\phi}\Big(\frac{w}{2}\Big),$$

where

$$p(w) = \frac{1}{2}\sum_{n\in Z} p_n\, e^{-inw} \tag{2}$$

is the two-scale symbol of the two-scale sequence $\{p_n\}_{n\in Z}$ of ϕ.

Define a subspace $V_j^u \subset L^2(R)$ by

$$V_j^u = \mathrm{Clos}_{L^2(R)}\langle \phi_{j,n}(x),\ n \in Z\rangle, \tag{3}$$

where $\phi_{j,n}(x) = 2^j\,\phi(2^j x - n)$.

As usual, $\phi(x)$ defined in (1) generates a uni-multiresolution analysis (u-MRA) $\{V_j^u\}_{j\in Z}$ of $L^2(R)$ if $\{V_j^u\}_{j\in Z}$ defined in (3) satisfies the following properties:

(1) $\cdots \subset V_0^u \subset V_1^u \subset V_2^u \cdots$;
(2) $\mathrm{Clos}_{L^2(R)}(\cup_{j\in Z} V_j^u) = L^2(R)$;
(3) $\cap_{j\in Z} V_j^u = \{0\}$;
(4) $\forall f(x) \in V_j^u \Leftrightarrow f(2x) \in V_{j+1}^u$;
(5) the family $\{\phi_{j,n}(x) : n \in Z\}$ is a Riesz basis for V_j^u.

We call $\phi(x)$ an orthogonal scaling function if $\langle \phi(x), \phi(x-n)\rangle = \delta_{0,n}$, $n \in Z$. The associated multiresolution analysis is said to be an orthogonal uni-multiresolution analysis.

Let W_j^u, $j \in Z$, denote the complementary subspace of V_j^u in V_{j+1}^u. According to the wavelet analysis theory (see [1], [2], [16]), there exists a function $\psi(x)$ whose translation and dilation form a Riesz basis for W_j^u, i.e.,

$$W_j^u = \mathrm{Clos}_{L^2(R)}\langle \psi_{j,n}(x) : n \in Z\rangle, \quad j \in Z. \tag{4}$$

From condition (4), it is clear that $\psi(x)$ is in $W_0 \subset V_1$. Hence there exists a sequence of $\{q_n\}_{n\in Z}$ such that

$$\psi(x) = \sum_{n\in Z} q_n\,\phi(2x - n). \tag{5}$$

By the two-scale relation (5) we have

$$\hat{\psi}(w) = q\Big(\frac{w}{2}\Big)\,\hat{\phi}\Big(\frac{w}{2}\Big),$$

where $q(w) = \frac{1}{2}\sum_{n\in Z} q_n e^{-inw}$.

Theorem A ([1], [2]). *Let $\phi(x)$ be an orthogonal scaling function, $\psi(x)$ be orthogonal wavelet associated with $\phi(x)$; $p(w)$ and $q(w)$, respectively, be the two-scale symbols of the two-scale sequences $\{p_n\}, \{q_n\}$; then $p(w)$ and $q(w)$ satisfy* *the following identities:*

$$p(w)\,\overline{p(w)} + p(w+\pi)\,\overline{p(w+\pi)} = 1, \tag{6}$$

$$p(w)\,\overline{q(w)} + p(w+\pi)\,\overline{q(w+\pi)} = 0, \tag{7}$$

$$q(w)\,\overline{q(w)} + q(w+\pi)\,\overline{q(w+\pi)} = 1. \tag{8}$$

Equivalently, the two-scale sequences $\{p_n\}$, $\{q_n\}$ *satisfy*

$$\sum_{i\in Z} p_i\,\overline{p}_{i+2k} = 2\delta_{0,k}, \tag{9}$$

$$\sum_{i\in Z} p_i\,\overline{q}_{i+2k} = 0, \tag{10}$$

$$\sum_{i\in Z} q_i\,\overline{q}_{i+2k} = 2\delta_{0,k}. \tag{11}$$

2.2. **Orthogonal multiwavelets.** Let $\Phi(x) = (\phi_1, \phi_2)^T, \phi_1, \phi_2 \in L^2(R)$, satisfy the equation

$$\Phi(x) = \sum_{k\in Z} P_k\,\Phi(2x-k) \tag{12}$$

for some 2×2 matrices sequence $\{P_k\}_{k\in Z}$ called the two-scale matrix sequence. Then $\Phi(x)$ is called a multiscaling function of multiplicity 2.

