ON YJ-INJECTIVITY AND ANNIHILATORS

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Abstract. This note contains the following results for a ring $A$: (1) $A$ is a quasi-Frobenius ring iff $A$ is a left and right YJ-injective, left Noetherian ring whose prime factor rings are right YJ-injective iff every non-zero one-sided ideal of $A$ is the annihilator of a finite subset of elements of $A$; (2) if $A$ is a right YJ-injective ring such that any finitely generated right ideal is either a maximal right annihilator or a projective right annihilator, then $A$ is either quasi-Frobenius or a right p.p. ring such that every non-zero left ideal of $A$ contains a non-zero idempotent; (3) a commutative YJ-injective Goldie ring is quasi-Frobenius; (4) if the Jacobson radical of $A$ is reduced, every simple left $A$-module is either YJ-injective or flat and every maximal left ideal of $A$ is either injective or a two-sided ideal of $A$, then $A$ is either strongly regular or left self-injective regular with non-zero socle.

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INTRODUCTION

Quasi-Frobenius rings introduced by T. Nakayama (1939) have since been extensively studied (cf. [8], [10], [11], [17], [20], [22]). We here consider quasi-Frobenius rings in terms of certain special annihilators. In [25] p-injective modules were introduced to study von Neumann regular rings, $V$-rings, self-injective rings and associated rings, have drawn the attention of various authors (cf. for example [2], [3], [9], [11], [15], [16], [21], [24]).

Throughout the paper, $A$ denotes an associative ring with identity and $A$-modules are unital. $J$, $Y$ will always stand respectively for the Jacobson radical and the right singular ideal of $A$. Recall that a right $A$-module $M$ is p-injective if, for every principal right ideal $P$ of $A$, any right $A$-homomorphism of $P$ into $M$ extends to one of $A$ into $M$ (cf. [11, p. 122], [20, p. 340], [21], [24], [25]). It is well known that $A$ is von Neumann regular iff every right (left) $A$-module is flat (M. Harada (1956); M. Auslander (1957)). This remains true if “flat” is replaced by “p-injective” (cf. [2], [15], [21], [24], [25]). The generalization of p-injectivity to YJ-injectivity is performed as follows: a right $A$-module $M$ is called YJ-injective if, for every $0 \neq a \in A$, there exist a positive integer $n$ such that $a^n \neq 0$ and any right $A$-homomorphism of $a^n A$ into $M$ extends to one of $A$ into $M$ ([7], [21], [28], [29]). $A$ is called a right p-injective (resp. YJ-injective) ring if $A_A$ is p-injective (resp. YJ-injective). Similarly, p-injectivity and YJ-injectivity are defined on the left side.
An ideal of \( A \) will always mean a two-sided ideal of \( A \). As usual, a right (resp. left) ideal \( I \) of \( A \) is called a right (resp. left) annihilator if \( I = r(S) \) (resp. \( 1(S) \)) for some subset \( S \) of elements of \( A \). In \cite[Lemma 3]{29}, it is proved that \( A \) is a right YJ-injective ring if for every \( 0 \neq a \in A \), there exist a positive integer \( n \) such that \( Aa^n \) is a non-zero left annihilator. Also, if \( A \) is right YJ-injective, then \( J = Y \) (cf. \cite[p. 222]{27} and \cite[p. 103]{28}) (where this notation was introduced).

It is well known that quasi-Frobenius rings are left and right self-injective, left and right Artinian rings whose one-sided ideals are annihilators. Note that certain special annihilators will impose the chain conditions on rings (cf. \cite[33]).

Rings whose one-sided ideals are annihilators of an element are studied by C. R. Yohe \cite{23}. The next result is motivated by the remark at the end of Yohe’s paper. The proof of \cite[Theorem 16]{33} shows that if every non-zero one-sided ideal of \( A \) is the annihilator of an element of \( A \), then \( A \) is a principal ideal ring. By a theorem of M. Ikeda and T. Nakayama, \( A \) is left and right self-injective. Consequently, \cite[Theorem 24.20]{10} yields

**Theorem 1.** The following conditions are equivalent:

1. Every factor ring of \( A \) is quasi-Frobenius;
2. \( A \) is a principal left and right ideal ring which is quasi-Frobenius;
3. Every non-zero one-sided ideal of \( A \) is the annihilator of an element of \( A \).

We here give an example of a commutative principal ideal quasi-Frobenius ring.

