

WAVELET FRAMES FOR DISTRIBUTIONS IN ANISOTROPIC BESOV SPACES

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Abstract. This paper deals with wavelet frames in anisotropic Besov spaces $B_{pq}^{s,a}(\mathbb{R}^n)$, $s \in \mathbb{R}$, $0 < p, q \leq \infty$, and $a = (a_1, \dots, a_n)$ is an anisotropy, with $a_i > 0$, $i = 1, \dots, n$, $a_1 + \dots + a_n = n$. We present sub-atomic and wavelet decompositions for a large class of distributions. To some extent our results can be regarded as anisotropic counterparts of those recently obtained in [38].

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INTRODUCTION

Let $1 < p < \infty$ and (s_1, \dots, s_n) be an n -tuple of natural numbers, then

$$W_p^{s,a}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{L_p(\mathbb{R}^n)} + \sum_{k=1}^n \left\| \frac{\partial^{s_k} f}{\partial x_k^{s_k}} \right\|_{L_p(\mathbb{R}^n)} < \infty \right\} \quad (0.1)$$

is the classical anisotropic Sobolev space on \mathbb{R}^n . It is obvious that, unlike the case of the usual (isotropic) Sobolev space ($s_1 = \dots = s_n$), the smoothness properties of an element from $W_p^{s,a}(\mathbb{R}^n)$ depend on a chosen direction in \mathbb{R}^n . The number s , defined by

$$\frac{1}{s} = \frac{1}{n} \left(\frac{1}{s_1} + \dots + \frac{1}{s_n} \right) \quad (0.2)$$

is usually called the *mean smoothness*, and $a = (a_1, \dots, a_n)$,

$$a_1 = \frac{s}{s_1}, \dots, a_n = \frac{s}{s_n} \quad (0.3)$$

characterizes the *anisotropy*. Similar to the isotropic situation, more general anisotropic Bessel potential spaces (fractional Sobolev spaces) $H_p^{s,a}(\mathbb{R}^n)$, where $1 < p < \infty$, $s \in \mathbb{R}$ and $a = (a_1, \dots, a_n)$ is a given anisotropy, fit in the scales of anisotropic Besov spaces $B_{pq}^{s,a}(\mathbb{R}^n)$ and anisotropic Triebel–Lizorkin spaces $F_{pq}^{s,a}(\mathbb{R}^n)$, respectively. The theory of anisotropic spaces was developed parallel to the theory for isotropic spaces; we refer in particular to the monographs [25] and [4] (and the papers referred to therein). It is well known that this theory is a more or less complete counterpart to the basic facts (definitions, description via differences and derivatives, elementary properties, embeddings for different metrics, interpolation) of isotropic spaces $B_{pq}^s(\mathbb{R}^n)$ and $F_{pq}^s(\mathbb{R}^n)$. We shall use the Fourier-analytical definition of $B_{pq}^{s,a}(\mathbb{R}^n)$, $F_{pq}^{s,a}(\mathbb{R}^n)$, where any function $f \in$

$\mathcal{S}'(\mathbb{R}^n)$ is decomposed in a sum of entire analytic functions $(\varphi_j \widehat{f})^\vee$ and this decomposition, measured in ℓ_q and $L_p(\mathbb{R}^n)$, respectively, is used to introduce these spaces. This concept goes back to [32] and [31], see [27, Chapter 4]. The theory of isotropic spaces was studied systematically in [33], [34], [35], and, more recently, in [36].

In the theory of isotropic function spaces it has turned out that other well-known types of decompositions in simpler building blocks like atoms, quarks or wavelets, are better adapted to some problems and applications connected with the study of function spaces. In contrast to sub-atomic decompositions (based on the so-called *quarks*) introduced not long ago in [35], see also [36], atomic decompositions have a history of some twenty years and a historical report on this topic is given in [34, 1.9], cf. also [1]. We only want to recall that (smooth) atoms in isotropic spaces $B_{pq}^s(\mathbb{R}^n)$ and $F_{pq}^s(\mathbb{R}^n)$ as defined in [14], [15] (cf. also [16]) proved to be a powerful tool in the theory of function spaces.

In the last years several authors have been concerned with the problem of obtaining useful decompositions of anisotropic function spaces, too. A construction of unconditional bases in spaces $B_{pq}^{s,a}(\mathbb{R}^n)$ and $F_{pq}^{s,a}(\mathbb{R}^n)$ using Meyer wavelets was obtained in [2], [3]; see, more recent works [18], [17], [19]; a different approach, involving the φ -transform of Frazier and Jawerth (see [15], [16]) was developed in [9], [10], see also [28]. The most recent contributions we know are made in [7], [21], [22]; see also [5], [6].

Our main aim in the present paper is to prove an anisotropic counterpart of some recent results on wavelet frames parallel to the isotropic case in [38]. Essential ingredients in our approach are representations by local means, atomic and sub-atomic decompositions according to [13], [12]; see also [8], [41], [42]. More precisely, let $a = (a_1, \dots, a_n)$ be a given anisotropy, $|\cdot|_a$ an anisotropic distance, and k be a non-negative C^∞ -function in \mathbb{R}^n such that $\text{supp } k \subset \{y \in \mathbb{R}^n : |y|_a < 2^J, y_j > 0, j = 1, \dots, n\}$ for some $J \in \mathbb{N}$, with

$$\sum_{m \in \mathbb{Z}^n} k(x - m) = 1, \quad x \in \mathbb{R}^n.$$

For $\beta \in \mathbb{N}_0^n$ and $x \in \mathbb{R}^n$, put $k^\beta(x) = (2^{-Ja}x)^\beta k(x)$, and let for $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$k^\beta(t, f)(x) = \int_{\mathbb{R}^n} k^\beta(y) f(x + t^a y) dy, \quad t > 0, \quad x \in \mathbb{R}^n, \quad (0.4)$$

be the corresponding (anisotropic) local means (see [34, 2.4.6/1] for the isotropic version and [13] for its anisotropic counterpart). Let $\omega \in \mathcal{S}(\mathbb{R}^n)$ with $\text{supp } \omega \subset (-\pi, \pi)^n$, $\omega(x) = 1$ if $|x|_a \leq 2$. Parallel to the isotropic case we introduce

$$\omega^\beta(x) = \frac{i^{|\beta|} 2^{Ja\beta}}{(2\pi)^n \beta!} x^\beta \omega(x) \quad \text{and} \quad \Omega^\beta(x) = \sum_{m \in \mathbb{Z}^n} (\omega^\beta)^\vee(m) e^{-imx},$$

where $x \in \mathbb{R}^n$, $\beta \in \mathbb{N}_0^n$. Assume that φ_0 is some C^∞ -function on \mathbb{R}^n with $\varphi_0(x) = 1$ if $|x|_a \leq 1$, and $\varphi_0(x) = 0$ if $|x|_a \geq \frac{3}{2}$; put $\varphi(x) = \varphi_0(x) - \varphi_0(2^a x)$. For $\beta \in \mathbb{N}_0^n$ we introduce (anisotropic) *father wavelets* $\Phi_F^\beta(x)$ and *mother wavelets*

$\Phi_M^\beta(x)$ by

$$\left(\Phi_F^\beta\right)^\vee(\xi) = \varphi_0(\xi)\Omega^\beta(\xi), \quad \left(\Phi_M^\beta\right)^\vee(\xi) = \varphi(\xi)\Omega^\beta(\xi), \quad \xi \in \mathbb{R}^n,$$

and use the abbreviation

$$\Phi_{jm}^\beta(x) = \begin{cases} \Phi_F^\beta(x-m), & \text{if } j=0, \\ \Phi_M^\beta(2^{ja}x-m), & \text{if } j \in \mathbb{N}, \end{cases} \quad (0.5)$$

$\beta \in \mathbb{N}_0^n$, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$. Our first main result now establishes a quarkonial (sub-atomic) decomposition in the sense that for given $0 < p \leq \infty$, $s > n \max\left(\frac{1}{p} - 1, 0\right)$, $\varrho \geq 0$, and an anisotropy a , $f \in \mathcal{S}'(\mathbb{R}^n)$ is an element of $B_{pp}^{s,a}(\mathbb{R}^n)$ if and only if it can be represented as

$$f = \sum_{\beta \in \mathbb{N}_0^n} \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}^\beta k^\beta(2^{ja}x-m), \quad x \in \mathbb{R}^n, \quad (0.6)$$

the series being absolutely convergent in $L_{\max(p,1)}(\mathbb{R}^n)$, with

$$\left(\sum_{\beta \in \mathbb{N}_0^n} \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{\varrho a \beta p + j(s-n/p)p} \left| \lambda_{jm}^\beta \right|^p \right)^{1/p} < \infty.$$