Applying the Fourier transform to both sides of (12) we have

$$\hat{\Phi}(w) = P\Big(\frac{w}{2}\Big)\,\hat{\Phi}\Big(\frac{w}{2}\Big), \tag{13}$$

where $P(w) = \frac{1}{2}\sum\limits_{k\in Z} P_k e^{-iwk}$ is called the two-scale matrix symbol of the two-scale matrix sequence $\{P_k\}_{k\in Z}$ of $\Phi(x)$.

By repeated applications of (13) we obtain

$$\hat{\Phi}(w) = \bigg(\prod_{j=1}^{\infty} P\Big(\frac{w}{2^j}\Big)\bigg)\hat{\Phi}(0). \tag{14}$$

If the infinite product $\prod\limits_{j=1}^{\infty} P(\frac{w}{2^j})$ converges, then $\hat{\Phi}(w)$ is well-defined and we say that $\hat{\Phi}(w)$ is generated by $P(w)$.

We now introduce the following theorem to ensure the convergence of the above infinite product.

Theorem B ([17]). *The infinite matrix product* $\big(\prod\limits_{j=1}^{\infty} P(\frac{w}{2^j})\big)$ *converges uniformly, on compact sets, to a continuous matrix-valued function if and only if* $P(0)$ *has eigenvalues* $\lambda_1 = 1$, $|\lambda_2| < 1$.

Define a subspace $V_j^m \subset L^2(R)$ by

$$V_j^m = \mathrm{clos}_{L^2(R)}\langle \phi_{\ell:j,k} : \ell = 1, 2,\ k \in Z\rangle,\ j \in Z. \tag{15}$$

Here and in what follows for $f_\ell \in L^2$ we denote $f_{\ell:j,k} = 2^{\frac{j}{2}} f_\ell(2^j x - k)$.

$\Phi(x)$ defined in (12) generates a multiresolution analysis of multiplicity 2 (m-MRA) $\{V_j^m\}_{j\in Z}$ of $L^2(R)$ if $\{V_j^m\}_{j\in Z}$ defined in (15) satisfies the following properties:

(1) $\cdots \subset V_0^m \subset V_1^m \subset V_2^m \cdots$;

(2) $\mathrm{clos}_{L^2(R)}(\bigcup_{j\in Z} V_j^m) = L^2(R)$;

(3) $\bigcap_{j\in Z} V_j^m = \{0\}$;

(4) $f(x) \in V_j^m \Leftrightarrow f(2x) \in V_{j+1}^m,\ j \in Z$;

(5) the family $\{\phi_{\ell:j,k} : \ell = 1,2;\ k \in Z\}$ is a Riesz basis for V_j^m.

Let W_j^m, $j \in Z$, denote the complementary subspace of V_j^m in V_{j+1}^m, and the vector-valued function $\Psi(x) = (\psi_1, \psi_2)^T$, $\psi_\ell \in L^2$, $\ell = 1, 2$, form a Riesz basis for W_j^m, i.e.,

$$W_j = \mathrm{clos}_{L^2(R)}\langle \psi_{\ell:j,k} : \ell = 1,2;\ k \in Z\rangle, \quad j \in Z. \tag{16}$$

Since $\psi_1(x), \psi_2(x) \in W_0 \subset V_1$, there exists a sequence of 2×2 matrices $\{Q_k\}_{k\in Z}$ such that

$$\Psi(x) = \sum_{k\in Z} Q_k\, \Phi(2x - k). \tag{17}$$

By the two-scale relation (17) we have

$$\hat{\Psi}(w) = Q\Big(\frac{w}{2}\Big)\, \hat{\Phi}\Big(\frac{w}{2}\Big), \tag{18}$$

where

$$Q(w) = \frac{1}{2}\sum_{k\in Z} Q_k\, e^{-ikw}. \tag{19}$$

Theorem C ([8], [9]). *Let $\Phi(x)$ be orthogonal multiscaling function of multiplicity 2, $\Psi(x)$ be orthogonal multiwavelet associated with the orthogonal multiscaling function $\Phi(x)$; $P(w)$ and $Q(w)$ be the two-scale matrix symbols, then*

$$P(w)\, P(w)^* + P(w+\pi)\, P(w+\pi)^* = I_2, \tag{20}$$

$$P(w)\, Q(w)^* + P(w+\pi)\, Q(w+\pi)^* = O_2, \tag{21}$$

$$Q(w)\, Q(w)^* + Q(w+\pi)\, Q(w+\pi)^* = I_2, \tag{22}$$

where O_2 and I_2 denote the zero matrix and the unity matrix, respectively.

Here and in what follows the asterisk denotes the conjugate transpose of the matrix.

3. Extension of Multiresolution Analysis

In this section, we introduce a procedure of extension of orthogonal MRA. The extension is from orthogonal u-MRA to m-MRA of multiplicity 2. The generated elements of the new orthogonal MRA of multiplicity 2 are two functions, one of which is the known uniscaling function, while the other is the function to be constructed. We give the construction method of the function.