**Example (Q).** Set \( A = \mathbb{Z}/4\mathbb{Z} \). Then \( M = \{0, 2\} \) is the unique non-trivial ideal of \( A \). Every non-zero ideal of \( A \) is the annihilator of an element of \( A \) (\( M \) is the annihilator of 2). \( A \) is a principal ideal ring (\( M \) is generated by 2). \( A \) is therefore Artinian, self-injective, and \( M^2 = 0 \). Consequently, \( M \) is not an injective \( A \)-module and \( A \) is not semi-prime. Note that every factor ring of \( A \) is quasi-Frobenius (\( A/M \) being a field).

Following \cite{11}, we say that “\( A \) is VNR” if \( A \) is a von Neumann regular ring. Since 1979 K. R. Goodearl’s classic \cite{14} has motivated numerous papers on VNR and associated rings. Following C. Faith, \( A \) is called a right (resp. left) \( V \)-ring if every simple right (resp. left) \( A \)-module is injective. A theorem of I. Kaplansky asserts that a commutative ring is a \( V \)-ring if it is VNR. In the non-commutative case, there is no implication between these two classes of rings. A vast amount of work on injective modules over non semi-simple Artinian rings and on flat modules over non-VNR rings motivate the study of p-injectivity and YJ-injectivity over rings which are not necessarily VNR (cf. for example, \cite{3}, \cite{11}, \cite{15}, \cite{16}, \cite{22}).

A result of P. Menal and P. Vamos asserts that any arbitrary ring can be embedded in a FP-injective ring \cite[p. 308]{11}. (This does not hold if “FP-injective” is replaced by “self-injective”.). Consequently, any ring can be embedded in a p-injective or YJ-injective ring. We are thus motivated to study YJ-injective
rings. A is called a right CF-ring if every cyclic right A-module embeds in a free module (cf. [21]). A semi-perfect, right CF, right YJ-injective ring is quasi-Frobenius [21, Corollary 8].

**Theorem 2.** The following conditions are equivalent:

1. A is quasi-Frobenius;
2. A is a right CF, right YJ-injective ring satisfying the maximum condition on left annihilators of elements of A;
3. A is a left and right YJ-injective, left Noetherian ring whose prime factor rings are right YJ-injective;
4. Every non-zero one-sided ideal of A is the annihilator of a finite subset of elements of A.

**Proof.** (1) Clearly implies (2), (3) and (4).

Assume (2). Since A is right CF, then it is left p-injective and therefore every principal right ideal of A is a right annihilator. Now A satisfies the descending chain condition on principal right ideals which means that A is left perfect. By [17, Corollary 11.6.2], and [21, Corollary 8], A is quasi-Frobenius and therefore, (2) implies (1).

Assume (3). Let B be a prime factor ring of A. Then B is left Noetherian and since B is right YJ-injective, every non-zero-divisor of B is invertible in B which implies that B coincides with a classical left (and right) quotient ring. By Goldie’s theorem, B is simple Artinian. If A is prime, then A is Artinian as just seen. If not, then every proper prime factor ring of A is simple Artinian and by [10, Lemma 18.34B], A is left Artinian. In any case, A is left Artinian. Since A is right YJ-injective, for any minimal left ideal U of A, U = Au, u ∈ A, there exist a positive integer n such that Au^n is a non-zero left annihilator [28, Lemma 3]. If U^2 = 0, then u^2 = 0 which implies that Au is a left annihilator. If U = Ae, e = e^2 ∈ A, then U is again a left annihilator. Similarly, every minimal right ideal of A is a right annihilator. A is therefore quasi-Frobenius by a result of H. H. Storrer [19]. Thus (3) implies (1).

Finally, assume (4). Let L be a non-zero proper left ideal of A. If L = l(F), where F = {x_1, . . . , x_m}, x_i ∈ A, with R = \sum_{i=1}^{m} x_i A, L = l(R) and by hypothesis, R = r(G), where G = {y_1, . . . , y_n} is a finite subset of elements of A. With K = \sum_{i=1}^{n} Ay_i, R = r(K) and K is also a left annihilator which implies that L = l(R) = l(r(K)) = K is a finitely generated left ideal of A. Therefore A is left Noetherian. Since every principal one-sided ideal of A is an annihilator, then A is quasi-Frobenius (a left Noetherian, left or right perfect ring is left Artinian ([10, Proposition 18.12])). Thus (4) implies (1).

□

**Remark 1.** It follows from Theorem 2 that if every factor ring of A is a left and right YJ-injective, left Noetherian ring, then A is quasi-Frobenius.

This remark, together with [28, Proposition 3.1(3)] motivate the following
**Question 1.** Is a left and right YJ-injective, left Noetherian ring quasi-Frobenius? (It is known that if $A$ is a left Noetherian ring whose minimal one-sided ideals are annihilators, then $A$ needs not be left Artinian.)