Moreover, for $f \in B_{pp}^{s,a}(\mathbb{R}^n)$, (0.6) with λ_{jm}^β replaced by $\lambda_{jm}^\beta(f) = 2^{jn}(f, \Phi_{jm}^\beta)$, $\beta \in \mathbb{N}_0^n$, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, is an optimal representation, i.e.,

$$\|f\|_{B_{pp}^{s,a}(\mathbb{R}^n)} \sim \left(\sum_{\beta \in \mathbb{N}_0^n} \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{\varrho a \beta p + j(s-n/p)p} \left| \lambda_{jm}^\beta(f) \right|^p \right)^{1/p}.$$

In the case of negative s we have a parallel “dual” result, namely that for $1 < p \leq \infty$, $s < 0$, and a given anisotropy a , $f \in \mathcal{S}'(\mathbb{R}^n)$ is an element of $B_{pp}^{s,a}(\mathbb{R}^n)$ if and only if it can be represented as

$$f = \sum_{\beta \in \mathbb{N}_0^n} \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}^\beta \Phi_{jm}^\beta, \quad x \in \mathbb{R}^n, \quad (0.7)$$

the series being unconditionally convergent in $\mathcal{S}'(\mathbb{R}^n)$, with

$$\left(\sum_{\beta \in \mathbb{N}_0^n} \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{j(s-n/p)p} \left| \lambda_{jm}^\beta \right|^p \right)^{1/p} < \infty.$$

Again, we have an optimal representation for $f \in B_{pp}^{s,a}(\mathbb{R}^n)$ if in (0.7) we replace the coefficients λ_{jm}^β by $k_{jm}^\beta(f) = k^\beta(2^{-j}f)(2^{-ja}m)$, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, $\beta \in \mathbb{N}_0^n$, see (0.4).

The paper is organized as follows. In Section 1 we collect fundamentals about anisotropic Besov spaces and recall the atomic decomposition in such spaces. Our main results are presented in Section 2, whereas all the proofs (and some auxiliary assertions) are given in Section 3.

1. PRELIMINARIES

1.1. General notation. As usual, \mathbb{R}^n denotes the n -dimensional real Euclidean space, \mathbb{N} the collection of all natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, \mathbb{C} stands for complex numbers, and \mathbb{Z}^n means the lattice of all points in \mathbb{R}^n with integer-valued components. We use the equivalence “ \sim ” in $\varphi(x) \sim \psi(x)$ always to mean that there are two positive numbers c_1 and c_2 such that

$$c_1 \varphi(x) \leq \psi(x) \leq c_2 \varphi(x)$$

for all admitted values of x , where φ, ψ are non-negative functions. If $a \in \mathbb{R}$, then $a_+ := \max(a, 0)$. Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ be a multi-index, then

$$|\alpha| = \alpha_1 + \dots + \alpha_n, \quad \alpha! = \alpha_1! \cdots \alpha_n!, \quad \alpha \in \mathbb{N}_0^n, \quad (1.1)$$

the derivatives D^α have the usual meaning, x^α means $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, and $\alpha\gamma = \alpha_1\gamma_1 + \dots + \alpha_n\gamma_n$, $\gamma \in \mathbb{R}^n$, stands for the scalar product in \mathbb{R}^n .

Given two quasi-Banach spaces X and Y , we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X in Y is continuous. All unimportant positive constants will be denoted by c , occasionally with additional subscripts within the same formula. We shall always deal with function spaces on \mathbb{R}^n ; so for convenience we shall usually omit the “ \mathbb{R}^n ” in their notation.

1.2. Anisotropic function spaces. Let $a = (a_1, \dots, a_n)$ be a fixed n -tuple of positive numbers with $a_1 + \dots + a_n = n$, then we call a an *anisotropy*. We denote $a_{\min} = \min\{a_i : 1 \leq i \leq n\}$ and $a_{\max} = \max\{a_i : 1 \leq i \leq n\}$. If $a = (1, \dots, 1)$ we speak about the “isotropic case”.

The action of $t \in [0, \infty)$ on $x \in \mathbb{R}^n$ is defined by the formula

$$t^a x = (t^{a_1} x_1, \dots, t^{a_n} x_n). \quad (1.2)$$

For $t > 0$ and $s \in \mathbb{R}$ we put $t^{sa} x = (t^s)^a x$. In particular we write $t^{-a} x = (t^{-1})^a x$ and $2^{-ja} x = (2^{-j})^a x$.

Definition 1.1. An *anisotropic distance function* is a continuous function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ with the properties $u(x) > 0$ if $x \neq 0$ and $u(t^a x) = tu(x)$ for all $t > 0$ and all $x \in \mathbb{R}^n$.

Remark 1.2. It is easy to see that $u_\lambda : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by

$$u_\lambda(x) = \left(\sum_{i=1}^n |x_i|^{\frac{\lambda}{a_i}} \right)^{1/\lambda} \quad (1.3)$$

is an anisotropic distance function for every $0 < \lambda < \infty$, u_2 is usually called an anisotropic distance from x to the origin, see [27, 4.2.1]. It is well known, see [10, 1.2.3] and [41, 1.4], that any two anisotropic distance functions u and u' are equivalent (in the sense that there exist constants $c, c' > 0$ such that $cu(x) \leq u'(x) \leq c'u(x)$ for all $x \in \mathbb{R}^n$) and that if u is an anisotropic distance function there exists a constant $c > 0$ such that $u(x+y) \leq c(u(x)+u(y))$ for all $x, y \in \mathbb{R}^n$. We are interested in using smooth anisotropic distance functions. Note that for appropriate values of λ one can obtain arbitrary (finite) smoothness of the

function u_λ from (1.3), cf. [10, 1.2.4]. A standard method concerning the construction of anisotropic distance functions in $C^\infty(\mathbb{R}^n \setminus \{0\})$ is given in [30].

For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $x \neq 0$, let $|x|_a$ be a unique positive number t such that

$$\frac{x_1^2}{t^{2a_1}} + \dots + \frac{x_n^2}{t^{2a_n}} = 1 \quad (1.4)$$

and let $|0|_a = 0$; then $|\cdot|_a$ is an anisotropic distance function in $C^\infty(\mathbb{R}^n \setminus \{0\})$, see [41, 1.4/3,8]. Plainly, in the isotropic case $|x|_a$ is the Euclidean distance from x to the origin.

Before introducing the function spaces to be considered we need to recall some notation. By \mathcal{S} we denote the Schwartz space of all complex-valued, infinitely differentiable and rapidly decreasing functions on \mathbb{R}^n and by \mathcal{S}' the dual space of all tempered distributions on \mathbb{R}^n . Furthermore, L_p with $0 < p \leq \infty$, stands for the usual quasi-Banach space with respect to the Lebesgue measure quasi-normed by

$$\|f\|_{L_p} := \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p},$$

with the obvious modification if $p = \infty$. If $\varphi \in \mathcal{S}$, then

$$\widehat{\varphi}(\xi) \equiv (\mathcal{F}\varphi)(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} \varphi(x) dx, \quad x \in \mathbb{R}^n, \quad (1.5)$$

denotes the Fourier transform of φ . As usual, $\mathcal{F}^{-1}\varphi$ or φ^\vee stands for the inverse Fourier transform given by the right-hand side of (1.5) with i in place of $-i$. Here $x\xi$ denotes the scalar product in \mathbb{R}^n . Both \mathcal{F} and \mathcal{F}^{-1} are extended to \mathcal{S}' in standard manner. Let $\varphi_0 \in \mathcal{S}$ be such that

$$\varphi_0(x) = 1 \quad \text{if } |x|_a \leq 1 \quad \text{and} \quad \text{supp } \varphi_0 \subset \{x \in \mathbb{R}^n : |x|_a \leq 2\}, \quad (1.6)$$

and for each $j \in \mathbb{N}$ let

$$\varphi_j^a(x) := \varphi_0(2^{-ja}x) - \varphi_0(2^{-(j+1)a}x), \quad x \in \mathbb{R}^n. \quad (1.7)$$

Then the sequence $(\varphi_j^a)_{j=0}^\infty$ forms a smooth anisotropic dyadic resolution of unity, cf. [27, 4.2]. Let $f \in \mathcal{S}'$, then the compact support of $\varphi_j^a \widehat{f}$ implies by the Paley–Wiener–Schwartz theorem that $(\varphi_j^a \widehat{f})^\vee$ is an entire analytic function on \mathbb{R}^n .

Definition 1.3. Assume $0 < p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, a be an anisotropy, and $(\varphi_j^a)_{j=0}^\infty$ a smooth anisotropic dyadic resolution of unity. Then

$$B_{pq}^{s,a} = \left\{ f \in \mathcal{S}' : \|f\|_{B_{pq}^{s,a}} = \left(\sum_{j=0}^\infty 2^{jsq} \|(\varphi_j^a \widehat{f})^\vee\|_{L_p}^q \right)^{1/q} < \infty \right\} \quad (1.8)$$

(with the usual modification if $q = \infty$).