Next we consider the extension of MRA of multiplicity 2. For simplicity, let $P(w)$ be the two-scale lower triangle matrix symbol, i.e.,

$$P(w) = \begin{bmatrix} A(w) & 0 \\ B(w) & C(w) \end{bmatrix}. \tag{23}$$

As in known (see [7] or [8]), if $P(w)$ is the two-scale matrix symbol associated with orthogonal multiscaling function of multiplicity 2, then it satisfies the equation

$$P(w)P(w)^* + P(w+\pi)P(w+\pi)^* = I_2. \tag{24}$$

This means that $A(w), B(w), C(w)$ satisfy the following three equations:

$$|A(w)|^2 + |A(w+\pi)|^2 = 1, \tag{25}$$

$$A(w)\overline{B(w)} + A(w+\pi)\overline{B(w+\pi)} = 0, \tag{26}$$

$$|B(w)|^2 + |B(w+\pi)|^2 + |C(w)|^2 + |C(w+\pi)|^2 = 1. \tag{27}$$

To construct $P(w)$, i.e., $A(w)$, $B(w)$, $C(w)$, we need

Lemma 1. *Let $s(w)$, $h(w)$ (where $s(w)h(w) \neq 0$) be π-periodic functions and satisfy the equation $|s(w)|^2 + |h(w)|^2 = 1$; then $|s(w)| < 1$, $|h(w)| < 1$.*

Theorem 1. *Let $\phi^1(x)$ be an orthogonal uniscaling function, $\psi^1(x)$ be orthogonal uniwavelet associated with the orthogonal uniscaling function $\phi^1(x)$; $p^1(w)$ and $q^1(w)$, respectively, be the two-scale symbols of $\phi^1(w)$, $\psi^1(w)$; $p^2(w)$ be the two-scale symbol of the orthogonal uniscaling function $\phi^2(x)$; $s(w)$, $h(w)$ be two functions satisfying the condition of Lemma 1. Define*

$$P(w) = \begin{bmatrix} p^1(w) & 0 \\ s(w)q^1(w) & h(w)p^2(w) \end{bmatrix}, \tag{28}$$

then

$$P(w)P(w)^* + P(w+\pi)P(w+\pi)^* = I_2. \tag{29}$$

Proof. Since $p^1(w), p^2(w)$ and $q^1(w)$ are two-scale symbols associated with $\phi^1(x)$, $\phi^2(w)$ and $\psi^1(w)$, respectively, by Theorem A we have

$$\begin{aligned} |p^1(w)|^2 + |p^1(w+\pi)|^2 &= 1, \\ p^1(w)\,\overline{q^1(w)} + p^1(w+\pi)\,\overline{q^1(w+\pi)} &= 0, \\ |q^1(w)|^2 + |q^1(w+\pi)|^2 &= 1, \\ |p^2(w)|^2 + |p^2(w+\pi)|^2 &= 1. \end{aligned}$$

Since $s(w)$, $h(w)$ are π-periodic functions, and satisfy $|s(w)|^2 + |h(w)|^2 = 1$, we obtain

$$\begin{aligned} &P(w)P(w)^* + P(w+\pi)P(w+\pi)^* \\ &= \begin{bmatrix} p^1(w) & 0 \\ s(w)q^1(w) & h(w)p^2(w) \end{bmatrix} \begin{bmatrix} \overline{p^1(w)} & \overline{s(w)}\,\overline{q^1(w)} \\ 0 & \overline{h(w)}\,\overline{p^2(w)} \end{bmatrix} \end{aligned}$$

$$+\begin{bmatrix} p^1(w+\pi) & 0 \\ s(w+\pi)q^1(w+\pi) & h(w+\pi)p^2(w+\pi) \end{bmatrix}$$

$$\times\begin{bmatrix} \overline{p^1(w+\pi)} & \overline{s(w+\pi)}\,\overline{q^1(w+\pi)} \\ 0 & \overline{h(w+\pi)}\,\overline{p^2(w+\pi)} \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{1}|p^1(w+k\pi)|^2 & J(w) \\ \overline{J(w)} & H(w) \end{bmatrix} = I_2,$$

where

$$J(w) = \sum_{k=0}^{1} \overline{s(w+k\pi)}p^1(w+k\pi)\overline{q^1(w+k\pi)},$$

$$H(w) = \sum_{k=0}^{1} \left[|s(w+k\pi)|^2|q^1(w+k\pi)|^2 + |h(w+k\pi)|^2|p^2(w+k\pi)|^2\right]. \qquad \square$$

Theorem 2. *Under the condition of Theorem* 1, $P(w)$ *defined in* (28) *is the matrix symbol, then the matrix* $P(0)$ *has two eigenvalues* $\lambda_1 = 1$, $|\lambda_2| < 1$.

