Georgian Mathematical Journal
Volume 12 (2005), Number 4, 607–618

CONVERGENCE IN MEASURE OF LOGARITHMIC MEANS OF DOUBLE WALSH–FOURIER SERIES

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Abstract. The main aim of this paper is to prove that the logarithmic means of the double Walsh–Fourier series do not improve the convergence in measure. In other words, we prove that for any Orlicz space, which is not a subspace of $L\log L(I^2)$, the set of functions for which quadratic logarithmic means of the double Walsh–Fourier series converge in measure is of first Baire category.

2000 Mathematics Subject Classification: 42C10.
Key words and phrases: Double Walsh–Fourier series, Orlicz space, convergence in measure.

1. Introduction

The partial sums $S_n(f)$ of the Walsh–Fourier series of a function $f \in L(I)$, $I = [0,1]$, converge in measure on I [4, Ch. 5]. The condition $f \in L\ln^+ L(I^2)$ provides the convergence in measure on I^2 of the rectangular partial sums $S_{n,m}(f)$ of the corresponding double Walsh–Fourier series [11, Ch. 3]. The first example of a function from classes wider than $L\ln^+ L(I^2)$ with $S_{n,n}(f)$ divergent in measure on I^2 was obtained in [3]. Moreover, in every Orlicz space wider than $L\ln^+ L(I^2)$ the set of functions for which quadratic Walsh–Fourier sums converge in measure on I^2 is of first Baire category [10]. We prove (Theorem 1) that a similar proposition is also true for the Nörlund logarithmic means. It is well-known that Cesàro and Abel–Poisson means of the double Walsh–Fourier series of a function $f \in L(I^2)$ converge in L-norm [11, Ch. 3]. Thus, in question of convergence in measure Nörlund logarithmic means differ from these means and are similar to the usual rectangular Walsh–Fourier sums.

The results for subsequences of partial sums of Walsh–Fourier series can be found in [5, 7, 10].

2. Definitions and the Notation

We denote by $L^0 = L^0(I^2)$ the Lebesque space of functions that are measurable and finite almost everywhere on $I^2 = [0,1) \times [0,1)$. $\operatorname{mes}(A)$ is the Lebesque measure of the set $A \subset I^2$. The constants appearing in the article are denoted by c.

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Let $L_\Phi = L_\Phi(I^2)$ be the Orlicz space [6] generated by a Young function Φ, i.e., Φ is a convex continuous even function such that $\Phi(0) = 0$ and

$$\lim_{u\to+\infty} \frac{\Phi(u)}{u} = +\infty, \quad \lim_{u\to 0} \frac{\Phi(u)}{u} = 0.$$

This space is endowed with the norm

$$\|f\|_{L_\Phi(I^2)} = \inf\left\{k > 0 : \int_{I^2} \Phi(|f(x,y)|/k)dxdy \le 1\right\}.$$

In particular, if $\Phi(u) = u\ln(1+u)$, $u > 0$, then the corresponding space is denoted by $L\ln L(I^2)$.

It is well-known [6] that $L_\Phi \subset L_\Psi \Leftrightarrow \varliminf_{u\to\infty} \frac{\Phi(u)}{\Psi(u)} > 0$.

Recall that the Rademacher system is defined by

$$r_n(x) = r_0(2^n x), \quad n \ge 1, \quad x \in [0,1),$$

where

$$r_0(x) = \begin{cases} 1, & \text{if } x \in [0,1/2), \\ -1, & \text{if } x \in [1/2,1), \end{cases} \qquad r_0(x+1) = r_0(x).$$

Let $w_0, w_1, \dots$ stand for the Walsh functions [8] (i.e., $w_0(x) = 1$ and if $k = 2^{n_1} + \cdots + 2^{n_s}$ is a positive integer with $n_1 > n_2 > \cdots > n_s \ge 0$, then $w_k(x) = r_{n_1}(x)\cdots r_{n_s}(x)$) and

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x)$$

for the Walsh–Dirichlet kernel. Recall that

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in [0,1/2^n), \\ 0, & \text{if } x \in [1/2^n,1). \end{cases} \tag{1}$$

We consider the double system $\{w_n(x) \times w_m(y) : n,m = 0,1,2,\dots\}$ on the unit square I^2.