Note that there is a parallel definition for spaces of type $F_{pq}^{s,a}$, $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, a an anisotropy, when interchanging the order of ℓ_q - and L_p -quasi-norms in (1.8). It is obvious, that the quasi-norm (1.8) depends on the chosen system $(\varphi_j^a)_{j \in \mathbb{N}_0}$, but not the space $B_{pq}^{s,a}$ (in the sense of equivalent quasi-norms); therefore in the sequel we omit the subscript φ in our notation. It is well known that $B_{pq}^{s,a}$ are quasi-Banach spaces (Banach spaces if $p \geq 1$ and $q \geq 1$), and, as in the isotropic case, $\mathcal{S} \hookrightarrow B_{pq}^{s,a} \hookrightarrow \mathcal{S}'$ for all admissible values of p, q, s , see [33, 2.3.3]. If $s \in \mathbb{R}$ and $0 < p < \infty$, $0 < q < \infty$, then \mathcal{S} is dense in $B_{pq}^{s,a}$, see [41, 3.5] and [10, 1.2.10]. Note that we have indicated the only (formal) difference from the isotropic counterparts of (1.8) by the additional superscript at the smooth anisotropic dyadic resolution of unity $(\varphi_j^a)_{j=0}^\infty$.

Remark 1.4. A systematic treatment of the theory of (isotropic) B_{pq}^s (and F_{pq}^s) spaces can be found in the monographs [33], [34], [35] and [36]; see also [11] and [26]. A survey on the basic results for the (anisotropic) spaces $B_{pq}^{s,a}$ (and $F_{pq}^{s,a}$) is given in [27, 4.2.1–4.2.4] and [20, 2.1–2.2]. In addition to the literature mentioned in our introduction, in the sequel we essentially rely on [13] and [12].

For convenience, in the case of $p = q$ we shall stick to the notation

$$B_p^{s,a} = B_{pp}^{s,a} \quad \text{where} \quad 0 < p \leq \infty, \quad s \in \mathbb{R}, \quad (1.9)$$

in the sequel.

To prove our main theorems, we need the following proposition, which is interesting in itself; thus we state it here separately.

Proposition 1.5. *Let $0 < p \leq \infty$, $s \in \mathbb{R}$, a be an anisotropy, and $(\varphi_j^a)_{j=0}^\infty$ a smooth anisotropic dyadic resolution of unity. Then for each $f \in B_p^{s,a}$,*

$$\left(\sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{j(s-n/p)p} |(\varphi_j^a \widehat{f})^\vee(2^{-ja}m)|^p \right)^{1/p} \sim \|f\|_{B_p^{s,a}} \quad (1.10)$$

(with the usual modification if $p = \infty$), where the equivalence constants are independent of s and f .

1.3. Decomposition of anisotropic function spaces. Our main results formulated in Section 2 concern sub-atomic and wavelet representations of anisotropic spaces. The arguments given there are essentially based on the (known) corresponding atomic decompositions. Thus we recall some basic facts about anisotropic atoms and mainly rely on [13] in that context; see also [12].

Let $a = (a_1, \dots, a_n)$, $\nu \in \mathbb{N}_0$, and $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$, then we denote by $R_{\nu m}^a$ the rectangle in \mathbb{R}^n centered at $2^{-\nu a}m = (2^{-\nu a_1}m_1, \dots, 2^{-\nu a_n}m_n)$ which has the sides parallel to the axes and side lengths $2^{-\nu a_1}, \dots, 2^{-\nu a_n}$, respectively. Note that R_{0m}^a is the cube with side length 1. If $R_{\nu m}^a$ is such a rectangle in \mathbb{R}^n and $c > 0$, then $cR_{\nu m}^a$ is the rectangle in \mathbb{R}^n concentric with $R_{\nu m}^a$ and with side lengths $c2^{-\nu a_1}, \dots, c2^{-\nu a_n}$, respectively.

Definition 1.6.

- (i) Let $K \in \mathbb{R}$, $c > 1$. A function $\varrho : \mathbb{R}^n \rightarrow \mathbb{C}$ for which there exist all derivatives $D^\gamma \varrho$ if $a\gamma \leq K$ (continuous if $K \leq 0$) is called an *anisotropic 1_K -atom* if

$$\text{supp } \varrho \subset cR_{0m}^a \quad \text{for some } m \in \mathbb{Z}^n, \quad (1.11)$$

$$|D^\gamma \varrho(x)| \leq 1 \quad \text{if } a\gamma \leq K, \quad \gamma \in \mathbb{N}_0^n, \quad x \in \mathbb{R}^n. \quad (1.12)$$

- (ii) Let $s \in \mathbb{R}$, $0 < p \leq \infty$, $K, L \in \mathbb{R}$. A function $\varrho : \mathbb{R}^n \rightarrow \mathbb{C}$ for which there exist all derivatives $D^\gamma \varrho$ if $a\gamma \leq K$ (continuous if $K \leq 0$) is called an *anisotropic $(s, p)_{K,L}$ -atom* if

$$\text{supp } \varrho \subset cR_{\nu m}^a \quad \text{for some } \nu \in \mathbb{N}, \quad m \in \mathbb{Z}^n, \quad (1.13)$$

$$|D^\gamma \varrho(x)| \leq 2^{-\nu(s-\frac{n}{p})+\nu a\gamma} \quad \text{if } a\gamma \leq K, \quad \gamma \in \mathbb{N}_0^n, \quad x \in \mathbb{R}^n, \quad (1.14)$$

$$\int_{\mathbb{R}^n} x^\beta \varrho(x) dx = 0 \quad \text{if } a\beta \leq L, \quad \beta \in \mathbb{N}_0^n. \quad (1.15)$$

If the atom ϱ is located at $R_{\nu m}^a$ (that means $\text{supp } \varrho \subset cR_{\nu m}^a$ with $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, $c > 1$), then we denote it by $\varrho_{\nu m}^a$.

Remark 1.7. The value of $c > 1$ in (1.11) and (1.13) is unimportant; it only indicates that at level ν some controlled overlapping of the supports of $\varrho_{\nu m}^a$ has to be allowed. The moment conditions (1.15) can be rewritten as

$$D^\beta \widehat{\varrho}(0) = 0 \quad \text{if } a\beta \leq L,$$

which shows that a sufficiently strong decay of $\widehat{\varrho}$ at the origin is required. If $L < 0$ then (1.15) should be interpreted in the sense that there are no moment conditions. The normalizing factors in (1.12), (1.14) imply that there exists a constant $c > 0$ such that for all these atoms we have $\|\varrho\|_{B_{pq}^{s,a}} \leq c$, see Theorem 1.9 below. Hence, as in the isotropic case, atoms are normalized building blocks satisfying certain moment conditions.

This construction generalizes isotropic atoms (cf. [14], [15] and the survey [16]). It is also slightly related to the concept of anisotropic building blocks (compactly supported and satisfying some norming and some moment conditions) used in [29] to define anisotropic Hardy spaces and to study the relation of these spaces to anisotropic Lipschitz and Campanato–Morrey spaces. As already mentioned, we use the presentation from [13], which itself was motivated by the isotropic counterparts in [34], [35].

Suitable anisotropic sequence spaces can be introduced as follows.

Definition 1.8. Let $0 < p \leq \infty$, $0 < q \leq \infty$, and a be an anisotropy. Then b_{pq} is the collection of all sequences $\lambda = \{\lambda_{\nu m} \in \mathbb{C} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$ such that

$$\|\lambda\|_{b_{pq}} = \left(\sum_{\nu=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^p \right)^{q/p} \right)^{1/q} \quad (1.16)$$

(with the usual modification if $p = \infty$ and/or $q = \infty$) is finite.

Again, note that there is a counterpart for spaces of type $F_{pq}^{s,a}$; the corresponding sequence spaces f_{pq}^a can be introduced similarly, but we do not need them in the sequel. One can easily check that b_{pq} are quasi-Banach spaces. For $0 < p \leq \infty$ we introduce the abbreviation

$$\sigma_p = n \left(\frac{1}{p} - 1 \right)_+ . \quad (1.17)$$

Theorem 1.9. *Let $0 < p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and let $K, L \in \mathbb{R}$ be such that*

$$K \geq a_{\max} + s \quad \text{if } s \geq 0, \quad (1.18)$$

$$L \geq \sigma_p - s. \quad (1.19)$$

Then $g \in \mathcal{S}'$ belongs to $B_{pq}^{s,a}$ if and only if it can be represented as

$$g = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \varrho_{\nu m}^a, \quad (1.20)$$

the series being unconditionally convergent in \mathcal{S}' , where $\varrho_{\nu m}^a$ are anisotropic 1_K -atoms ($\nu = 0$) or anisotropic $(s, p)_{K,L}$ -atoms ($\nu \in \mathbb{N}$) and $\lambda \in b_{pq}$ with $\lambda = \{\lambda_{\nu m} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$. Moreover,

$$\inf \|\lambda|b_{pq}\|,$$

where the infimum is taken over all admissible representations (1.20), is an equivalent quasi-norm in $B_{pq}^{s,a}$.