Proof. Since $p^1(w)$, $p^2(w)$ are two-scale symbols associated with $\phi^1(x)$, $\phi^2(w)$, respectively, we have $p^1(0) = p^2(0) = 1$. Hence

$$P(0) = \begin{bmatrix} p^1(0) & 0 \\ s(0)q^1(0) & h(0)p^2(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & h(0) \end{bmatrix}.$$

This means that $P(0)$ has two eigenvalues $\lambda_1 = 1$, $\lambda_2 = h(0)$. By Lemma 1, $|h(0)| < 1$. $\square$

Remark 1. In Theorem 1, $p^1(w)$, $p^2(w)$ may be either equal or not equal each other, i.e., $p^1(w)$, $p^2(w)$ may be related the two-scale symbols associated with the same orthogonal uniscaling function or by two varying two-scale symbols associated with two varying orthogonal uniscaling functions. Additionally, in Theorem 1, $s(w)$, $h(w)$ are required to satisfy the following conditions: (1): $s(w)$, $h(w)$ are π-periodic functions; (2): $s(w)h(w) \neq 0$; (3): $|s(w)|^2 + |h(w)|^2 = 1$. There exist a lot of functions satisfying the above conditions, for example $h(w) = \frac{2-\sin 2w}{4}$, $s(w) = \frac{\sqrt{12+4\sin(2w)-\sin^2 2w}}{4}$.

Since $P(0) = \begin{bmatrix} p^1(0) & 0 \\ s(0)q^1(0) & h(0)p^2(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & h(0) \end{bmatrix}$, we know that the infinite product $\prod_{j=1}^{\infty} P(\frac{w}{2^j})$ converges by Theorem B. Thus a vector function $\hat{\Phi}(w)$ is well-defined and satisfies

$$\hat{\Phi}(w) = \left[\hat{\phi}_1(w), \hat{\phi}_2(w)\right]^T = \begin{bmatrix} p^1(w) & 0 \\ s(w)q^1(w) & h(w)p^2(w) \end{bmatrix} \left[\hat{\phi}_1\Big(\frac{w}{2}\Big), \hat{\phi}_2\Big(\frac{w}{2}\Big)\right]^T.$$

Therefore

$$\hat{\phi}_1(w) = p^1\Big(\frac{w}{2}\Big)\hat{\phi}_1\Big(\frac{w}{2}\Big), \quad \hat{\phi}_2(w) = s\Big(\frac{w}{2}\Big)q^1\Big(\frac{w}{2}\Big)\hat{\phi}_1\Big(\frac{w}{2}\Big) + h\Big(\frac{w}{2}\Big)p^2\Big(\frac{w}{2}\Big)\hat{\phi}_2\Big(\frac{w}{2}\Big).$$

We summarize the above discussion and have the following theorem.

Theorem 3. *Let $p^1(w)$, $p^2(w)$ be two two-scale symbols associated with two orthogonal uniscaling functions $\phi^1(x)$, $\phi^2(x)$; $\psi^1(x)$, $\psi^2(x)$ be two orthogonal uniwavelets associated with two orthogonal uniscaling functions $\phi^1(x)$, $\phi^2(x)$; $q^1(w)$, $q^2(w)$ be two two-scale symbols associated with two orthogonal uniwavelets $\psi^1(x)$, $\psi^2(x)$, respectively; $s(w)$, $h(w)$ be two functions satisfying the conditions of Lemma* 1. *Then the refinable vector function $\Phi(x) = [\phi_1(x), \phi_2(x)]^T$ generated by the two-scale matrix symbol $P(w)$ defined in* (28) *is an orthogonal multiscaling function of multiplicity* 2.

Remark 2. Theorem 3 tells us how to use an orthogonal uniwavelet to construct orthogonal multiwavelets of multiplicity 2. At the same time, it also introduces a method of extension of the orthogonal multiresolution analysis. Obviously, the orthogonal multiresolution analysis of multiplicity 2 includes the corresponding orthogonal uni-multiresolution analysis.