Under the rectangular partial sums of the double Fourier series with respect to the Walsh system the following expansions are meant:

$$S_{M,N}(f,x,y) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \hat{f}(m,n)\, w_m(x)w_n(y),$$

where the number

$$\hat{f}(m,n) = \int_{I^2} f(x,y)\, w_m(x)w_n(y)dxdy$$

is the (m,n)-th Walsh–Fourier coefficient.

The logarithmic means of the double Walsh–Fourier series is defined as follows:

$$t_{n,m}(f,x,y) = \frac{1}{l_n l_m} \sum_{i=1}^{n-1} \sum_{j=1}^{m-1} \frac{S_{i,j}(f,x,y)}{(n-i)(m-j)},$$

where

$$l_n = \sum_{k=1}^{n-1} \frac{1}{k}.$$

It is evident that

$$t_{n,m}(f,x,y) - f(x,y) = \int_0^1 [f(x \oplus t, y \oplus s) - f(x,y)] F_n(t) F_m(s)\, dt\, ds,$$

where

$$F_n(t) = \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{D_k(t)}{n-k}$$

and $\oplus$ denotes the dyadic addition [1, 4, 9].

3. Main Results

The main results of this paper are presented in the following proposition.

Theorem 1. *Let $L_\Phi(I^2)$ be an Orlicz space such that*

$$L_\Phi(I^2) \nsubseteq L \log L(I^2).$$

Then the set of functions from the Orlicz space $L_\Phi(I^2)$ with quadratic logarithmic means of double Walsh–Fourier series convergent in measure on I^2 is of first Baire category in $L_\Phi(I^2)$.

Thus, for a function from the Orlicz space, in contrast to other means (for example Cesàro or Abel means [11]), the logarithmic means of their double Walsh–Fourier series, in general, do not improve the convergence in measure.

Corollary 1. *Let $\varphi : [0,\infty[\to [0,\infty[$ be a nondecreasing function satisfying, for $x \to +\infty$, the condition*

$$\varphi(x) = o(x \log x).$$

Then there exists a function $f \in L(I^2)$ such that

a) $$\int_{I^2} \varphi(|f(x,y)|) dx dy < \infty;$$

b) *quadratic logarithmic means of the double Walsh–Fourier series of f diverge in measure on I^2.*

4. Auxiliary Results

It is well-known for the Dirichlet kernel function that

$$|D_n(x)| < \frac{1}{x}$$

for any $0 < x < 1$, which implies that for such x's we get

$$|F_n(x)| < \frac{1}{x}\,,$$

where $n \in \mathbb{N}$ is a nonnegative integer. As is well-known, for Walsh–Paley–Dirichlet kernel functions we have the following lower bound. Let $p_A = 2^{2A} + \cdots + 2^2 + 2^0$ $(A \in \mathbb{N})$. Then for any $2^{-2A-1} \le x < 1$ and $A \in \mathbb{N}$ we have

$$|D_{p_A}(x)| \ge \frac{1}{4x}\,.$$

Since this inequality plays a prominent role in the proofs of some divergence results concerning the partial sums of Fourier series, then it seems that it would be useful to get a similar inequality for the logarithmic kernels as well. We prove the inequality

$$|F_{p_A}(x)| \ge c\,\frac{\log(1/x)}{x\log p_A}$$

for all $1 \le A \in \mathbb{N}$, but not for every x in the interval $(0,1)$. We have an exceptional set such that it is "rare around zero". For $t = t_0, t_0+1, \dots, 2A$, $t_0 = \inf\{t : \left\lfloor \frac{l_{p_{[t/2]-1}}}{16} - 2^{15} \right\rfloor > 1\}$ the set $\tilde{t} := \left\lfloor \frac{l_{p_{[t/2]-1}}}{16} - 2^{15} \right\rfloor$ ($\lfloor u \rfloor$ denotes the lower integer part of u), and we take a "small part" of the interval $I_t \setminus I_{t+1} = [2^{-t-1}, 2^{-t})$. Thus we have the intervals

$$J_t := \left[\frac{1}{2^{t+1}}, \frac{1}{2^{t+1}} + \frac{1}{2^{t+\tilde{t}}}\right).$$