Remark 1.10. The proof of this theorem – and of its counterpart for spaces $F_{pq}^{s,a}$ – is given in [13, Section 5.1]. The convergence in \mathcal{S}' can be obtained as a by-product of the proof using the same method as for its isotropic counterpart in [35, 13.9]. As already mentioned, it generalizes atomic decomposition results in [14], [15], [35] to anisotropic function spaces.

Our main object is to study the anisotropic counterpart of the results from [38]; hence we closely follow that presentation, adapting it to our context when necessary, and keeping similar notation if possible. Let

$$\mathbb{R}_{++}^n = \{y \in \mathbb{R}^n : y = (y_1, \dots, y_n), y_j > 0 \text{ for all } j\} \quad (1.21)$$

and let k be a non-negative C^∞ -function in \mathbb{R}^n with

$$\text{supp } k \subset \{y \in \mathbb{R}^n : |y|_a < 2^J\} \cap \mathbb{R}_{++}^n, \quad (1.22)$$

for some $J \in \mathbb{N}$, and

$$\sum_{m \in \mathbb{Z}^n} k(x - m) = 1, \quad x \in \mathbb{R}^n. \quad (1.23)$$

Recall that $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$ where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\beta \in \mathbb{N}_0^n$, and put

$$k^\beta(x) = (2^{-Ja}x)^\beta k(x) \geq 0, \quad x \in \mathbb{R}^n, \beta \in \mathbb{N}_0^n. \quad (1.24)$$

Let

$$\lambda = \{\lambda_{jm}^\beta \in \mathbb{C} : j \in \mathbb{N}_0, m \in \mathbb{Z}^n, \beta \in \mathbb{N}_0^n\}. \quad (1.25)$$

For $s \in \mathbb{R}$, $0 < p \leq \infty$, and $\varrho \geq 0$, we introduce $b_p^{s,\varrho}$ by

$$\|\lambda|b_p^{s,\varrho}\| = \left(\sum_{\beta \in \mathbb{N}_0^n} \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{\varrho a \beta p + j(s-n/p)p} |\lambda_{jm}^\beta|^p \right)^{1/p}. \quad (1.26)$$

Let

$$\omega \in \mathcal{S}, \quad \text{supp } \omega \subset (-\pi, \pi)^n, \quad \omega(x) = 1 \quad \text{if } |x|_a \leq 2, \quad (1.27)$$

and, taking into account (1.1), let

$$\omega^\beta(x) = \frac{i^{|\beta|} 2^{Ja\beta}}{(2\pi)^n \beta!} x^\beta \omega(x) \quad \text{for } x \in \mathbb{R}^n, \quad \beta \in \mathbb{N}_0^n. \quad (1.28)$$

Finally, let

$$\Omega^\beta(x) = \sum_{m \in \mathbb{Z}^n} (\omega^\beta)^\vee(m) e^{-imx}, \quad x \in \mathbb{R}^n, \quad \beta \in \mathbb{N}_0^n. \quad (1.29)$$

Definition 1.11. Let φ_0 be a C^∞ -function in \mathbb{R}^n with

$$\varphi_0(x) = 1 \quad \text{if } |x|_a \leq 1, \quad \text{and} \quad \varphi_0(x) = 0 \quad \text{if } |x|_a \geq \frac{3}{2}, \quad (1.30)$$

and let $\varphi(x) = \varphi_0(x) - \varphi_0(2^a x)$ and $\beta \in \mathbb{N}_0^n$. *Father wavelets* $\Phi_F^\beta(x)$ and *mother wavelets* $\Phi_M^\beta(x)$ are given by

$$\left(\Phi_F^\beta \right)^\vee(\xi) = \varphi_0(\xi) \Omega^\beta(\xi), \quad \xi \in \mathbb{R}^n, \quad (1.31)$$

$$\left(\Phi_M^\beta \right)^\vee(\xi) = \varphi(\xi) \Omega^\beta(\xi), \quad \xi \in \mathbb{R}^n. \quad (1.32)$$

Remark 1.12. Our assumption $\omega^\beta \in \mathcal{S}$ implies that $\left(\Phi_F^\beta \right)^\vee$, $\left(\Phi_M^\beta \right)^\vee$, and hence Φ_F^β , Φ_M^β , too, are elements of \mathcal{S} . Moreover, Φ_F^β and Φ_M^β are entire analytic functions with vanishing moments of arbitrary order for Φ_M^β because (1.30) implies $\text{supp } \varphi \subset \{x \in \mathbb{R}^n : \frac{1}{2} \leq |x|_a \leq \frac{3}{2}\}$, and thus (1.32) yields $D^\alpha \left(\Phi_M^\beta \right)^\vee(0) = 0$, $\alpha \in \mathbb{N}_0^n$, which can be rewritten as

$$\int_{\mathbb{R}^n} \Phi_M^\beta(\xi) \xi^\alpha d\xi = 0, \quad \alpha \in \mathbb{N}_0^n. \quad (1.33)$$

By construction, we have

$$\Phi_F^\beta(x) = \sum_{m \in \mathbb{Z}^n} (\omega^\beta)^\vee(m) \widehat{\varphi}_0(x+m), \quad x \in \mathbb{R}^n, \quad (1.34)$$

$$\Phi_M^\beta(x) = \sum_{m \in \mathbb{Z}^n} (\omega^\beta)^\vee(m) \widehat{\varphi}(x+m), \quad x \in \mathbb{R}^n. \quad (1.35)$$

For fundamentals about wavelets we refer, for instance, to [24] and [40].

2. MAIN RESULTS

Let Φ_F^β and Φ_M^β , $\beta \in \mathbb{N}_0^n$, be given by Definition 1.11, and introduce for $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, the wavelets

$$\Phi_{jm}^\beta(x) = \begin{cases} \Phi_F^\beta(x - m), & \text{if } j = 0, \\ \Phi_M^\beta(2^{ja}x - m), & \text{if } j \in \mathbb{N}. \end{cases} \quad (2.1)$$

According to the dual pairing $(\mathcal{S}, \mathcal{S}')$ we put, for given $f \in \mathcal{S}'$,

$$\lambda_{jm}^\beta(f) = 2^{jn}(\Phi_{jm}^\beta, f), \quad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n, \quad \beta \in \mathbb{N}_0^n. \quad (2.2)$$

Finally, let

$$B_p^{+,a} = \bigcup_{s>0} B_p^{s,a}, \quad 0 < p \leq \infty.$$

Recall our notation (1.17) and (1.9).

Theorem 2.1. *Let $0 < p \leq \infty$, $s > \sigma_p$, $\varrho \geq 0$, and a be an anisotropy.*

(i) *Then $f \in \mathcal{S}'$ is an element of $B_p^{s,a}$ if and only if it can be represented as*

$$f = \sum_{\beta \in \mathbb{N}_0^n} \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}^\beta k^\beta(2^{ja}x - m), \quad x \in \mathbb{R}^n, \quad (2.3)$$

with $\|\lambda|b_p^{s,\varrho}\| < \infty$, the series being absolutely convergent in $L_{\max(1,p)}$. Moreover,

$$\|f|B_p^{s,a}\| \sim \inf \|\lambda|b_p^{s,\varrho}\|, \quad (2.4)$$

where the infimum is taken over all admissible representations (2.3).

(ii) *Let $\lambda_{jm}^\beta(f)$ be given by (2.2). Then $f \in B_p^{+,a}$ can be represented as*

$$f = \sum_{\beta \in \mathbb{N}_0^n} \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}^\beta(f) k^\beta(2^{ja}x - m), \quad (2.5)$$

the series being absolutely convergent in $L_{\max(1,p)}$, and, in addition, $f \in B_p^{s,a}$ if and only if $\|\lambda(f)|b_p^{s,\varrho}\| < \infty$.

(iii) *Let $f \in B_p^{s,a}$, then (2.5) is an optimal representation, i.e.,*

$$\|f|B_p^{s,a}\| \sim \|\lambda(f)|b_p^{s,\varrho}\|, \quad (2.6)$$

where the equivalence constants are independent of f .

Remark 2.2. The isotropic version of this result can be found in [38] which in turn is a specification and modification of [36, Theorem 2.9], where further references and approaches are discussed. Note that (i) represents the so-called sub-atomic (or quarkonial) decomposition in $B_p^{s,a}$; we refer also to [13, Theorem 3.7].

