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SOME BASIC d-ORTHOGONAL POLYNOMIAL SETS

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Abstract. The purpose of this paper is to study the class of polynomial sets which are at the same time d-orthogonal and q-Appell. By a linear change of variable, the resulting set reduces to q-Al-Salam–Carlitz polynomials, for $d = 1$. Various properties of the obtained polynomials are singled out: a generating function, a recurrence relation of order $d + 1$. We also explicitly express a d-dimensional functional for which the d-orthogonality holds.

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1. Introduction and Preliminaries

During the past few years, there has been a growing interest in multiple orthogonal polynomials (see, for instance, [4], [5] ,[6], [13], [27], [28]). However, it is only recently that examples of multiple orthogonal polynomials have appeared in the literature. A convenient framework to discuss such examples consists in considering a subclass of multiple orthogonal polynomials known as d-orthogonal polynomials (see, for instance, [8–12], [14–18], [24], [29]). Many of these papers generalized some known characterization theorems for orthogonal polynomials [2] to the d-orthogonality, especially, for continuous and discrete cases. As far as we know, the basic polynomials are not considered. So, it is significant to study this case. In this paper, we investigate some d-orthogonal polynomials related to the q-difference operator (1.4).

Next, we present some basic definitions which we need below.

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$, the set of complex numbers, and let $\mathcal{P}'$ be its dual. We denote by $\langle u, f\rangle$ the effect of a functional $u \in \mathcal{P}'$ on a polynomial $f \in \mathcal{P}$. Let $\{P_n\}_{n\geq 0}$ be a sequence of polynomials in $\mathcal{P}$ such that $\deg P_n(x) = n$ for all n. In this case, we also call $\{P_n\}_{n\geq 0}$ a polynomial set. The corresponding monic polynomial sequence $\{\widehat{P}_n\}_{n\geq 0}$ is given by $P_n = \lambda_n \widehat{P}_n$, $n \geq 0$, where λ_n is the normalization coefficient and its dual sequence $\{u_n\}_{n\geq 0}$ is defined by $\langle u_n, \widehat{P}_m\rangle = \delta_{n,m}$, $n, m \geq 0$.

Definition 1.1. Let d be an arbitrary positive integer. A polynomial sequence $\{P_n\}_{n\geq 0}$ is called a d-orthogonal polynomial sequence (d-OPS, shortly) with respect to a d-dimensional functional $\mathcal{U} = {}^t(u_0, \dots, u_{d-1})$ if it satisfies [24], [29]

$$\begin{cases} \langle u_k, P_m P_n\rangle = 0, & m > dn + k\ , n \geq 0, \\ \langle u_k, P_n P_{dn+k}\rangle \neq 0, & n \geq 0, \end{cases} \tag{1.1}$$

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for each integer k belonging to $\{0, 1, \ldots, d-1\}$.

The orthogonality conditions (1.1) are equivalent to the fact that the sequence $\{P_n\}_{n\geq 0}$ satisfies a $(d+1)$-order recurrence relation [29] which we write in the *monic* form as

$$\widehat{P}_{m+d+1}(x) = (x - \beta_{m+d})\widehat{P}_{m+d}(x) - \sum_{\nu=0}^{d-1} \gamma_{m+d-\nu}^{d-1-\nu}\, \widehat{P}_{m+d-1-\nu}(x)\,, \quad m \geq 0, \tag{1.2}$$

with the initial conditions

$$\begin{cases} \widehat{P}_0(x) = 1, \quad \widehat{P}_1(x) = x - \beta_0 \quad \text{and} \quad \text{if} \quad d \geq 2, \\ \widehat{P}_n(x) = (x - \beta_{n-1})\widehat{P}_{n-1}(x) - \displaystyle\sum_{\nu=0}^{n-2} \gamma_{n-1-\nu}^{d-1-\nu} \widehat{P}_{n-2-\nu}(x)\,,\ 2 \leq n \leq d, \end{cases} \tag{1.3}$$

and the regularity conditions

$$\gamma_{n+1}^{0} \neq 0\,, \quad n \geq 0.$$

When $d = 1$, recurrence (1.2) with (1.3) is the well-known second-order recurrence relation

$$\begin{cases} \widehat{P}_{n+2}(x) = (x - \beta_{n+1})\widehat{P}_{n+1}(x) - \gamma_{n+1}\widehat{P}_n(x),\ n \geq 0, \\ \widehat{P}_0(x) = 1,\ \widehat{P}_1(x) = x - \beta_0. \end{cases}$$

In the remainder of this paper, all the polynomial sets are assumed monic.