We next discuss a special setting. Let $p^1(w)$, $p^2(w)$ be the two-scale symbol of the well-known Daubechies' orthogonal uniscaling function $\phi_N(x)$, i.e.,

$$p^1(w) = p^2(w) = \left(\frac{1+e^{-iw}}{2}\right)^N S_N(w), \tag{30}$$

where $S_N(w)$ is the Laurent polynomial satisfying

$$|S_N(w)|^2 = P_N\big(\sin^2 w/2\big), \quad P_N(y) = \sum_{k=0}^{N-1} \binom{N-1+k}{k} y^k. \tag{31}$$

Theorem 4. *Let $p^1(w)$, $p^2(w)$ defined in* (30) *be the two-scale symbols of the well-known Daubechies' orthogonal uniscaling function $\phi_N(x)$; $s(w)$, $h(w)$ be two functions satisfying the condition of Lemma* 1. *Suppose $|h(w)| \le \frac{1}{2^N}|\sin w|^N$. Then the refinable function $\Phi(x) = [\phi_1(x), \phi_2(x)]^T$ generated by $P(w)$ defined in* (28) *is an orthogonal multiscaling function of multiplicity* 2 *and its Fourier transform has a decay estimate*

$$|\hat{\phi}_i(w)| \le \frac{C_i}{(1+|w|)^{(1-\frac{\log 3}{2\log 2})N}}, \quad i = 1, 2,$$

where C_i is a constant.

Proof. By Theorem 3, it is easy to show that $\Phi(x) = [\phi_1(x), \phi_2(x)]^T$ generated by $P(w)$ defined in (28) is an orthogonal multiscaling function. Next we prove that the Fourier transform of $\Phi(x) = [\phi_1(x), \phi_2(x)]^T$ satisfies the above inequality.

Since $\phi_1(x)$ is the well-known Daubechies' scaling function $\phi_N(x)$, we have $|\hat{\phi}_1(w)| \le \frac{C_1}{(1+|w|)^{(1-\frac{\log 3}{2\log 2})N}}$.

Now let us consider a decay estimate for $\hat{\phi}_2(w)$. Note that $|s(w)q^1(w)|^2 \le |s(w)q^1(w)|^2 + |h(w)p^2(w)|^2 + |s(w+\pi)q^1(w+\pi)|^2 + |h(w+\pi)p^2(w+\pi)|^2 = 1$. Thus we obtain

$$|\hat{\phi}_2(w)| = \left|s\Big(\frac{w}{2}\Big)q^1\Big(\frac{w}{2}\Big)\hat{\phi}_1\Big(\frac{w}{2}\Big) + h\Big(\frac{w}{2}\Big)p^2\Big(\frac{w}{2}\Big)\hat{\phi}_2\Big(\frac{w}{2}\Big)\right|$$

$$
\begin{aligned}
&\le \Big|\hat{\phi}_1\Big(\frac{w}{2}\Big)\Big| + \Big|h\Big(\frac{w}{2}\Big)p^2\Big(\frac{w}{2}\Big)\hat{\phi}_2\Big(\frac{w}{2}\Big)\Big| \\
&\le \Big|\hat{\phi}_1\Big(\frac{w}{2}\Big)\Big| + \Big|h\Big(\frac{w}{2}\Big)p^2\Big(\frac{w}{2}\Big)\Big|\Big|\hat{\phi}_1\Big(\frac{w}{4}\Big)\Big| + \Big|h\Big(\frac{w}{4}\Big)p^2\Big(\frac{w}{4}\Big)\hat{\phi}_2\Big(\frac{w}{4}\Big)\Big| \le \cdots \\
&\le \Big|\hat{\phi}_1\Big(\frac{w}{2}\Big)\Big| + \sum_{j=1}^{\infty}\Big(\prod_{k=1}^{j}\Big|h\Big(\frac{w}{2^k}\Big)p^2\Big(\frac{w}{2^k}\Big)\Big|\Big|\hat{\phi}_1\Big(\frac{w}{2^{j+1}}\Big)\Big|\Big),
\end{aligned}
$$