We define the exceptional set as

$$J := \bigcup_{t=t_0}^{\infty} J_t.$$

Lemma 1. *For $x \in \left(2^{-2A-1}, 1\right) \setminus J$ we have*

$$|F_{p_A}(x)| \ge c\frac{\log(1/x)}{x\log p_A}.$$

Before proving the lemma we remark that the exceptional set J does not seem to be a too important set, since for the upper bound function $1/x$ on the set J we have

$$\int_J \frac{1}{x}\,dx = \sum_{t=t_0}^{\infty} \int_{2^{-t-1}}^{2^{-t-1}+2^{-t-\tilde{t}}} \frac{1}{x}\,dx \le \sum_{t=t_0}^{\infty} \frac{2^{t+1}}{2^{t+\tilde{t}}} < \infty.$$

Proof. Suppose that $x \in \left(2^{-2A-1}, 2^{-t_0}\right) \setminus J$. Then we have a unique $t = t_0, t_0 + 1, \ldots, 2A$ such that $x \in \left[\frac{1}{2^{t+1}} + \frac{1}{2^{t+\bar{t}}}, \frac{1}{2^t}\right)$.

Denote

$$G_{p_A}(x) = \sum_{k=1}^{p_A - 1} \frac{D_{p_A - k}(x)}{k}.$$

Then we have

$$\begin{aligned} G_{p_A}(x) &= \sum_{k=1}^{p_{A-1}-1} \frac{1}{k} D_{p_A - k}(x) + \sum_{k=p_{A-1}}^{p_A - 1} \frac{1}{k} D_{p_A - k}(x) \\ &=: B_{1,A}(x) + B_{2,A}(x). \end{aligned}$$

Let us first discuss $B_{1,A}(x)$. Since $k < p_{A-1}$,

$$D_{p_A - k}(x) = D_{2^{2A}}(x) + r_{2A}(x) D_{p_{A-1} - k}(x).$$

This gives

$$B_{1,A}(x) = D_{2^{2A}}(x)\, l_{p_{A-1}} + r_{2A}(x)\, G_{p_{A-1}}(x). \tag{2}$$

We apply (2) recursively. This gives

$$\begin{aligned} G_{p_A}(x) &= D_{2^{2A}}(x)\, l_{p_{A-1}} + r_{2A}(x)\, G_{p_{A-1}}(x) + B_{2,A}(x) = \cdots \\ &= \sum_{j=\lfloor t/2 \rfloor}^{A} \left[\prod_{l=j+1}^{A} r_{2l}(x)\right] D_{2^{2j}}(x) l_{p_{j-1}} + \sum_{j=\lfloor t/2 \rfloor}^{A} \left[\prod_{l=j+1}^{A} r_{2l}(x)\right] B_{2,j}(x) \\ &\quad + \left[\prod_{l=\lfloor t/2 \rfloor}^{A} r_{2l}(x)\right] G_{p_{\lfloor t/2 \rfloor - 1}}(x) \\ &=: \tilde{B}_1(x) + \tilde{B}_2(x) + \tilde{B}_3(x). \end{aligned}$$

Since $2\lfloor t/2 \rfloor \le t \le 2\lfloor t/2 \rfloor + 1$, from (1) we have

$$\tilde{B}_1(x) = r_{2A}(x) \ldots r_{2\lfloor t/2 \rfloor + 2}(x) 2^{2\lfloor t/2 \rfloor} l_{p_{\lfloor t/2 \rfloor - 1}}$$

and

$$\tilde{B}_3(x) = r_{2A}(x) \ldots r_{\lfloor t/2 \rfloor}(x) G_{p_{\lfloor t/2 \rfloor - 1}}(0).$$

Consequently,

$$|\tilde{B}_1(x) + \tilde{B}_3(x)| \ge 2^{2\lfloor t/2 \rfloor} l_{p_{\lfloor t/2 \rfloor - 1}} - G_{p_{\lfloor t/2 \rfloor - 1}}(0) \ge 2^{t-2} l_{p_{\lfloor t/2 \rfloor - 1}}.$$