Let q be a real number, Hahn [21] defined a linear operator L_q by

$$L_q(f)(x) = \frac{f(qx) - f(x)}{(q-1)x}, \qquad |q| \neq 1, \tag{1.4}$$

where f is a suitable function for which the second member of this equality exists. This operator tends to the derivative operator D as $q \longrightarrow 1$.

Let $\{P_n\}_{n\geq 0}$ be a d-OPS. Put $Q_n(x) = L_q P_{n+1}(x),\ n \geq 0$. According to Hahn's property [21], if the sequence $\{Q_n\}_{n\geq 0}$ is also d-orthogonal, the sequence $\{P_n\}_{n\geq 0}$ is called L_q-*classical* d-OPS.

Throughout this paper, we shall use the following notation, definitions and formulas related to the q-theory. For details the reader is referred to [19] or [23]. If n, k are positive integers, we use the notation

$$(a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k), \tag{1.5}$$

$$[n] := [n]_q := \frac{q^n - 1}{q - 1}, \quad [n]_q! := [n][n-1]\cdots[1], [0]! = 1. \tag{1.6}$$

We define the Gaussian polynomial or the q-binomial coefficient by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} = \frac{(q^n - 1)\cdots(q^{n-k+1} - 1)}{(q^k - 1)\cdots(q - 1)}\,. \tag{1.7}$$

So

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q = \frac{[n][n-1]\cdots[n-k+1]}{[k]!} = \frac{[n]!}{[k]![n-k]!}. \tag{1.8}$$

It is clear that when $q \longrightarrow 1$, $\left[\begin{smallmatrix} n \\ j \end{smallmatrix}\right] \longrightarrow \binom{n}{j}$ and $[n]_q \longrightarrow n$.

For positive integers $m_1, \ldots, m_r$, we define the Gaussian multinomial coefficient or the q-multinomial coefficient by

$$\begin{bmatrix} m_1 + m_2 + \cdots + m_r \\ m_1, m_2, \ldots, m_r \end{bmatrix}_q := \frac{(q;q)_{m_1+m_2+\cdots m_r}}{(q;q)_{m_1}(q;q)_{m_2}\cdots(q;q)_{m_r}}. \tag{1.9}$$

A q-analogue of the exponential function is defined by

$$e_q(t) := \sum_{n \geq 0} \frac{t^n}{[n]!} = e((1-q)t) \tag{1.10}$$

with

$$e(t) = e(q,t) = \sum_{n\geq 0} \frac{t^n}{(q;q)_n} = \begin{cases} \prod_{m=0}^{\infty} (1-tq^m)^{-1} & \text{if} \quad |q| < 1, \\ \prod_{m=1}^{\infty} \left(1 - tq^{-m}\right) & \text{if} \quad |q| > 1, \end{cases} \tag{1.11}$$

$$e(t)^{-1} = \sum_{n \geq 0} (-1)^n q^{\frac{n(n-1)}{2}} \frac{t^n}{(q;q)_n}. \tag{1.12}$$

One easily verifies that

$$L_q e_q(xt) = x e_q(xt) \tag{1.13}$$

for a fixed real x and

$$e_q(t) \longrightarrow e^t \quad if \quad (q \longrightarrow 1). \tag{1.14}$$

For given two functions f and g, we have:

$$L_q^n(f)(0) := L_q(L_q^{n-1})(f)(0) = \frac{[n]!}{n!} f^{(n)}(0). \tag{1.15}$$

$$L_q(fg)(t) = g(t)L_q(f)(t) + f(qt)L_q(g)(t). \tag{1.16}$$

A polynomial sequence $\{P_n\}_{n\geq 0}$ is called an Appell polynomial set if $DP_{n+1}(x) = (n+1)P_n(x)$, $n \geq 0$. A natural generalization of this definition with the operator L_q is given by the following

Definition 1.2. A polynomial set $\{P_n\}_{n\geq 0}$ is called a L_q-Appell or a q-Appell polynomial set if

$$L_q P_{n+1}(x) = [n+1]_q P_n(x), \quad n \geq 0. \tag{1.17}$$

When $q \longrightarrow 1$, we deal with the Appell polynomials.

Now, let us consider the following problem:

P: *Find all polynomial sets which are at the same time d-OPS and q-Appell.*

Such a characterization takes into account the fact that polynomial sets which are obtainable from one another by a linear change of variable are assumed equivalent.