where

$$
\begin{aligned}
&\Big(\prod_{k=1}^{j}\Big|h\Big(\frac{w}{2^k}\Big)p^2\Big(\frac{w}{2^k}\Big)\Big|\Big|\hat{\phi}_1\Big(\frac{w}{2^{j+1}}\Big)\Big|\Big) \\
&= \Big(\prod_{k=1}^{j-1}\Big|h\Big(\frac{w}{2^k}\Big)p^2\Big(\frac{w}{2^k}\Big)\Big|\Big)\Big|h\Big(\frac{w}{2^j}\Big)p^2\Big(\frac{w}{2^j}\Big)\Big|\Big|\hat{\phi}_1\Big(\frac{w}{2^{j+1}}\Big)\Big| \\
&= \Big(\prod_{k=1}^{j-1}\Big|h\Big(\frac{w}{2^k}\Big)p^1\Big(\frac{w}{2^k}\Big)\Big|\Big)\Big|h\Big(\frac{w}{2^j}\Big)p^1\Big(\frac{w}{2^j}\Big)\Big|\Big|\hat{\phi}_1\Big(\frac{w}{2^{j+1}}\Big)\Big| \\
&\le \Big(\frac{1}{2^N}\Big)^{j-1}\Big(\prod_{k=1}^{j-1}\Big|p^1\Big(\frac{w}{2^k}\Big)\Big|\Big)\frac{1}{2^N}\Big|\sin\frac{w}{2^j}\Big|^N\Big|p^1\Big(\frac{w}{2^j}\Big)\Big|\Big|\hat{\phi}_1\Big(\frac{w}{2^{j+1}}\Big)\Big| \\
&\le \Big(\frac{1}{2^N}\Big)^{j-1}\Big(\prod_{k=1}^{j-1}\Big|p^1\Big(\frac{w}{2^k}\Big)\Big|\Big)\Big|\cos\frac{w}{2^{j+1}}\Big|^N\Big|p^1\Big(\frac{w}{2^j}\Big)\Big|\Big|\hat{\phi}_1\Big(\frac{w}{2^{j+1}}\Big)\Big|.
\end{aligned}
$$

Since $|S_N(w)| \ge 1$ and, by (30), we have $|p^1(\frac{w}{2^{j+1}})| = |\cos\frac{w}{2^{j+1}}|^N|S_N(e^{\frac{iw}{2^{j+1}}})| \ge |\cos\frac{w}{2^{j+1}}|^N$, we have

$$
\begin{aligned}
&\Big(\prod_{k=1}^{j}\Big|h\Big(\frac{w}{2^k}\Big)p^2\Big(\frac{w}{2^k}\Big)\Big|\Big|\hat{\phi}_1\Big(\frac{w}{2^{j+1}}\Big)h\Big|\Big) \\
&\qquad\le \Big(\frac{1}{2^N}\Big)^{j-1}\Big(\prod_{k=1}^{j-1}\Big|p^1\Big(\frac{w}{2^k}\Big)\Big|\Big)\Big|\cos\frac{w}{2^{j+1}}\Big|^N\Big|p^1\Big(\frac{w}{2^j}\Big)\Big|\Big|\hat{\phi}_1\Big(\frac{w}{2^{j+1}}\Big)\Big| \\
&\qquad\le \Big(\frac{1}{2^N}\Big)^{j-1}\Big(\prod_{k=1}^{j+1}\Big|p^1\Big(\frac{w}{2^k}\Big)\Big|\Big)\Big|\hat{\phi}_1\Big(\frac{w}{2^{j+1}}\Big)\Big| = \Big(\frac{1}{2^N}\Big)^{j-1}\big|\hat{\phi}_1(w)\big|.
\end{aligned}
$$

Hence $\hat{\phi}_2(w)$ can be estimated as follows:

$$
\begin{aligned}
\big|\hat{\phi}_2(w)\big| &\le \Big|\hat{\phi}_1\Big(\frac{w}{2}\Big)\Big| + \sum_{j=1}^{\infty}\Big(\frac{1}{2^N}\Big)^{j-1}\big|\hat{\phi}_1(w)\big| \\
&= \hat{\phi}_1\Big(\frac{w}{2}\Big)\Big| + \big|\hat{\phi}_1(w)\big|\sum_{j=1}^{\infty}\Big(\frac{1}{2^N}\Big)^{j-1} \\
&= \hat{\phi}_1\Big(\frac{w}{2}\Big)\Big| + \frac{1}{2^N-1}\big|\hat{\phi}_1(w)\big| \le \frac{C_2}{(1+|w|)^{(1-\frac{\log 3}{2\log 2})N}},
\end{aligned}
$$

where C_2 is a constant. This completes the proof of Theorem 4. □

4. An Explicit Formula of Orthogonal Multiwavelets

In the above section, we have given the method of construction of an orthogonal multiscaling function of multiplicity 2 by the matrix symbol $P(w)$. In this section, we discuss the construction of multiwavelets associated with an orthogonal multiscaling function generated by $P(w)$.

To give an explicit formula of orthogonal multiwavelets of multiplicity 2, construct the matrices $Q(w), M(w)$ by

$$Q(w) = \begin{bmatrix} 0 & q^2(w) \\ h(w)q^1(w) & -s(w)p^2(w) \end{bmatrix}, \tag{32}$$

$$M(w) = \begin{bmatrix} P(w) & P(w+\pi) \\ Q(w) & Q(w+\pi) \end{bmatrix}, \tag{33}$$

respectively.