As already mentioned in the introduction, we study the “dual” situation, i.e., spaces $B_p^{s,a}$ with $s < 0$, too. For that purpose, denote the counterpart of $B_p^{+,a}$ by

$$B_p^{-,a} = \bigcup_{s<0} B_p^{s,a}, \quad 0 < p \leq \infty. \quad (2.7)$$

Let k and k^β be given by (1.23) and (1.24), and consider the corresponding local means,

$$k^\beta(t, f)(x) = \int_{\mathbb{R}^n} k^\beta(y) f(x + t^a y) \, dy, \quad t > 0, \quad x \in \mathbb{R}^n, \quad (2.8)$$

where $x + t^a y = (x_1 + t^{a_1} y_1, \dots, x_n + t^{a_n} y_n)$, and put

$$k_{jm}^\beta(f) = k^\beta(2^{-j}, f)(2^{-ja} m), \quad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n. \quad (2.9)$$

We use the norm given by (1.26) with $\varrho = 0$, denoted simply by $b_p^s = b_p^{s,0}$ for convenience. Let $k(f) = \{k_{jm}^\beta(f) : j \in \mathbb{N}_0, m \in \mathbb{Z}^n, \beta \in \mathbb{N}_0^n\}$, hence

$$\|k(f)|b_p^s\| = \left(\sum_{\beta \in \mathbb{N}_0^n} \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{j(s-n/p)p} |k_{jm}^\beta(f)|^p \right)^{\frac{1}{p}}. \quad (2.10)$$

Recall our notation (2.1).

Theorem 2.3. *Let $1 < p \leq \infty$, $s < 0$.*

(i) *Then $f \in \mathcal{S}'$ is an element of $B_p^{s,a}$ if and only if it can be represented as*

$$f = \sum_{\beta \in \mathbb{N}_0^n} \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}^\beta \Phi_{jm}^\beta, \quad x \in \mathbb{R}^n, \quad (2.11)$$

with $\|\lambda|b_p^s\| < \infty$, the series being unconditionally convergent in \mathcal{S}' . Moreover,

$$\|f|B_p^{s,a}\| \sim \inf \|\lambda|b_p^s\|, \quad (2.12)$$

where the infimum is taken over all admissible representations (2.11).

(ii) *Any $f \in B_p^{-,a}$ can be represented as*

$$f = \sum_{\beta \in \mathbb{N}_0^n} \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} k_{jm}^\beta(f) \Phi_{jm}^\beta, \quad (2.13)$$

the series being unconditionally convergent in \mathcal{S}' , and, in addition, $f \in B_p^{s,a}$ if and only if $\|k(f)|b_p^s\| < \infty$.

(iii) *Let $f \in B_p^{s,a}$, then (2.13) is an optimal representation, i.e.,*

$$\|f|B_p^{s,a}\| \sim \|k(f)|b_p^s\|, \quad (2.14)$$

where the equivalence constants are independent of f .

Remark 2.4. Parallel to Remark 2.2 we refer to the isotropic version of the above result in [38] with further discussions (about local means) in [34] and [36].

3. PROOFS

We now collect all proofs and make use of the notation and concepts introduced before.

Proof of Proposition 1.5. We want to show that

$$\left(\sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{j(s-n/p)p} |(\varphi_j^a \widehat{f})^\vee(2^{-ja}m)|^p \right)^{1/p} \sim \|f|B_p^{s,a}\|. \quad (3.1)$$

Taking into account definition (1.8),

$$\|f|B_p^{s,a}\| = \left(\sum_{j=0}^{\infty} 2^{jsp} \|(\varphi_j^a \widehat{f})^\vee|L_p\|^p \right)^{1/p}, \quad (3.2)$$

the assertion reduces to

$$\sum_{m \in \mathbb{Z}^n} |(\varphi_j^a \widehat{f})^\vee(2^{-ja}m)|^p \sim 2^{jn} \left\| (\varphi_j^a \widehat{f})^\vee|L_p \right\|^p \quad (3.3)$$

with equivalence constants independent of $j \in \mathbb{N}_0$ and $f \in \mathcal{S}'$. Here we use an isotropic result given in [33, Section 1.3.3]: adapted to our above notation, it states that for $0 < p \leq \infty$ there exist some numbers $\nu_0 > 0$ and $c_2 > c_1 > 0$ such that for all $\nu \geq \nu_0$, and all $\varphi \in \mathcal{S}$ with $\text{supp } \mathcal{F}\varphi \subset \Omega$ there holds

$$c_1 \sum_{m \in \mathbb{Z}^n} |\varphi(2^{-\nu}m)|^p \leq 2^{\nu n} \|\varphi|L_p\|^p \leq c_2 \sum_{m \in \mathbb{Z}^n} |\varphi(2^{-\nu}m)|^p, \quad (3.4)$$

where $\Omega \subset \mathbb{R}^n$ is compact. (This should be properly modified when $p = \infty$.) In addition, it is known, cf. [33, Remark 1.3.3] or [32, 1.3.5], that if for some suitably chosen $y^0 \in \mathbb{R}^n$ and $b > 0$,

$$\Omega \subset \{y \in \mathbb{R}^n : |y_j - y_j^0| \leq b, j = 1, \dots, n\}, \quad (3.5)$$

then ν_0 can be taken such that $2^{\nu_0} \sim b$. Thus, with $\varphi = (\varphi_j \widehat{f})^\vee$ this implies $b \sim 2^j$ in the isotropic case, i.e., (3.4) with $\varphi = (\varphi_j \widehat{f})^\vee$ and $\nu = j$ yields the desired (isotropic) result. In order to prove (3.3) we slightly modify the above argument: let $\psi \in \mathcal{S}$ with

$$\text{supp } \mathcal{F}\psi \subset \Omega^a \subset \{y \in \mathbb{R}^n : |y_j - y_j^1| \leq b^{a_j}, j = 1, \dots, n\},$$

for suitably chosen $y^1 \in \mathbb{R}^n$ and $b > 0$, then we define

$$\varphi(tx) := \psi(t^a x), \quad t > 0, \quad x \in \mathbb{R}^n. \quad (3.6)$$

Obviously, $\varphi \in \mathcal{S}$, $\mathcal{F}\varphi(t\xi) = \mathcal{F}\psi(t^a \xi)$, $t > 0$, $\xi \in \mathbb{R}^n$, and consequently

$$\text{supp } \mathcal{F}\varphi \subset \{y \in \mathbb{R}^n : |y_j - y_j^1| \leq b, j = 1, \dots, n\}$$

(with $y^0 = b^{1-a}y^1$). Hence the application of (3.4) and (3.6) leads to

$$c_1 \sum_{m \in \mathbb{Z}^n} |\psi(2^{-\nu a}m)|^p \leq 2^{\nu n} \|\varphi|L_p\|^p \leq c_2 \sum_{m \in \mathbb{Z}^n} |\psi(2^{-\nu a}m)|^p. \quad (3.7)$$

On the other hand, with $a_1 + \dots + a_n = n$,

$$\begin{aligned} \|\varphi|_{L_p}\|^p &= \int_{\mathbb{R}^n} |\varphi(x)|^p dx = t^n \int_{\mathbb{R}^n} |\varphi(ty)|^p dy \\ &= t^n \int_{\mathbb{R}^n} |\psi(t^a y)|^p dy = \int_{\mathbb{R}^n} |\psi(z)|^p dz = \|\psi|_{L_p}\|^p. \end{aligned} \quad (3.8)$$

Finally, (3.7) and (3.8) with $\psi = (\varphi_j^a \widehat{f})^\vee$, $b \sim 2^j$ and $\nu = j$ finish the proof. \square

Proof of Theorem 2.1.

Step 1. We assume that f is given by (2.3) with $\|\lambda|b_p^{s,\varrho}\| < \infty$ for some $\varrho \geq 0$. We want to show that $f \in B_p^{s,a}$ and that there exists a constant $c > 0$ such that

$$\|f|B_p^{s,a}\| \leq c \|\lambda|b_p^{s,\varrho}\|. \quad (3.9)$$

We rewrite (2.3) as

$$f = \sum_{\beta} f^{\beta} \quad \text{with} \quad f^{\beta} = \sum_{j,m} \lambda_{j,m}^{\beta} k^{\beta}(2^{ja}x - m). \quad (3.10)$$

By definition (1.22), the support of k is contained in an open ball centered at the origin and of radius $2^{J-\varepsilon}$ for some $\varepsilon > 0$. Using the atomic decomposition for the spaces $B_p^{s,a}$ with $0 < p \leq \infty$ and $s > \sigma_p$ described in Theorem 1.9, we find by Definition 1.6 that

$$\{2^{\varepsilon a \beta} 2^{-j(s-n/p)} k^{\beta}(2^{ja}x - m) : j \in \mathbb{N}_0, m \in \mathbb{Z}^n\}, \quad \beta \in \mathbb{N}_0^n, \quad (3.11)$$

are admissible systems of anisotropic atoms, and hence $f^{\beta} \in B_p^{s,a}$ for all $\beta \in \mathbb{N}_0^n$, with

$$\|f^{\beta}|B_p^{s,a}\| \leq c 2^{-\varepsilon a \beta} \left(\sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{j(s-n/p)p} |\lambda_{j,m}^{\beta}|^p \right)^{1/p}, \quad (3.12)$$

and c independent of $\beta \in \mathbb{N}_0^n$. Summation over β proves $f \in B_p^{s,a}$ and (3.9). The absolute convergence follows in the same way as discussed in [38] and [36, 1.4, 2.7].