Note that the polynomial sets obtained as solutions of this problem must be L_q-classical d-OPS according to (1.17) and Hahn's property [21].

This problem, for the limiting case $(d, q) = (1, 1)$, was solved by many authors. The obtained solution is the Hermite polynomial set. It should be mentioned that Hahn, Al-Salam, Carlitz and Chihara are among the authors who treated the case $d = 1$ for arbitrary q. The obtained q-Al-Salam–Carlitz polynomials are a unique solution [1], [2], [3]. Later, Douak [14] treated the limiting case ($q \longrightarrow 1$) for a general positive integer d and obtained certain generalizations of the Hermite polynomials, containing among others the Gould–Hopper polynomials [20]. In connection with this problem, Khériji and Maroni [23] discussed the L_q classical orthogonal polynomials. In this paper, we solve the problem when d is a positive integer. The main result is

Theorem 1.1. *The only polynomial sets which are at the same time d-OPS and q-Appell are given by*

$$\sum_{n \geq 0} P_n(q; x) \frac{t^n}{[n]!} = \frac{e_q(xt)}{e_q(x_0 t) e_q(x_1 t) \cdots e_q(x_d t)}, \tag{1.18}$$

where $x_0, x_1, \ldots, x_d \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$.

The outline of the paper is as follows. In Section 2, we give some properties of q-Appell polynomials, we recall a characterization of these polynomials by means of a generating function from which a recurrence relation is deduced. In Section 3, we prove Theorem 1.1 and derive other results related to the particular case: $q \longrightarrow 1$.

In Section 4, we explicitly express the d-dimensional functional $\mathcal{U} = {}^t(u_0, \cdots, u_{d-1})$ for which we have the d-orthogonality.

2. q-Appell Polynomial Sets

At first, we mention that the polynomial set $\{x^n\}_{n \geq 0}$ is a q-Appell polynomial set since

$$L_q(x^{n+1}) = [n+1]_q x^n.$$

Such a polynomial set is generated by

$$e_q(xt) = \sum_{n \geq 0} \frac{x^n t^n}{[n]!}. \tag{2.1}$$

The polynomial set defined by (2.1) can be used to characterize all q-Appell polynomial sets. We have in fact

Theorem 2.1 ([25]). *Let $\{P_n(q; .)\}_{n \geq 0}$ be a polynomial set. The following assertions are equivalent:*

(i) $\{P_n(q; .)\}_{n \geq 0}$ *is a q-Appell polynomial set.*

(ii) *There exists a sequence $(a_k)_{k \geq 0}$, independent of n; $a_0 = 1$, such that*

$$P_n(q; x) = \sum_{k=0}^{n} a_k \frac{[n]!}{[n-k]!} x^{n-k}.$$

(iii) $\{P_n(q;\cdot)\}_{n\geq 0}$ *is generated by*

$$A(t)e_q(xt) = \sum_{n\geq 0} P_n(q;x)\frac{t^n}{[n]!}\,, \tag{2.2}$$

where

$$A(t) = \sum_{k\geq 0} a_k t^k, \quad a_0 = 1.$$

Theorem 2.1 allows us to recognize some well known polynomial sets as q-Appell. For instance, the polynomial set $\{H_n(x)\}_{n\geq 0}$, generated by

$$\sum_{n\geq 0} H_n(x)\frac{t^n}{[n]!} = e_q(t)e_q(xt), \tag{2.3}$$

is q-Appell. Szegö [26] proved that this set is orthogonal over the unit circle with respect to the weight function: $f(\alpha) = \sum_{-\infty}^{+\infty} q^{\frac{n^2}{2}} e^{in\alpha}$, $|q| < 1$.

Another example of a q-Appell polynomial set is q-Al-Salam and Carlitz polynomials denoted by $\{U_n^{(a)}(x)\}_{n\geq 0}$ and generated by

$$\sum_{n\geq 0} U_n^{(a)}(x)\frac{t^n}{[n]!} = \frac{e_q(xt)}{e_q(t)e_q(at)}\,. \tag{2.4}$$

As shown in [3], $\{U_n^{(a)}(x)\}_{n\geq 0}$ are orthogonal over the real line.