Theorem 5. *Let $P(w)$, $Q(w)$, $M(w)$ defined in* (28), (32) *and* (33), *respectively, be three matrices, under the condition of Theorem* 3. *Then the matrix $M(w)$ is a unitary matrix.*

Further, let $\Phi(x) = [\phi_1(x), \phi_2(x)]^T$ be an orthogonal multiscaling function generated by $P(w)$, $\Psi(x) = [\psi_1(x), \psi_2(x)]^T$ be the corresponding orthogonal multiwavelet. Then

$$\hat{\Psi}(w) = Q(w)\hat{\Phi}\Big(\frac{w}{2}\Big) \tag{34}$$

and therefore

$$\hat{\psi}_1(w) = q^2\Big(\frac{w}{2}\Big)\hat{\phi}_2\Big(\frac{w}{2}\Big), \quad \hat{\psi}_2(w) = h\Big(\frac{w}{2}\Big)q^1\Big(\frac{w}{2}\Big)\hat{\phi}_1\Big(\frac{w}{2}\Big) - s\Big(\frac{w}{2}\Big)p^2\Big(\frac{w}{2}\Big)\hat{\phi}_2\Big(\frac{w}{2}\Big).$$

Proof. According to the construction theorem of wavelets we only need to prove that $M(w)$ is a unitary matrix. Under the conditions of Theorem 3 it is easy to verify that the following matrix

$$\begin{bmatrix} p^1(w) & 0 & p^1(w+\pi) & 0 \\ s(w)q^1(w) & h(w)p^2(w) & s(w+\pi)q^1(w+\pi) & h(w+\pi)p^2(w+\pi) \\ 0 & q^2(w) & 0 & q^2(w+\pi) \\ h(w)q^1(w) & -s(w)p^2(w) & h(w+\pi)q^1(w+\pi) & -s(w+\pi)p^2(w+\pi) \end{bmatrix}$$

ia a unitary matrix, i.e., $M(w)$ is a unitary matrix. □

5. A Construction Example

We will illustrate by an example how to make use of orthogonal uniwavelet to construct orthogonal multiwaveltets.

Example. Let $\phi^1(x) = \phi^2(x)$ be Daubechies' orthogonal scaling function with $N = 2$, $\psi^1(x) = \psi^2(x)$ be an orthogonal wavelet associated with orthogonal

Daubechies' scaling function with $N = 2$; $p^1(w)$, $p^2(w)$, $q^1(w)$, $q^2(w)$ be the two-scale symbols associated with $\phi^1(x)$, $\phi^2(x)$, $\psi^1(x)$, $\psi^2(x)$, respectively, where

$$\begin{cases} p^1(w)=p^2(w)=\dfrac{1}{2}\left(\dfrac{1+\sqrt{3}}{4}+\dfrac{3+\sqrt{3}}{4}e^{-iw}+\dfrac{3-\sqrt{3}}{4}e^{-2iw}+\dfrac{1-\sqrt{3}}{4}e^{-3iw}\right), \\ q^1(w)=q^2(w)=\dfrac{1}{2}\left(\dfrac{1+\sqrt{3}}{4}e^{-iw}-\dfrac{3+\sqrt{3}}{4}+\dfrac{3-\sqrt{3}}{4}e^{iw}-\dfrac{1-\sqrt{3}}{4}e^{2iw}\right). \end{cases}$$

Suppose $h(w) = \frac{1}{2}$, $s(w) = \frac{\sqrt{3}}{2}$, by Theorem 4 we obtain that the orthogonal multiscaling functions are

$$\begin{cases} \phi_1(x) = \dfrac{1+\sqrt{3}}{4}\phi_1(2x) + \dfrac{3+\sqrt{3}}{4}\phi_1(2x-1) + \dfrac{3-\sqrt{3}}{4}\phi_1(2x-2) \\ \qquad\qquad + \dfrac{1-\sqrt{3}}{4}\phi_1(2x-3), \\ \phi_2(x) = \dfrac{3+\sqrt{3}}{8}\phi_1(2x-1) - \dfrac{3+3\sqrt{3}}{8}\phi_1(2x) + \dfrac{3\sqrt{3}-3}{8}\phi_1(2x+1) \\ \qquad\qquad - \dfrac{\sqrt{3}-3}{8}\phi_1(2x+2) + \dfrac{1+\sqrt{3}}{8}\phi_2(2x) + \dfrac{3+\sqrt{3}}{8}\phi_2(2x-1) \\ \qquad\qquad + \dfrac{3-\sqrt{3}}{8}\phi_2(2x-2) + \dfrac{1-\sqrt{3}}{8}\phi_2(2x-3). \end{cases}$$