Step 2. Let $f \in B_p^{s,a}$ with $0 < p \leq \infty$ and $s > \sigma_p$; we shall show that we can decompose it as (2.5). Note that this covers then (i) as well.

Let R_j^a , $j \in \mathbb{N}_0$, be a rectangle in \mathbb{R}^n centered at the origin with side-length $2\pi 2^{ja}$ where $2^{ja} = (2^{ja_1}, \dots, 2^{ja_n})$. Let φ_j^a be given by (1.7), that is, with $\text{supp } \varphi_j^a \subset R_j^a$. Now we can write that

$$\widehat{f}(x) = \sum_{j=0}^{\infty} \varphi_j^a(x) \widehat{f}(x), \quad x \in \mathbb{R}^n,$$

and expand $\varphi_j^a \widehat{f}$ in R_j^a into a Fourier series,

$$(\varphi_j^a \widehat{f})(\xi) = \sum_{m \in \mathbb{Z}^n} b_{jm} \exp(-i 2^{-ja} m \xi), \quad \xi \in R_j^a. \quad (3.13)$$

We calculate b_{jm} , $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, by

$$\begin{aligned} (\varphi_j^a \widehat{f})^\vee(2^{-ja}m) &= (2\pi)^{-\frac{n}{2}} \int_{R_j^a} (\varphi_j^a \widehat{f})(y) \exp(i2^{-ja}my) \, dy \\ &= (2\pi)^{-\frac{n}{2}} \sum_{k \in \mathbb{Z}^n} b_{jk} \int_{R_j^a} \exp(i2^{-ja}(m-k)y) \, dy \\ &= (2\pi)^{-\frac{n}{2}} \sum_{k \in \mathbb{Z}^n} b_{jk} \int_{R_j^a} \exp\left(i \sum_{l=1}^n 2^{-ja_l}(m_l - k_l)y_l\right) \, dy. \end{aligned}$$

Substitute in the last formula $\xi_l = 2^{-ja_l}y_l$, then $d\xi = 2^{-j(a_1+\dots+a_n)} dy = 2^{-jn} dy$, and we arrive at

$$(\varphi_j^a \widehat{f})^\vee(2^{-ja}m) = 2^{jn} (2\pi)^{-\frac{n}{2}} \sum_{k \in \mathbb{Z}^n} b_{jk} \int_{Q_\pi} e^{i(m-k)\xi} \, d\xi, \quad (3.14)$$

where Q_π is the cube of side-length 2π in each direction. For the last term in (3.14) we thus have

$$\int_{Q_\pi} e^{i(m-k)\xi} \, d\xi = \begin{cases} (2\pi)^n, & m = k, \\ 0, & m \neq k, \end{cases}$$

which by (3.14) finally leads to

$$\begin{aligned} b_{jm} &= (2\pi)^{-\frac{n}{2}} 2^{-jn} (\varphi_j^a \widehat{f})^\vee(2^{-ja}m) \\ &= (2\pi)^{-n} 2^{-jn} \int_{R_j^a} (\varphi_j^a \widehat{f})(\xi) \exp(i2^{-ja}m\xi) \, d\xi. \end{aligned} \quad (3.15)$$

By Proposition 1.5 and (3.15) we thus have for $0 < p \leq \infty$,

$$\|f\|_{B_p^{s,a}} \sim \left(\sum_{j=0}^{\infty} 2^{j(s-n/p)p} 2^{jnp} \sum_{m \in \mathbb{Z}^n} |b_{jm}|^p \right)^{1/p}, \quad (3.16)$$

(with the usual modification if $p = \infty$). Let ω be given by (1.27) and $\omega_j(x) = \omega(2^{-ja}x)$; then ω_j has a compact support in R_j^a and it follows by (3.13) that

$$\begin{aligned} (\varphi_j^a \widehat{f})^\vee(x) &= \sum_{m \in \mathbb{Z}^n} b_{jm} \omega_j^\vee(x - 2^{-ja}m) \\ &= 2^{jn} \sum_{m \in \mathbb{Z}^n} b_{jm} \omega^\vee(2^{ja}x - m), \quad x \in \mathbb{R}^n. \end{aligned} \quad (3.17)$$

Let k be given by (1.22), (1.23). We expand the analytic function $\omega^\vee(2^{ja}x - m)$ at $2^{-ja}l$, $l \in \mathbb{Z}^n$, and obtain

$$k(2^{ja}x - l)\omega^\vee(2^{ja}x - m) = \sum_{\beta \in \mathbb{N}_0^n} \frac{2^{ja\beta}}{\beta!} (D^\beta \omega^\vee)(l - m)(x - 2^{-ja}l)^\beta k(2^{ja}x - l)$$

$$\begin{aligned}
 &= \sum_{\beta \in \mathbb{N}_0^n} \frac{2^{ja\beta}}{\beta!} (D^\beta \omega^\vee)(l-m) 2^{-ja\beta} (2^{ja}x-l)^\beta k(2^{ja}x-l) \\
 &= \sum_{\beta \in \mathbb{N}_0^n} \frac{2^{Ja\beta}}{\beta!} (D^\beta \omega^\vee)(l-m) k^\beta(2^{ja}x-l), \tag{3.18}
 \end{aligned}$$

where we applied (1.24) to the last line. By (1.23), (3.17) and (3.18) we obtain

$$\begin{aligned}
 (\varphi_j^a \widehat{f})^\vee(x) &= \sum_{m \in \mathbb{Z}^n} 2^{jn} b_{jm} \sum_{l \in \mathbb{Z}^n} k(2^{ja}x-l) \omega^\vee(2^{ja}x-m) \\
 &= \sum_{\beta \in \mathbb{N}_0^n} \sum_{l \in \mathbb{Z}^n} k^\beta(2^{ja}x-l) \sum_{m \in \mathbb{Z}^n} 2^{jn} b_{jm} \frac{2^{Ja\beta}}{\beta!} (D^\beta \omega^\vee)(l-m).
 \end{aligned}$$

Hence, as $(\varphi_j^a)_{j \in \mathbb{N}_0}$ is a resolution of unity,

$$f = \sum_{j=0}^{\infty} \sum_{\beta \in \mathbb{N}_0^n} \sum_{l \in \mathbb{Z}^n} k^\beta(2^{ja}x-l) \lambda_{jl}^\beta \tag{3.19}$$

with

$$\lambda_{jl}^\beta = \sum_{m \in \mathbb{Z}^n} 2^{jn} b_{jm} \frac{2^{Ja\beta}}{\beta!} (D^\beta \omega^\vee)(l-m). \tag{3.20}$$

We first check that λ_{jl}^β are optimal coefficients, and verify their representation (2.2) afterwards. We thus claim that for $\varrho \geq 0$ we can find a constant c such that with $\lambda = \{\lambda_{jl}^\beta : \beta \in \mathbb{N}_0^n, j \in \mathbb{N}_0, l \in \mathbb{Z}^n\}$ given by (3.20),

$$\|\lambda|b_p^{s,\varrho}\| \leq c \|f|B_p^{s,a}\| \quad \text{for all } f \in B_p^{s,a}. \tag{3.21}$$

We use an isotropic result [37, 3.1.1], which states that for any $\varepsilon > 0$ there are constants $c > 0$ and $c_\varepsilon > 0$ such that

$$|D^\beta \omega^\vee(x)| \leq c_\varepsilon 2^{c|\beta|} (1+|x|^2)^{-\varepsilon} \quad \text{for } x \in \mathbb{R}^n, \quad \beta \in \mathbb{N}_0^n, \tag{3.22}$$

where c is independent of x , ε , and β , and c_ε independent of x , β . Furthermore, note that there are constants $c_2 > c_1 > 0$ such that for all $\xi \in \mathbb{R}^n$,

$$c_1 (1+|\xi|)^{1/a_{\max}} \leq 1+|\xi|_a \leq c_2 (1+|\xi|)^{1/a_{\min}},$$

cf. [23]. On the other hand, we have $a_{\min}|\beta| \leq a\beta \leq a_{\max}|\beta|$, thus (3.22) implies

$$|D^\beta \omega^\vee(x)| \leq c'_\varepsilon 2^{c'a\beta} (1+|x|_a)^{-\varepsilon} \quad \text{for } x \in \mathbb{R}^n, \quad \beta \in \mathbb{N}_0^n. \tag{3.23}$$