**Remark 2.** If $A$ is left p-injective with maximum condition on left annihilators of elements of $A$ and $A$ contains no nilpotent minimal right ideal, then $A$ is semi-simple Artinian.

The singular submodule of a module is a fundamental concept in ring theory (a standard reference is K.R. Goodearl [13]). Recall that for a right $A$-module $M$, the right singular submodule of $M$ is $Z(M) = \{y \in M \mid r(y) \text{ is an essential right ideal of } A\}$ and $M$ is called right non-singular if $Z(M) = 0$. In this note, we write $Y = Z(A_A)$ and $Y$ is called the right singular ideal of $A$.

A well-known result asserts that $A$ is right non-singular (i.e. $Y = 0$) iff $A$ has VNR maximal right quotient ring $Q$. In that case, $Q_A$ is the injective hull of $A_A$ and $Q$ is a right self-injective VNR ring. Another result on non-singular rings is due to Y. Utumi [13, Theorem 2.38]: If $A$ is right and left non-singular, then the maximal right and left quotient rings of $A$ coincide iff every complement one-sided ideal of $A$ is an annihilator (the terms “complement” and “annihilator” in [11, p. 181] should be permuted).

The next result is motivated by Example (Q) and depends mainly on the right singular ideal of $A$.

**Proposition 3.** Let $A$ be a right YJ-injective ring such that any finitely generated right ideal is either a maximal right annihilator or a projective right annihilator. Then $A$ is either quasi-Frobenius or a right p.p. ring such that every non-zero left ideal contains a non-zero idempotent.

**Proof.** Suppose that $Y$, the right singular ideal of $A$, is non-zero. For any $0 \neq y \in Y$, $yA_A$ is not projective which implies that $yA$ is a maximal right annihilator. If $w \neq yA$, then $yA + wA = A$. This proves that $Y = yA$. Therefore $Y$ is a minimal right ideal of $A$. But $Y$ is also a maximal right ideal of $A$. Since $Y$ contains no non-zero idempotent, then $Y$ is an essential right ideal of $A$. For any proper non-zero right ideal $R$ of $A$, $R \cap Y \neq 0$, which implies that $R \cap Y = Y$ by the minimality of $Y$. Then $Y \subseteq R$ which yields $Y = R$ by the maximality of $Y$. We have proved that $Y$ is the unique non-zero proper right ideal of $A$. $A$ is therefore right Artinian local ring. Since $A$ is right YJ-injective, then every minimal left ideal of $A$ is a left annihilator as in the proof of Theorem 2. Also, every minimal right ideal of $A$ is a right annihilator by hypothesis, which proves that $A$ is quasi-Frobenius. Now suppose that $Y = 0$. If $0 \neq a \in A$ such that $aA$ is a maximal right annihilator, then $aA$ is not an essential right ideal (in as much as $Y = 0$). If $0 \neq b \in A$ such that $aA \cap bA = 0$, then $A = aA + bA$ since $aA$ is a maximal right annihilator. This proves that every principal right ideal of $A$ is projective and $A$ is therefore a right p.p. ring. For any $0 \neq c \in A$, by [28, Lemma 3], there exists a positive integer $n$ such that $Ac^n$ is a non-zero left annihilator. Since $r(c^n)$ is a direct summand of $A_A$, then $Ac^n = l(r(Ac^n))$ is a direct summand of $A_A$. Therefore $Ac$ contains a non-zero idempotent, which completes the proof. □
The next proposition is motivated by [12, Proposition 3.3(ii)].

**Proposition 4.** The following conditions are equivalent for a left YJ-injective ring $A$:

1. $A$ is right Artinian;
2. $A$ is a semi-perfect ring with maximum condition on left annihilators and finite right Goldie dimension.

**Proof.** It is clear that (1) implies (2).
Assume (2). Since $A$ satisfies the maximum conditions on left annihilators, then $Z$, the left singular of $A$, is nilpotent [13, Proposition 3.31]. From [28, p. 103], $J = Z$ is nilpotent. Since $A$ is semi-perfect, $A/J$ is semi-simple Artinian which implies that $A$ is semi-primary. Since $A$ is left YJ-injective, every minimal right ideal of $A$ is a right annihilator (cf. the proof of Theorem 2). Since $A$ has finite right Goldie dimension, $\text{Soc}(A_A)$, the right socle of $A$, is a finitely generated right ideal. Also $\text{Soc}(A_A)$ is an essential right ideal of $A$ (because $A$ is left perfect). Then, $\text{Soc}(A_A)$ coincides with $\text{Soc}(A_A)$, the left socle of $A$, by [4, Theorem 3.1]. Now $A$, being semi-primary with maximum condition on left annihilators such that $\text{Soc}(A_A) = \text{Soc}(A_A)$ is finite-dimensional as a right $A$-module must imply that $A$ is right Artinian by [5, Lemma 6]. Thus (2) implies (1).