Now, we are to give an upper bound for the absolute value of $\tilde{B}_2(x)$ in the same way as we give an upper bound for the absolute values of its addends, i.e., for $B_{2,j}(x)$. Next, we are to apply

$$|G_{p_A}(x)| \ge 2^{t-2} l_{p_{\lfloor t/2 \rfloor - 1}} - \sum_{j=\lfloor t/2 \rfloor}^{A} |B_{2,j}(x)|. \tag{3}$$

Let us discuss $B_{2,j}$. We recall its definition

$$B_{2,j}(x) = \sum_{k=p_{j-1}}^{p_j-1} \frac{1}{k} D_{p_j-k}(x).$$

Define k' as $k = p_{j-1} + k'$. This means $k' \in [0, 2^{2j})$. Since

$$D_{p_j-k}(x) = D_{2^{2j}-k'}(x) = D_{2^{2j}}(x) - \omega_{2^{2j}-1}(x) D_{k'}(x), \tag{4}$$

we get

$$\begin{aligned} B_{2,j}(x) &= \sum_{k'=0}^{2^{2j}-1} \frac{1}{p_{j-1}+k'} \left(D_{2^{2j}}(x) - \omega_{2^{2j}-1}(x) D_{k'}(x)\right) \\ &= D_{2^{2j}}(x)\left(l_{p_j} - l_{p_{j-1}}\right) - \omega_{2^{2j}-1}(x) \sum_{k'=1}^{2^{2j}-1} \frac{1}{p_{j-1}+k'} D_{k'}(x) \\ &=: B_{2,1,j}(x) + B_{2,2,j}(x). \end{aligned} \tag{5}$$

Hence follows

$$\sum_{j=\lfloor t/2 \rfloor}^{A} |B_{2,1,j}(x)| \le 2 \sum_{j=\lfloor t/2 \rfloor}^{A} D_{2^{2j}}(x) = 2 \cdot 2^{2\lfloor t/2 \rfloor} \le 2^{t+1}. \tag{6}$$

We give an upper bound for $|B_{2,2,j}|$. By means of the Abel transform we have

$$\begin{aligned} |B_{2,2,j}(x)| &\le \sum_{k=1}^{2^{2j}-2} \left(\frac{1}{p_{j-1}+k} - \frac{1}{p_{j-1}+k+1}\right) k|K_k(x)| \\ &\quad + \frac{1}{p_j-1}\left(2^{2j}-1\right)|K_{2^{2j}-1}(x)| \\ &\le \frac{1}{2^{4j-4}} \sum_{k=1}^{2^{2j}} |kK_k(x)| + |K_{2^{2j}-1}(x)|. \end{aligned}$$

Since [1]

$$|kK_k(x)| < \frac{4}{\left(x-2^{-t-1}\right)x} + \frac{4}{x^2}, \quad x \in \left(2^{-t-1}, 2^{-t}\right),$$

for $x \in \left[\frac{1}{2^{t+1}} + \frac{1}{2^{t+\tilde{t}}}, \frac{1}{2^t}\right)$ we get

$$|kK_k(x)| < 2^{2t+\tilde{t}+3} + 2^{2t+4} < 2^{2t+\tilde{t}+4}.$$

Consequently,

$$|B_{2,2,j}(x)| \le \frac{2^{2j}}{2^{4j-4}} 2^{2t+\tilde{t}+4} + \frac{1}{2^{2j}-1} 2^{2t+\tilde{t}+4} < 2^{2t+\tilde{t}-2j+9}.$$

This inequality gives

$$\sum_{j=\left\lfloor (t+\tilde{t})/2 \right\rfloor+1}^{A} |B_{2,2,j}(x)| \le 2^9 2^t \sum_{j=\left\lfloor (t+\tilde{t})/2 \right\rfloor+1}^{A} 2^{t+\tilde{t}-2j} < 2^{10} 2^t. \tag{7}$$

On the other hand, $|D_k(x)| \le 2^t$ gives

$$|B_{2,2,j}(x)| \le \sum_{k=1}^{2^{2j}-1} \frac{1}{p_{j-1}+k} |D_k(x)| \le 2^t \frac{2^{2j}}{2^{2j-2}} = 4 \cdot 2^t. \tag{8}$$