By Theorem 5, the corresponding orthogonal multiwavelets are

$$\begin{cases} \psi_1(x) = \dfrac{1+\sqrt{3}}{4}\phi_2(2x-1) - \dfrac{3+\sqrt{3}}{4}\phi_2(2x) + \dfrac{3-\sqrt{3}}{4}\phi_2(2x+1) \\ \qquad\qquad - \dfrac{1-\sqrt{3}}{4}\phi_2(2x+2), \\ \psi_2(x) = \dfrac{1+\sqrt{3}}{8}\phi_1(2x-1) - \dfrac{3+\sqrt{3}}{8}\phi_1(2x) + \dfrac{3-\sqrt{3}}{8}\phi_1(2x+1) \\ \qquad\qquad - \dfrac{1-\sqrt{3}}{8}\phi_1(2x+2) - \dfrac{3+\sqrt{3}}{8}\phi_2(2x) - \dfrac{3+3\sqrt{3}}{8}\phi_2(2x-1) \\ \qquad\qquad - \dfrac{3\sqrt{3}-3}{8}\phi_2(2x-2) - \dfrac{\sqrt{3}-3}{8}\phi_2(2x-3). \end{cases}$$

In the above example, we use one orthogonal uniscaling function to construct orthogonal multiwavelets. For the setting with two varying orthogonal uniscaling functions, the corresponding orthogonal multiwavelets are constructed similarly.

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References

1. I. Daubechies, Ten lectures on wavelets. *CBMS-NSF Regional Conference Series in Applied Mathematics,* 61. *Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA,* 1992.
2. I. Daubechies, Orthonormal bases of compactly supported wavelets. *Comm. Pure Appl. Math.* **41**(1988), No. 7, 909–996.
3. C. K. Chui and J. Z. Wang, On compactly supported spline wavelets and a duality principle. *Trans. Amer. Math. Soc.* **330**(1992), No. 2, 903–915.
4. He Wenjie and Lai Ming-Jun, Construction of bivariate compactly supported biorthogonal box spline wavelets with arbitrarily high regularities. *Appl. Comput. Harmon. Anal.* **6**(1999), No. 1, 53–74.
5. W. Lawton, S. L. Lee, and Z. Shen, An algorithm for matrix extension and wavelet construction. *Math. Comp.* **65**(1996), No. 214, 723–737.
6. T. N. T. Goodman, S. L. Lee, and W. S. Tang, Wavelets in wandering subspaces. *Trans. Amer. Math. Soc.* **338**(1993), No. 2, 639–654.
7. C. K. Chui and J. Z. Wang, A cardinal spline approach to wavelets. *Proc. Amer. Math. Soc.* **113**(1991), No. 3, 785–793.
8. C. K. Chui and J. Lian, A study of orthonormal multi-wavelets. *Selected keynote papers presented at* 14*th IMACS World Congress* (*Atlanta, GA,* 1994). *Appl. Numer. Math.* **20**(1996), No. 3, 273–298.
9. J. Lian, Orthogonality criteria for multi-scaling functions. *Appl. Comput. Harmon. Anal.* **5**(1998), No. 3, 277–311.
10. J. S. Geronimo, D. P. Hardin, and P. R. Massopust, Fractal functions and wavelet expansions based on several scaling functions. *J. Approx. Theory* **78**(1994), No. 3, 373–401.
11. G. C. Donovan, J. S. Geronimo, D. P. Hardin, and P. R. Massopust, Construction of orthogonal wavelets using fractal interpolation functions. *SIAM J. Math. Anal.* **27**(1996), No. 4, 1158–1192.
12. Yang Shouzhi and Cheng Zhengxing, Orthonormal multi-wavelets on the interval $[0, 1]$ with multiplicity r. (Chinese) *Acta Math. Sinica* **45**(2002), No. 4, 789–796.
13. Yang Shouzhi, Cheng Zhengxing, and Wang Hongyong, Construction of biorthogonal multiwavelets. *J. Math. Anal. Appl.* **276**(2002), No. 1, 1–12.
14. Q. Jiang, Orthogonal multiwavelets with optimum time-frequency resolution. *IEEE Trans. Signal Process.* **46**(1998), No. 4, 830–844.
15. Q. Jiang, On the design of multifilter banks and orthonormal multiwavelet bases. *IEEE Trans. Signal Process.* **46**(1998), 3292–3303.
16. C. K. Chui, An introduction to wavelets. *Wavelet Analysis and its Applications,* 1. *Academic Press, Inc., Boston, MA,* 1992.
17. C. Cabrelli, C. Heil, and U. Molter, Accuracy of lattice translates of several multidimensional refinable functions. *J. Approx. Theory* **95**(1998), No. 1, 5–52.

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Author's address:

Dept. of Maths., Shantou University
Shantou, 515063
P. R. China
E-mail: szyang@stu.edu.cn