Let $p \geq 1$. We interpret λ_{jl}^β as a convolution in ℓ_p : let $l \in \mathbb{Z}^n$ and

$$a_l = \sum_{m \in \mathbb{Z}^n} c_m d_{l-m},$$

then $\|a_l|_{\ell_p}\| \leq \|d_k|_{\ell_1}\| \|c_m|_{\ell_p}\|$. Put $d_k = \frac{2^{Ja\beta}}{\beta!} (D^\beta \omega^\vee)(k)$, then (3.23) leads to

$$\|d_k|_{\ell_1}\| \leq C_\varepsilon \frac{2^{(J+c')a\beta}}{\beta!} \leq c(\varrho) 2^{-(\varrho+1)a\beta},$$

if $\varepsilon > 0$ is chosen appropriately. The last inequality results from an estimate of $\beta! = \beta_1! \cdots \beta_n!$ by Stirling's formula, $n! = \Gamma(n+1) \sim \sqrt{n} \left(\frac{n}{e}\right)^n$, $n \in \mathbb{N}$. Consequently, (3.20) with $a_l = \lambda_{jl}^\beta$ and $c_m = 2^{jn} b_{jm}$ implies, that for $\varrho \geq 0$ there is a constant $c(\varrho)$ such that

$$\left(\sum_{l \in \mathbb{Z}^n} |\lambda_{jl}^\beta|^p \right)^{1/p} \leq c(\varrho) 2^{-(\varrho+1)a\beta} \left(\sum_{l \in \mathbb{Z}^n} |2^{jn} b_{jl}|^p \right)^{1/p}. \quad (3.24)$$

If $p < 1$, then one uses the p -triangle inequality. Now (3.21) follows from (3.24) and (3.16).

Step 3. We need to prove that λ_{jl}^β can be represented as (2.2). By (3.15) and the properties of the Fourier transform we have

$$2^{jn} b_{jm} = (2\pi)^{-n} \int_{\mathbb{R}^n} (\varphi_j^a)^\vee (2^{-ja} m - y) f(y) \, dy, \quad j \in \mathbb{N}_0, \quad (3.25)$$

where φ is given by Definition 1.11. Now $\varphi_j^a(x) = \varphi(2^{-ja}x)$ leads to

$$2^{jn} b_{jm} = (2\pi)^{-n} 2^{jn} \int_{\mathbb{R}^n} \varphi^\vee(m - 2^{-ja}y) f(y) \, dy, \quad j \in \mathbb{N}. \quad (3.26)$$

Recall that $(D^\beta \omega^\vee)(\xi) = i^{|\beta|} (x^\beta \omega(x))^\vee(\xi)$. Thus (1.28) implies for $j \in \mathbb{N}$,

$$\begin{aligned} \lambda_{jl}^\beta &= \sum_{m \in \mathbb{Z}^n} 2^{jn} b_{jm} \frac{2^{Ja\beta}}{\beta!} (D^\beta \omega^\vee)(l - m) \\ &= \sum_{m \in \mathbb{Z}^n} 2^{jn} b_{jm} \frac{2^{Ja\beta}}{\beta!} i^{|\beta|} (x^\beta \omega(x))^\vee(l - m) \\ &= \sum_{m \in \mathbb{Z}^n} 2^{jn} b_{jm} \frac{2^{Ja\beta}}{\beta!} i^{|\beta|} \frac{(2\pi)^n \cdot \beta!}{i^{|\beta|} 2^{Ja\beta}} (\omega^\beta)^\vee(l - m) \\ &= 2^{jn} \int_{\mathbb{R}^n} f(y) \sum_{m \in \mathbb{Z}^n} (\omega^\beta)^\vee(l - m) \varphi^\vee(m - 2^{ja}y) \, dy. \end{aligned} \quad (3.27)$$

Replacing $l - m$ by m and using $\varphi^\vee(z) = \widehat{\varphi}(-z)$ we get

$$\lambda_{jl}^\beta = 2^{jn} \int_{\mathbb{R}^n} f(y) \sum_{m \in \mathbb{Z}^n} (\omega^\beta)^\vee(m) \widehat{\varphi}(2^{ja}y - l + m) \, dy = 2^{jn} (\Phi_{jl}^\beta, f), \quad j \in \mathbb{N}, \quad (3.28)$$

where we used (1.35) and (2.1). The argument for $j = 0$ works analogously, so the proof is finished. \square

We begin the proof of Theorem 2.3 with some preparation. Let $l \in \mathbb{Z}^n$, $K, L \in \mathbb{R}$, and $a_l \in C^K$ be anisotropic atoms given by Definition 1.6 with $\text{supp } a_l \subset \{y \in \mathbb{R}^n : |y|_a \leq c\}$ for some appropriate $c > 0$. We know by (1.15) that

$$\int_{\mathbb{R}^n} x^\beta a_l(x) \, dx = 0 \quad \text{if} \quad a\beta \leq L. \quad (3.29)$$

Let $\mu = (\mu_l)_{l \in \mathbb{Z}^n}$ denote the decay factors,

$$|D^\gamma a_l(x)| \leq \mu_l, \quad a\gamma \leq K, \quad x \in \mathbb{R}^n. \quad (3.30)$$

We define now (special) anisotropic *molecules*

$$b(x) = \sum_{l \in \mathbb{Z}^n} a_l(x - l), \quad x \in \mathbb{R}^n, \quad (3.31)$$

and

$$b_{j,m}(x) = 2^{-j(s-\frac{n}{p})} b(2^{ja}x - m), \quad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n, \quad x \in \mathbb{R}^n. \quad (3.32)$$

Remark 3.1. The normalized (isotropic) molecules share the decay properties and moment conditions with the normalized (isotropic) atoms, but lack the assumption concerning a compact support, see (the isotropic counterparts of) (1.11), (1.13). This assumption is replaced by sufficiently strong decay assumption. In the isotropic case the counterpart of Theorem 1.9 remains valid if one uses molecular decompositions instead of atomic ones, cf. [16, Section 5]. There are also anisotropic versions of that result in the literature, see [7]. However, because of the special property of building blocks we shall use, we do not need this assertion in its full generality, but only a special case which is simpler to prove. Hence we consider this case below separately and give its direct proof.

Proposition 3.2. *Let $s \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$, and $\mu = (\mu_l)_{l \in \mathbb{Z}^n} \in \ell_{\min(1,p)}$. Let $f \in \mathcal{S}'$ be given by*

$$f = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{j,m} b_{j,m}, \quad (3.33)$$

where $\lambda = \{\lambda_{j,m} \in \mathbb{C} : j \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$, $\lambda \in b_{pq}$, and $b_{j,m}$, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, are given by (3.32). Then

$$\|f|B_{pq}^{s,a}\| \leq c \|\lambda|b_{pq}\|. \quad (3.34)$$

Proof. We owe the idea of this proof to some discussions with Prof. H. Triebel.

By definition and (3.31), (3.32),

$$\begin{aligned} f &= \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{j,m} b_{j,m} \\ &= \sum_{j=0}^{\infty} 2^{-j(s-\frac{n}{p})} \sum_{m \in \mathbb{Z}^n} \lambda_{j,m} b(2^{ja}x - m) \\ &= \sum_{j=0}^{\infty} 2^{-j(s-\frac{n}{p})} \sum_{m \in \mathbb{Z}^n} \lambda_{j,m} \sum_{l \in \mathbb{Z}^n} a_l(2^{ja}x - m - l) \\ &= \sum_{j=0}^{\infty} 2^{-j(s-\frac{n}{p})} \sum_{k \in \mathbb{Z}^n} \sum_{l \in \mathbb{Z}^n} \lambda_{j,k-l} a_l(2^{ja}x - k) \\ &= \sum_{j=0}^{\infty} 2^{-j(s-\frac{n}{p})} \sum_{k \in \mathbb{Z}^n} \gamma_{j,k} d_{j,k}(x), \end{aligned} \quad (3.35)$$

where we have put

$$\gamma_{j,k} = \sum_{l \in \mathbb{Z}^n} \mu_l |\lambda_{j,k-l}| > 0, \quad j \in \mathbb{N}_0, \quad k \in \mathbb{Z}^n,$$

and

$$d_{j,k}(x) = \gamma_{j,k}^{-1} \sum_{l \in \mathbb{Z}^n} \lambda_{j,k-l} a_l (2^{ja} x - k), \quad j \in \mathbb{N}_0, \quad k \in \mathbb{Z}^n, \quad x \in \mathbb{R}^n. \quad (3.36)$$