From (6) and (7) we have

$$\sum_{j=\lfloor t/2 \rfloor}^{A} |B_{2,2,j}(x)| = \sum_{j=\lfloor t/2 \rfloor}^{\lfloor (t+\widetilde{t})/2 \rfloor} |B_{2,2,j}(x)| + \sum_{j=\lfloor (t+\widetilde{t})/2 \rfloor+1}^{A} |B_{2,2,j}(x)|$$

$$< 4 \cdot 2^t \left(\frac{\widetilde{t}}{2} + 3 \right) + 2^{10} \cdot 2^t < 2^{t-2} \left(8\widetilde{t} + 2^{13} \right). \tag{9}$$

Combining (3), (5), (6) and (9) we have

$$|G_{p_A}(x)| \ge 2^{t-2} l_{p_{\lfloor t/2 \rfloor -1}} - 2^{t-2} \left(8\widetilde{t} + 2^{13} \right) - 2^{t+1} \ge 2^{t-3} l_{p_{\lfloor t/2 \rfloor -1}} \ge c 2^t t.$$

Consequently,

$$|F_{p_A}(x)| = \frac{|G_{p_A}(x)|}{l_{p_A}} \ge c \frac{\log(1/x)}{x \log p_A}, \quad x \in \left(2^{-2A-1}, 1 \right) \setminus J.$$

This completes the proof of Lemma 1. $\square$

Corollary 2. *For* $x \in \left(2^{-2A-1}, 2^{-A} \right) \setminus J$ *we have an estimation* $|F_{p_A}(x)| \ge \frac{c}{x}$.

We apply the reasoning of [2] formulated as the following proposition in a particular case.

Theorem A. *Let* $H : L^1(I^2) \to L^0(I^2)$ *be a linear continuous operator, which commutes with the family of translations* $\mathcal{E}$, *i.e.,* $\forall E \in \mathcal{E}\ \forall f \in L^1(I^2)$ $HEf = EHf$. *Let* $\|f\|_{L^1(I^2)} = 1$ *and* $\lambda > 1$. *Then, for any* $1 \le r \in \mathbb{N}$, *under the condition* $\operatorname{mes}\{(x,y) \in I^2 : |Hf| > \lambda\} \ge \frac{1}{r}$ *there exist* $E_1, \dots, E_r, E_1', \dots, E_r' \in \mathcal{E}$ *and* $\varepsilon_i = \pm 1$, $i = 1, \dots, r$ *such that*

$$\operatorname{mes}\{(x,y) \in I^2 : |H(\sum_{i=1}^{r} \varepsilon_i f(E_i x, E_i' y)| > \lambda\} \ge \frac{1}{8}.$$

Lemma 2. *Let* $\{H_m\}_{m=1}^{\infty}$ *be a sequence of linear continuous operators, acting from the Orlicz space* $L_\Phi(I^2)$ *to the space* $L^0(I^2)$. *Suppose that there exist a sequence of functions* $\{\xi_k\}_{k=1}^{\infty}$ *from the unit ball* $S_\Phi(0,1)$ *of the space* $L_\Phi(I^2)$ *and increasing sequences of integers* $\{m_k\}_{k=1}^{\infty}$ *and* $\{\nu_k\}_{k=1}^{\infty}$ *tending to infinity such that*

$$\varepsilon_0 = \inf_k \operatorname{mes}\{(x,y) \in I^2 : |H_{m_k} \xi_k(x,y)| > \nu_k\} > 0.$$

Then B, *the set of functions* f *from the space* $L_\Phi(I^2)$, *for which the sequence* $\{H_m f\}$ *converges in measure to an a.e. finite function, is of first Baire category in the space* $L_\Phi(I^2)$.

Proof. Let a function $f \in B$. Then the sequence $\{H_m f\}$ converges, on I^2, in measure to an a.e. finite function denoted by Hf.