Two interesting properties of q-Appell polynomial sets are given by

Theorem 2.2. *Let* $\{P_n(q;)\}_{n\geq 0}$ *be a* q*-Appell polynomial set generated by* (2.2). *Put*

$$\frac{L_q A(t)}{A(qt)} = \sum_{n\geq 0} \alpha_n t^n. \tag{2.5}$$

Then we have:

(i) *The polynomial set* $\{P_n(q;\cdot)\}_{n\geq 1}$ *satisfies the recurrence relation*

$$P_{n+1}(x) = (x+\alpha_0 q^n)P_n(x) + \sum_{k=1}^{n} \alpha_k q^{n-k}\frac{[n]!}{[n-k]!}P_{n-k}(x). \tag{2.6}$$

(ii) *The polynomial set* $\{P_n(q;\cdot)\}_{n\geq 2}$ *satisfies the* q*-difference equation*

$$\left(\sum_{k=2}^{n} \alpha_{k-1}q^{n-k}L_q^k + (x+\alpha_0 q^{n-1})L_q - [n]_q\right)P_n(x) = 0. \tag{2.7}$$

Proof. (i) According to (1.13) and (1.16), if we apply L_q to the two members of (2.2) viewed as functions of the variable t, we obtain

$$\sum_{n\geq 0} P_{n+1}(x)\frac{t^n}{[n]!} = L_q A(t)e_q(qxt) + xA(t)e_q(xt), \tag{2.8}$$

which, according to (2.5), may be rewritten as

$$\sum_{n\geq 0} P_{n+1}(x)\frac{t^n}{[n]!} = A(qt)e_q(qxt)\sum_{n\geq 0}\alpha_n t^n + x\sum_{n\geq 0} P_n(x)\frac{t^n}{[n]!}$$

$$= \sum_{n\geq 0} q^n P_n(x) \frac{t^n}{[n]!} \sum_{n\geq 0} \alpha_n t^n + x \sum_{n\geq 0} P_n(x) \frac{t^n}{[n]!}. \tag{2.9}$$

Finally, by comparing the coefficients of $\frac{t^n}{[n]!}$ in (2.9), we obtain (2.6).

(ii) By shifting $n \to n-1$ and multiplying both sides of (2.6) by $[n]_q$, we can deduce that $\{P_n(x)\}_{n\geq 2}$ satisfies (2.7) since $L_q^k P_n(x) = \frac{[n]!}{[n-k]!} P_{n-k}$. □

Remark. When $q \longrightarrow 1$, we have the He-Ricci result [22]: Appell polynomials $\{P_n(x)\}_{n\geq 0}$ generated by $A(t)e^{xt}$ satisfy the differential equation

$$\left(\sum_{k=2}^{n} \alpha_{k-1} D^k + (x+\alpha_0) D - n \right) P_n(x) = 0.$$

3. d-Orthogonal q-Appell Polynomial Sets

In order to characterize all d-OPS and q-Appell polynomials, we first state the following lemma.

Lemma 3.1. *The polynomial sequence $\{P_n(q;.)\}_{n\geq 0}$ generated by (2.2) is a d-OPS if and only if the coefficients $(\alpha_k)_{k\geq 0}$; given by (2.5) satisfy the conditions*

$$\alpha_k = 0 \quad \textit{for} \quad k \geq d+1 \quad \textit{and} \quad \alpha_d \neq 0. \tag{3.1}$$

Proof. According to Definition 1.1, the polynomial set generated by (2.2) is a d-OPS if and only if these polynomials satisfy a recurrence relation of type (1.2) – (1.3). Such conditions, by virtue of Theorem 2.2, are equivalent to the fact that the coefficients $(\alpha_k)_{k\geq 0}$, given by (2.5) satisfy conditions (3.1). □

Proof of Theorem 1.1. Let $\{P_n(q,.)\}_{n\geq 0}$ be a polynomial set generated by (2.2). If $\{P_n(q,.)\}_{n\geq 0}$ is d-orthogonal, then by using Lemma 3.1, condition (2.5) becomes

$$\frac{L_q A(t)}{A(qt)} = \sum_{k=0}^{d} \alpha_k t^k = P(t), \qquad \alpha_d \neq 0. \tag{3.2}$$

Therefore, taking into account that $\deg P = d$ and $L_q A(t) = \frac{1}{(q-1)t}\left(A(qt) - A(t)\right)$, we deduce from (3.2) that there exist $d+1$ complex numbers $x_0, x_1, \ldots, x_d \in \mathbb{C}^*$ such that

$$A(t) = (1-(1-q)x_0 t)(1-(1-q)x_1 t)\cdots(1-(1-q)x_d t) A(qt). \tag{3.3}$$

Iterate relation (3.3) m times for $|q| < 1$ and let $m \longrightarrow \infty$ to obtain

$$A(t) = \frac{1}{e_q(x_0 t)\cdots e_q(x_d t)}. \tag{3.4}$$

Notice that the same result holds when $|q| > 1$.