Since there exist left and right Artinian rings whose right ideals are annihilators which are not quasi-Frobenius, the ring considered in Proposition 4 needs not be quasi-Frobenius (not even right YJ-injective).

**Corollary 5.** $A$ is quasi-Frobenius iff $A$ is a semi-perfect, left and right YJ-injective, left and right Goldie ring.

**Corollary 6.** If $A$ is a left and right YJ-injective, left and right Noetherian ring, then $A$ is quasi-Frobenius iff $A/J$ is a right YJ-injective ring.

Theorem 2(3) motivates the next remark.

**Remark 3.** The following conditions are equivalent: (a) Every factor ring of $A$ is quasi-Frobenius, (b) Every factor ring of $A$ is left and right YJ-injective, left Noetherian.

**Remark 4.** If $A$ is a left YJ-injective ring with maximum condition on left annihilators, then idempotents can be lifted mod $J$.

**Remark 5.** A commutative YJ-injective ring with maximum condition on annihilators is semi-primary (cf. [11, Theorem 16.31] and [28, Remark 11]).

A is called a right (resp. left) FPF ring (finite pseudo-Frobenius) if every finitely generated faithful right (resp. left) $A$-module generates mod-$A$ (resp. $A$-mod). Such rings are closely connected with self-injective rings.

**Remark 6.** A commutative YJ-injective FPF ring is self-injective (cf. [11, Theorem 5.42]).
Remark 7. If $A$ is left Noetherian, then $A$ is left Artinian iff every prime factor ring of $A$ is left-injective iff every prime factor ring of $A$ is right YJ-injective.

Applying [11, Theorem 12.4D] and [31, Corollary 7], we get

Remark 8. If $A$ is right Noetherian with $J^2 = 0$ and every essential right ideal of $A$ is an idempotent ideal of $A$, then $A$ is right Artinian. (Such rings need not be right duo.)

A theorem of S. Page [11, Theorem 5.49] yields the following characterization.

Remark 9. $A$ is right and left self-injective regular ring of a bounded index iff $A$ is a right YJ-injective, right non-singular, right FPF ring.

Remark 10. $A$ is a right pseudo-Frobeniusean ring iff $A$ is a semi-perfect, right YJ-injective, right FPF ring with essential right socle.

We now characterize commutative quasi-Frobenius rings in terms of Goldie rings ([28, Corollary 3.2] is improved).

**Theorem 7.** The following conditions are equivalent for a commutative ring $A$:

1. $A$ is quasi-Frobenius;
2. $A$ is YJ-injective Goldie ring.

**Proof.** It is obvious that (1) implies (2).

Assume (2). Then $Y$, the singular ideal of $A$, is nilpotent. Since $A$ is YJ-injective, $Y = J$, the Jacobson radical of $A$ [28, p. 103]. Therefore $J$ is nilpotent. Again, since $A$ is YJ-injective, $A$ coincides with its classical quotient ring. By [11, Theorem 9.4], $A$ is Artinian. Every minimal ideal of $A$ is an annihilator (in as much as $A$ is YJ-injective). $A$ is therefore quasi-Frobenius and (2) implies (1). 

The study of rings whose simple modules are injective or projective is initiated in [1]. Such rings are called GV-rings by V. S. Ramamurthy and K. M. Rangaswamy (cf. [2], [18]).

The next example motivates our last propositions.

**Example (GV).** If $A$ denotes the $2 \times 2$ upper triangular matrix ring over a field, $A$ is a P.I. left and right Artinian, left and right hereditary, left and right quasi-duo ring whose singular left and right modules are injective but $A$ is not semi-prime (indeed, the Jacobson radical $J$ is non-zero with $J^2 = 0$). Also, all non-singular left and right modules are projective and the maximal left and right quotient rings of $A$ coincide (cf. [13, Theorems 5.21 and 5.23] and [34]). Note that $A$ is neither left nor right p-injective.