Since Hf is measurable, for ε_0 there exists an integer $\lambda = \lambda(\varepsilon_0)$ such that

$$\operatorname{mes}\{(x,y) \in I^2 : |Hf\,(x,y)\,| > \nu_\lambda\} < \frac{\varepsilon_0}{8}. \tag{10}$$

On the other hand, from the convergence in measure of $\{H_m f\}$ it follows that there exists an n such that for $m \geq m_n$ we have

$$\operatorname{mes}\{(x,y) \in I^2 : |H_m f\,(x,y) - Hf\,(x,y)\,| > \nu_\lambda\} < \frac{\varepsilon_0}{8}. \tag{11}$$

From (10) and (11) we conclude that for $m \geq m_n$

$$\begin{aligned}
&\operatorname{mes}\{(x,y) \in I^2 : |H_m f\,(x,y)\,| > 2\nu_\lambda\} \\
&\qquad \leq \operatorname{mes}\{(x,y) \in I^2 : |H_m f\,(x,y) - Hf\,(x,y)\,| > \nu_\lambda\} \\
&\qquad\qquad + \operatorname{mes}\{(x,y) \in I^2 : |Hf\,(x,y)\,| > \nu_\lambda\} \leq \frac{\varepsilon_0}{8} + \frac{\varepsilon_0}{8} = \frac{\varepsilon_0}{4}.
\end{aligned}$$

Therefore for $m \geq m_n$

$$f \in \left\{ g \in L_\Phi(I^2) : \operatorname{mes}\{(x,y) \in I^2 : |H_m g\,(x,y)\,| > 2\nu_\lambda\} \leq \frac{\varepsilon_0}{4} \right\}.$$

Thus, $B \subset \bigcup_{n=1}^\infty \bigcup_{\lambda=1}^\infty D_{n\lambda}$, where $D_{n\lambda} = \bigcap_{k=n}^\infty C_{k\lambda}$, and

$$C_{k\lambda} = \{g : \operatorname{mes}\{(x,y) \in I^2 : |H_{m_k} g\,(x,y)\,| > 2\nu_\lambda\} \leq \frac{\varepsilon_0}{4}\}.$$

The set $C_{k\lambda}$ is closed in the space $L_\Phi(I^2)$. Indeed, let for

$$g_i \in C_{k\lambda}, \quad i = 1, 2, \dots, \tag{12}$$

there exists a function $g \in L_\Phi(I^2)$ such that

$$\lim_{i\to\infty} \|g - g_i\|_{L_\Phi(I^2)} = 0. \tag{13}$$

From (13), under the continuity in measure of the operator H_{m_k} follows the convergence in measure of the sequence $\{H_{m_k} g_i\}$ to the function $H_{m_k} g$. Hence also follows the existence of a subsequence $\{H_{m_k} g_{i_q}\}$ which converges to $H_{m_k} g$ a.e. on I^2. Then by virtue of

$$\begin{aligned}
&\{(x,y) \in I^2 : |H_{m_k} g\,(x,y)\,| > 2\nu_\lambda\} \\
&\qquad\qquad \subseteq \bigcup_{p=1}^\infty \bigcap_{q=p}^\infty \{(x,y) \in I^2 : |H_{m_k} g_{i_q}\,(x,y)\,| > 2\nu_\lambda\}
\end{aligned}$$

and taking into account (12), we have $g \in C_{k\lambda}$.

Thus the set $D_{n\lambda}$ is closed.

We fix arbitrary $1 \leq n \in \mathbb{N}$, $1 \leq \lambda \in \mathbb{N}$, $f \in D_{n\lambda}$ and $\alpha > 0$. We put $k_0 = \max\{n, \inf_k\{\frac{4\nu_\lambda}{\alpha} < \nu_k\}\}$ and $h = f + \alpha\xi_{k_0}$.

The conditions of the lemma imply that

$$\begin{aligned}
&\operatorname{mes}\{(x,y) \in I^2 : |H_{m_{k_0}} h\,(x,y)\,| > 2\nu_\lambda\} \\
&\qquad \geq \operatorname{mes}\{(x,y) \in I^2 : |H_{m_{k_0}} \xi_{k_0}\,(x,y)\,| > \frac{4\nu_\lambda}{\alpha}\}
\end{aligned}$$

$$- \operatorname{mes}\{(x,y) \in I^2 : |H_{m_{k_0}} f(x,y)| > 2\nu_\lambda\}$$
$$\geq \operatorname{mes}\{(x,y) \in I^2 : |H_{m_{k_0}} \xi_{k_0}(x,y)| > \nu_{k_0}\} - \frac{\varepsilon_0}{4} \geq \frac{3\varepsilon_0}{4}$$

and thus $h \in S(f,\alpha) \setminus D_{n\lambda}$, i.e., f is not an inner point of the set $D_{n\lambda}$. □