Conversely, assume that the polynomial sequence $\{P_n(q,.)\}_{n\geq 0}$ satisfies (1.18). By virtue of the identity $e_q((1-q)qt) = (1-(1-q)t)e_q(t)$, one can easily verify that

$$A(t) = (1-(1-q)x_0 t)\cdots(1-(1-q)x_d t) A(qt), \tag{3.5}$$

from which we deduce

$$\frac{L_q A(t)}{A(qt)} = \frac{(1-(1-q)x_0 t)\cdots(1-(1-q)x_d t) - 1}{(1-q)t} = P(t), \qquad (3.6)$$

where $P(t)$ is a polynomial of degree d. It follows then, according to Lemma 3.1, that the polynomial set $\{P_n\}_{n\geq 0}$ is d-orthogonal. □

As a consequence of this characterization, we mention that a d-orthogonal and q-Appell polynomial set $\{P_n\}_{n\geq 0}$ satisfies a $(d+1)$-order q-difference equation of the type

$$\left(\sum_{k=2}^{d+1} \alpha_{k-1} q^{n-k} L_q^k + (x + \alpha_0 q^{n-1} - [n]_q) L_q\right) P_n(q,x) = 0. \qquad (3.7)$$

Remarks:

1) For the particular case: $d=1$, if $\{P_n(q;.)\}_{n\geq 0}$ is an orthogonal and also q-Appell polynomial set generated by (2.2) with (2.5), we have

$$P_{n+1}(x) = (x + \alpha_0 q^n) P_n(x) + \alpha_1 q^{n-1} [n]_q P_{n-1}(x). \qquad (3.8)$$

Note that we can show the existence of two constants c and a in $\mathbb{C}^*$ as solutions of the system

$$\begin{cases} \dfrac{\alpha_1}{(1-q)c^2} = a, \\ \dfrac{\alpha_0}{c} = -(1+a) \end{cases}$$

such that $U_n^{(a)}(q;x) = \frac{1}{c^n} P_n(q;cx)$.

We again have the result shown by Al-Salam in ([1, Theorem 4.1]): By a linear change of variable, the polynomial set $\{U_n^{(a)}(q;.)\}_{n\geq 0}$ is the only orthogonal polynomial set and also q-Appell.

2) If $a<0$ and $0<q<1$, it is shown in [3] that there exists a distribution function $\alpha(x)$ such that

$$\int_{-\infty}^{\infty} x^k U_n^{(a)}(x) d\alpha(x) = 0 \quad (k = 0,1,2,\ldots,n-1) \quad \text{with} \quad \int_{-\infty}^{\infty} d\alpha(x) = 1.$$

The corresponding moments of the distribution function $\alpha(x)$ are given by

$$C_n = \int_{-\infty}^{\infty} x^n d\alpha(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} a^k. \qquad (3.9)$$

4. d-Dimensional Functionals

We are now interested in determining an d-dimensional functional for which we have the d-orthogonality of a q-Appell polynomial set. It is convenient to adopt the technique used in [7] to express the dual sequence of a given polynomial set. The main result of this section is given in our next theorem.

Theorem 4.1. *The q-Appell polynomial set* $\{P_n(q;.)\}_{n\geq 0}$ *generated by*

$$G(x,t)=\sum_{n=0}^{\infty}P_n(x)\frac{t^n}{[n]!}=\frac{e_q(xt)}{e_q(x_0t)e_q(x_1t)\cdots e_q(x_dt)}\,, \tag{4.1}$$

is a d-OPS with respect to the d-dimensional functional $\mathcal{U}={}^t(u_0,u_1\cdots u_{d-1})$ *given by*

$$\langle u_0,f\rangle=\sum_{m\geq 0}H_m(x_0,x_1,\ldots,x_d)\frac{f^{(m)}(0)}{m!},\qquad f\in\mathcal{P}, \tag{4.2}$$

$$\langle u_r,f\rangle=\frac{1}{[r]!}\langle u_0,L_q^r f\rangle,\qquad r=1,\ldots,d-1, \tag{4.3}$$

where

$$H_m(x_0,x_1,\ldots,x_d)=\sum_{(i_0+i_1+\ldots+i_d)=m}\begin{bmatrix} m\\ i_0,i_1,\ldots,i_d\end{bmatrix}x_0^{i_0}x_1^{i_1}\ldots x_d^{i_d}. \tag{4.4}$$

To prove Theorem 4.1, we need the following lemma.