As usual, a left (right) ideal of $A$ is called reduced if it contains no non-zero nilpotent element. Kaplansky’s theorem on commutative $V$-rings has motivated several generalizations of non-commutative $V$-rings. In [7, Lemma 1], it is proved that if every simple left $A$-module is YJ-injective, then the Jacobson radical $J$ is reduced.
Proposition 8. Let $A$ be a ring with reduced Jacobson radical $J$ such that every simple left $A$-module is either YJ-injective or flat. Then $J = 0$.

Proof. Suppose there exist $0 \neq u \in J$. For any positive integer $m$, $l(u^m) = l(u) = r(u) = r(u^n)$ (because $J$ is reduced). If $AuA + l(u) \neq A$, let $M$ be a maximal left ideal of $A$ containing $AuA + l(u)$. Then $A/M$ is simple and therefore either YJ-injective or flat. First suppose that $A/M$ is YJ-injective. There exist a positive integer $n$ such that any left $A$-homomorphism of $Au^n \to A/M$ extends to one of $A$ into $A/M$. Define a left $A$-homomorphism $f : Au^n \to A/M$ by $f(au^n) = a + M$ for all $a \in A$ ($f$ is well-defined because $l(u^n) = l(u)$). Then $1 + M = f(u^n) = u^n y + M$ for some $y \in A$. Now $1 - u^n y \in M$ and since $u^n y \in AuA \subseteq M$, then $1 \in M$, which contradicts $M \neq A$. Now suppose that $A/M$ is flat. Then $u \in M$ implies that $u = uw$ for some $w \in M$ (cf. [6, p. 458]). Therefore $1 - w \in r(u) = l(u) \subseteq M$, which implies that $1 \in M$, again a contradiction! We thus have $AuA + l(u) = A$. If $1 = b + c$, $b \in AuA$, $c \in l(u)$, then $u = bu + cu = bu$ and $(1 - b)u = 0$. Since $b \in AuA \subseteq J$, $1 - b$ is left invertible in $A$ which yield $u = 0$, contradicting $u \neq 0$. We have proved that $J = 0$. □

Corollary 9. If every simple left $A$-module is YJ-injective, then $J = 0$. (Apply [7, Lemma 1].)

Corollary 10. The following conditions are equivalent for a right self-injective ring $A$: (1) $A$ is VNR; (2) Every simple right $A$-module is either YJ-injective or projective; (3) Every simple left $A$-module is YJ-injective.

Question 2. If $A$ is a right self-injective ring whose simple left modules are either YJ-injective or projective, is $A$ VNR?

The next proposition is again motivated by Example (GV).

Proposition 11. Let $A$ be a ring with reduced Jacobson radical $J$ such that every simple left $A$-module is either YJ-injective or flat and every maximal left ideal of $A$ is either injective or an ideal of $A$. Then $A$ is either strongly regular or left self-injective regular with non-zero socle.

Proof. By proposition 8, $J = 0$.

First suppose that each maximal left ideal of $A$ is an ideal of $A$. Then $A$ is semi-primitive, left quasi-duo which is therefore reduced (cf. the proof of “(2) implies (3)” in [26, Theorem 2.1]). Following the proof of Proposition 8, we see that for any $0 \neq a \in A$, any positive integer $m$, $l(a^m) = l(a) = r(a) = r(a^m)$. Since every simple left $A$-module is either YJ-injective or flat, we must have $AaA + l(a) = AaA + r(a) = A$ which implies that $A$ is fully left and right idempotent. Since $A$ is left quasi-duo, by [3, Proposition 9], $A$ is VNR and is therefore strongly regular. Now suppose that there exist a maximal left ideal $M$ of $A$ which is not an ideal of $A$. Then $AM$ is injective by [3, Lemma 4], $A$ is left self-injective and since $J = 0$, $A$ is VNR with non-zero socle. □

Note that Proposition 11 remains valid if we replace “every simple left $A$-module” by “every simple right $A$-module”.

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Our last result is motivated by a question raised in [32] (cf. U.Q.1(c)).

**Proposition 12.** If every principal left ideal of $A$ is the flat left annihilator of an element of $A$, then $A$ is VNR.

**Proof.** Let $0 \neq a \in A$. Then $Aa = l(c)$ for some $c \in A$. Since $A/l(c) \approx Ac$ is a flat left $A$-module and $l(c) = Aa$, $A/Aa$ is a finitely related flat left $A$-module which implies that $A/Aa$ is projective. It follows that $Aa$ is a direct summand of $AA$, which proves that $A$ is VNR. \qed 

**Question 3.** Is $A$ VNR if every principal left ideal of $A$ is a flat complement left ideal of $A$? (The answer is positive if $A$ is commutative.)

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**References**


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