Lemma 3. *Let L_Φ be an Orlicz space and let $\varphi : [0,\infty) \to [0,\infty)$ be a measurable function with the condition $\varphi(x) = o(\Phi(x))$ as $x \to \infty$. Then there exists an Orlicz space L_ω such that $\omega(x) = o(\Phi(x))$ as $x \to \infty$, and $\omega(x) \geq \varphi(x)$ for $x \geq c \geq 0$.*

Proof. Put $y_0 = 0$ and for $k = 0,1,2,\ldots$

$$x_{k+1} = \max\left\{y_k, \sup\left\{x \geq 0 : \varphi(x) \geq \frac{\Phi(x)}{k+2}\right\}\right\},$$

y_k $(1 \leq k \in \mathbb{N})$ is defined from

$$\frac{1}{k}[\Phi'_+(x_k)(y_k - x_k) + \Phi(x_k)] = \frac{\Phi(y_k)}{k+1}.$$

Consider the function $\omega(x)$

$$\omega(x) = \begin{cases} \frac{\Phi(x)}{k}, & \text{if } y_{k-1} < x \leq x_k, \\ \frac{1}{k}\left[\Phi'_+(x_k)(x - x_k) + \Phi(x_k)\right], & \text{if } x_k < x \leq y_k, \quad k = 1,2,\ldots. \end{cases}$$

It is evident that $\omega(x) \geq \varphi(x)$ for $x \geq x_1 = c$ and the function $\omega(x)$ is convex. Moreover, $\omega(x) \leq \frac{\Phi(x)}{k}$ for $x_k \leq x \leq x_{k+1}$, $k = 1,2,\ldots$, and consequently $\omega(x) = o(\Phi(x))$ as $x \to \infty$. □

5. Proof of the Theorem

Proof of Theorem 1. By Lemma 2 the proof of Theorem 1 will be complete if we show that there exist increasing sequences of integers $\{m_k : k \geq 1\}$ and $\{\nu_k : k \geq 1\}$ tending to infinity, and a sequence of functions $\{\xi_k : k \geq 1\}$ from the unit bull $S_\Phi(0,1)$ of the Orlicz space $L_\Phi(I^2)$, such that for all k

$$\operatorname{mes}\{(x,y) \in I^2 : \left|t_{p_{m_k}, p_{m_k}}(\xi_k; x, y)\right| > \nu_k\} \geq \frac{1}{8}. \tag{14}$$

First we prove that there exists $c > 0$ such that

$$\operatorname{mes}\{(x,y) \in I^2 : |t_{p_n,p_n}(D_{2^{2n+1}} \otimes D_{2^{2n+1}}; x,y)| > 2^{3n}\} > c\frac{n}{2^{3n}}. \tag{15}$$

Denote

$$a_n = 2^{-n-1} + 2^{-n-\widetilde{n}}, \quad b_n = 2^{-n}, \quad G_n = \bigcup_{k=n}^{2n} (a_k, b_k).$$

Since

$$t_{p_n}(D_{2^{2n+1}}, x) = S_{2^{2n+1}}(F_{p_n}, x) = F_{p_n}(x),$$

for $(x,y)\in G_n\times G_n$ we have from Corollary 2 for quadratic logarithmic means of $D_{2^{2n+1}}(x)D_{2^{2n+1}}(y)$ the estimation

$$|F_{p_n}(x)F_{p_n}(y)|=|t_{p_n,p_n}(D_{2^{2n+1}}\otimes D_{2^{2n+1}};x,y)|\geq \frac{c}{xy}.$$

By virtue of $n_0\geq 2$, for $n\geq n_0+1$ we have

$$b_k-a_k\geq a_{k-1}-b_k.$$

Thus, for $k_1,k_2\geq n_0+1$, we get

$$\operatorname{mes}(a_{k_1},b_{k_1})\times(a_{k_2},b_{k_2})\geq\frac{1}{4}\operatorname{mes}(a_{k_1},a_{k_1-1})\times(a_{k_2},a_{k_2-1}).$$