Lemma 4.1. *The polynomial set* $\{P_n\}_{n\geq 0}$, *generated by*

$$\sum_{n\geq 0}P_n(x)\frac{t^n}{[n]!}=\frac{e_q(xt)}{e_q(x_0t)e_q(x_1t)\cdots e_q(x_dt)}$$

is a d-OPS with respect to the d-dimensional functional $\mathcal{U}={}^t(u_0,\ldots,u_{d-1})$ *given by*

$$\langle u_r,f\rangle=\frac{1}{[r]!}B(L_q)L_q^r(f)(0),\qquad r=1,\ldots,d-1,\qquad f\in\mathcal{P}, \tag{4.5}$$

where

$$B(t)=e(x_0t)\cdots e(x_dt).$$

Proof. Put $B(t):=\sum_{k\geq 0}b_kt^k$.

From the relation $e_q(xt)=B(t)G(x,t)$ we deduce that

$$\begin{aligned}x^n&=\sum_{k=0}^{n}b_k\frac{[n]!}{[n-k]!}P_{n-k}(x)=\sum_{k=0}^{n}b_kL_q^kP_n(x)\\&=\Big(\sum_{k\geq 0}b_kL_q^k\Big)P_n(x)=B(L_q)P_n(x).\end{aligned} \tag{4.6}$$

On the other hand, we have

$$\left[L_q^nx^m\right]_{x=0}=[n]!\,\delta_{mn}, \tag{4.7}$$

(4.6) and (4.7) allow us to express the dual sequence $\{u_n\}_{n\geq 0}$ of $\{P_n(q;.)\}_{n\geq 0}$ by

$$\langle u_n,f\rangle=\frac{1}{[n]!}B(L_q)L_q^n(f)(0),\quad f\in\mathcal{P},\quad n=0,1,\ldots\,.$$ □

Proof of Theorem 4.1. Observe that $B(t)$ can be written in the form

$$\begin{aligned} B(t) &= \sum_{m\geq 0}\left(\sum_{(i_0+\ldots+i_d)=m}\begin{bmatrix} m \\ i_0,\ldots,i_d\end{bmatrix} x_0^{i_0}x_1^{i_1}\ldots x_d^{i_d}\right)\frac{t^m}{[m]!} \\ &= \sum_{m\geq 0} H_m(x_0,x_1,\ldots,x_d)\frac{t^m}{[m]!}\,. \end{aligned} \tag{4.8}$$

Replacing t by L_q in (4.8) and using (1.15) and (4.5) we derive (4.2). $\square$

We remark that equation (4.3) follows directly from (4.5).

Next, we consider two examples from Theorem 4.1.

Example 1. Consider the case: $d = 1$, $B(t) = e_q(t)e_q(at)$, $a \neq 0$. That corresponds to q-Al-Salam-Carlitz polynomials.

For this case, according to (4.2), the moments of u_0 are given by

$$C_n = \langle u_0, x^n\rangle = \sum_{m\geq 0}\frac{H_m(1,a)}{m!}\left[(x^n)^{(m)}\right]_{x=0} = H_n(1,a) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix} a^k.$$

We again have the result obtained by Al-Salam-Carlitz in [3] as mentioned in (3.9).

Example 2. Consider now the case $d = 2$. Then the polynomial set $\{P_n\}_{n\geq 0}$ generated by

$$\sum_{n\geq 0} P_n(x)\frac{t^n}{[n]!} = \frac{e_q(xt)}{e_q(t)e_q(at)e_q(bt)}, \quad a, b \in \mathbb{C}^*,$$

is a 2-OPS with respect to the 2-dimensional functional $\mathcal{U} = {}^t(u_0, u_1)$, where the moments of u_0 and u_1 are given by

$$\langle u_0, x^n\rangle = H_n(1,a,b) = \sum_{0\leq n_1,n_2\leq n}\begin{bmatrix} n \\ n_1, n_2, n-n_1-n_2\end{bmatrix} a^{n_1}b^{n_2}$$

and

$$\begin{cases} \langle u_1, x^n\rangle = \langle u_0, L_q x^n\rangle = [n]_q H_{n-1}(1,a,b) & \text{if} \quad n\geq 1, \\ \langle u_1, 1\rangle = 0. \end{cases}$$

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