Since the function $\frac{1}{xy}$ is decreasing with respect to x and y, we have

$$\begin{aligned}
&\operatorname{mes}\{(x,y)\in I^2:|t_{p_n,p_n}(D_{2^{2n+1}}\otimes D_{2^{2n+1}};x,y)|>2^{3n}\}\\
&>\operatorname{mes}\left\{(x,y)\in G_n\times G_n:\frac{c}{xy}>2^{3n}\right\}\\
&\geq\frac{1}{4}\operatorname{mes}\left\{(x,y)\in(2^{-2n},2^{-n})\times(2^{-2n},2^{-n}):\frac{c}{xy}>2^{3n}\right\}-O\left(\frac{1}{2^{3n}}\right)\\
&\geq\frac{1}{4}\int\limits_{2^{-2n}}^{2^{-n}}\left(\frac{c}{2^{3n}x}-\frac{1}{2^{2n}}\right)dx-O\left(\frac{1}{2^{3n}}\right)\\
&=\frac{cn}{2^{3n+2}}-O\left(\frac{1}{2^{3n}}\right)\\
&>c\frac{n}{2^{3n}}.
\end{aligned}$$

Hence (15) is proved.

From the condition of the theorem we write [6, Ch.2]

$$\liminf_{u\to\infty}\frac{\Phi(u)}{u\log u}=0.$$

Consequently, there exists an increasing sequence of integers $\{m_k\}_{k=1}^{\infty}$ tending to infinity, such that

$$\lim_{k\to\infty}\Phi(2^{4m_k})2^{-4m_k}m_k^{-1}=0,\quad \Phi(2^{4m_k})\geq 2^{4m_k+2},\quad\forall k.$$

By (15) we write

$$\operatorname{mes}\{(x,y)\in I^2:|t_{p_{m_k},p_{m_k}}(D_{2^{2m_k+1}}\otimes D_{2^{2m_k+1}};x,y)|>2^{3m_k}\}>c\frac{m_k}{2^{3m_k}}.$$

Then, by virtue of Theorem A, there exist $e_1,\dots,e_r,e_1',\dots,e_r'\in[0,1]$ and $\varepsilon_1,\dots,\varepsilon_r=\pm1$ such that

$$\operatorname{mes}\left\{(x,y)\in I^2:|\sum_{i=1}^{r}\varepsilon_i t_{p_{m_k},p_{m_k}}(D_{2^{2m_k+1}}\otimes D_{2^{2m_k+1}};e_i\oplus x,e_i'\oplus y)|>2^{3m_k}\right\}>\frac{1}{8},$$

where $r=\left[\frac{2^{3m_k}}{cm_k}\right]+1$.

Denote

$$\nu_k = \frac{2^{7m_k-1}}{r\Phi\left(2^{4m_k}\right)}$$

and

$$\xi_k\left(x,y\right) = \frac{2^{4m_k-1}}{\Phi\left(2^{4m_k}\right)} M_k\left(x,y\right),$$

where

$$M_k\left(x,y\right) = \frac{1}{r}\sum_{i=1}^{r} \varepsilon_i D_{2^{2m_k+1}}\left(e_i \oplus x\right) D_{2^{2m_k+1}}\left(e_i' \oplus y\right).$$

Thus we obtain (14).

Moreover, since Φ is convex, we have $\xi_k \in S_\Phi(0,1)$. Indeed, the estimations $\|M_k\|_{L^\infty(I^2)} \leq 2^{4m_k+2}$ and $\|M_k\|_{L^1(I^2)} \leq 1$ imply that

$$\|\xi_k\|_{L_\Phi(I^2)} \leq \frac{1}{2}\left[1 + \int_{I^2} \Phi\left(\frac{2^{4m_k}\left|M_k\left(x,y\right)\right|}{\Phi(2^{4m_k})}\right) dxdy\right] \leq 1.$$

Theorem 1 is proved. □

The validity of Corollary 1 follows immediately from Theorem 1 and Lemma 3.

Acknowledgement

The first author was supported by the Hungarian National Foundation for Scientific Research (OTKA), grant nos. M 36511/2001 and T 048780, and by the Széchenyi fellowship of the Hungarian Ministry of Education Szö 184/2003.

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(Received 17.03.2005)

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