We claim that $2^{-j(s-\frac{n}{p})} d_{j,k}$ are anisotropic atoms according to Definition 1.6. Assume first $j = 0$, then

$$|d_{0,k}(x)| \leq \gamma_{0,k}^{-1} \sum_{l \in \mathbb{Z}^n} |\lambda_{0,k-l}| \cdot \mu_l = 1,$$

and likewise for all derivatives $D^\gamma d_{0,k}$, $\gamma \in \mathbb{N}_0^n$, $a\gamma \leq K$, according to (3.30). In the case of $j \in \mathbb{N}$ and $\gamma \in \mathbb{N}_0^n$ we conclude similarly that

$$\begin{aligned} \left| D^\gamma \left(2^{-j(s-\frac{n}{p})} d_{j,k}(x) \right) \right| &\leq 2^{-j(s-\frac{n}{p})} \gamma_{j,k}^{-1} \sum_{l \in \mathbb{Z}^n} |\lambda_{j,k-l}| |(D^\gamma a_l)(2^{ja} x - k)| 2^{ja\gamma} \\ &\leq 2^{-j(s-\frac{n}{p})+ja\gamma} \gamma_{j,k}^{-1} \sum_{l \in \mathbb{Z}^n} |\lambda_{j,k-l}| \mu_l = 2^{-j(s-\frac{n}{p})+ja\gamma}, \end{aligned}$$

where $a\gamma \leq K$. The corresponding moment conditions (1.15) are satisfied by (3.29), and condition (1.13) is guaranteed by construction (3.36). Hence $2^{-j(s-\frac{n}{p})} d_{j,k}$ are anisotropic atoms and Theorem 1.9 gives the anisotropic atomic decomposition

$$f = \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^n} \gamma_{j,k} 2^{-j(s-\frac{n}{p})} d_{j,k} \quad \text{with} \quad \|f|B_{pq}^{s,a}\| \leq c \|\gamma|b_{pq}\|.$$

It remains to estimate $\|\gamma|b_{pq}\|$ by $\|\lambda|b_{pq}\|$, where the assumption on the decay factors, i.e., $\mu \in \ell_{\min(p,1)}$ is now involved. We first assume that $p \geq 1$, then interpret $\gamma_{j,k}$ as a convolution in ℓ_p and obtain

$$\left(\sum_{k \in \mathbb{Z}^n} |\gamma_{j,k}|^p \right)^{\frac{1}{p}} \leq \left(\sum_{l \in \mathbb{Z}^n} \mu_l \right) \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{j,m}|^p \right)^{\frac{1}{p}} = \|\mu|_{\ell_1}\| \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{j,m}|^p \right)^{\frac{1}{p}}.$$

If $0 < p < 1$, we have by the p -triangle inequality

$$\sum_{k \in \mathbb{Z}^n} |\gamma_{j,k}|^p \leq \sum_{k,l \in \mathbb{Z}^n} \mu_l^p |\lambda_{j,k-l}|^p \leq \|\mu|_{\ell_p}\|^p \sum_{m \in \mathbb{Z}^n} |\lambda_{j,m}|^p.$$

This results in

$$\|\gamma|b_{pq}\| \leq c \|\mu|_{\ell_{\min(1,p)}}\| \|\lambda|b_{pq}\|$$

as desired, i.e., $\|f|B_{pq}^{s,a}\| \leq c \|\gamma|b_{pq}\| \leq c' \|\mu|_{\ell_{\min(1,p)}}\| \|\lambda|b_{pq}\|$. \square

Proof of Theorem 2.3.

Step 1. We assume that f is given by (2.11) with $\|\lambda|b_p^s\| < \infty$. We want to show that $f \in B_p^{s,a}$ and that there exists a constant $c > 0$ such that

$$\|f|B_p^{s,a}\| \leq c \|\lambda|b_p^s\|. \quad (3.37)$$

Recall our particular construction (1.34), (1.35), (2.1) with $\varphi \in \mathcal{S}$ given by Definition 1.11. Then for each $\beta \in \mathbb{N}_0^n$,

$$\{2^{\varepsilon a \beta} 2^{-j(s-\frac{n}{p})} \tilde{\Phi}_{jm}^\beta : j \in \mathbb{N}_0, m \in \mathbb{Z}^n\} \quad (3.38)$$

where $\tilde{\Phi}_{jm}^\beta = \sum_{l \in \mathbb{Z}^n} k(x-l) \Phi_{jm}^\beta$, are admissible anisotropic special molecules in $B_p^{s,a}$ in the sense of (3.29)–(3.32), satisfying, in addition, the decay assumption $\mu \in \ell_{\min(p,1)}$. Thus we can apply Proposition 3.2 with $q = p$ and then (3.37) follows in the same way as in *Step 1* in the proof of Theorem 2.1. This covers the unconditional convergence, too.

Step 2. Let $k(f) = \{k_{jm}^\beta(f) : \beta \in \mathbb{N}_0^n, j \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$ be given by (2.8), (2.9), i.e.,

$$\begin{aligned} k_{jm}^\beta(f) &= k^\beta(2^{-j}, f)(2^{-ja}m) = \int_{\mathbb{R}^n} k^\beta(y) f(2^{-ja}m + 2^{-ja}y) dy \\ &= 2^{jn} \int_{\mathbb{R}^n} k^\beta(2^{ja}y - m) f(y) dy = 2^{jn} (k^\beta(2^{ja} \cdot -m), f). \end{aligned} \quad (3.39)$$

We are to show that there is some $c > 0$ such that for all $f \in B_p^{s,a}$,

$$\|k(f)|b_p^s\| \leq c \|f|B_p^{s,a}\|. \quad (3.40)$$

Using (2.10) with $\varrho = 0$ (recall our convention $b_p^s = b_p^{s,0}$), (3.39) can be rewritten as

$$\begin{aligned} \|k(f)|b_p^s\| &= \left\| \sum_{\beta,j,m} 2^{j(s-\frac{n}{p})} k_{jm}^\beta \right\|_{\ell_p} \\ &= \left\| \sum_{\beta,j,m} 2^{j(s-\frac{n}{p})+jn} (k^\beta(2^{ja} \cdot -m), f) \right\|_{\ell_p}. \end{aligned} \quad (3.41)$$

By duality, $\ell_p = (\ell_{p'})'$ with $\frac{1}{p} + \frac{1}{p'} = 1$ for $1 < p \leq \infty$. Taking additionally into account that $2^{j(s-\frac{n}{p})+jn} = 2^{j(s+\frac{n}{p'})}$, we are led to

$$\begin{aligned} \|k(f)|b_p^s\| &= \left\| \sum_{\beta,j,m} 2^{j(s+\frac{n}{p'})} (k^\beta(2^{ja} \cdot -m), f) \right\|_{\ell_p} \\ &= \sup_{\lambda|b_{p'}^{-s} \leq 1} \sum_{\beta,j,m} \lambda_{jm}^\beta (k^\beta(2^{ja} \cdot -m), f), \end{aligned} \quad (3.42)$$

where the supremum in (3.42) is taken over all sequences $\lambda = \{\lambda_{jm}^\beta : \beta \in \mathbb{N}_0^n, j \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$ such that the right-hand side of (3.42) is non-negative, and $\|\lambda|b_{p'}^{-s}\| \leq 1$. Consequently,

$$\|k(f)|b_p^s\| \leq \sup_{\lambda|b_{p'}^{-s} \leq 1} |(g, f)|, \quad \text{where } g(x) = \sum_{\beta,j,m} \lambda_{jm}^\beta k^\beta(2^{ja}x - m). \quad (3.43)$$

Note that our assumptions $s < 0$ and $1 < p \leq \infty$ imply $-s > 0 = \sigma_{p'}$ such that Theorem 2.1 can be applied to $g \in B_{p'}^{-s,a}$. Thus we arrive at

$$\|k(f)|b_p^s\| \leq \sup \{ |(g, f)| : g \in B_{p'}^{-s,a}, \|g|B_{p'}^{-s,a}\| \leq c \}, \quad (3.44)$$

where $c > 0$ is independent of $g \in B_{p'}^{-s,a}$. Now we use the duality

$$(B_{p'}^{-\sigma,a}(\mathbb{R}^n))' = B_p^{\sigma,a}(\mathbb{R}^n), \quad 1 \leq p' < \infty, \quad \sigma \in \mathbb{R}, \quad (3.45)$$

see [39], and obtain

$$\|k(f)|b_p^s\| \leq c' \|f|B_p^{s,a}\|. \quad (3.46)$$

Step 3. Let $1 < p \leq \infty$, $s < 0$ and $f \in B_p^{s,a}$. It is sufficient to verify (2.13) in order to complete the proof. Let $\psi \in \mathcal{S}$ be arbitrary, then Theorem 2.1, in particular (2.5) with (2.2), yields

$$\psi = \sum_{\beta,j,m} 2^{jn} (\Phi_{jm}^\beta, \psi) k^\beta(2^{ja}x - m) \quad (3.47)$$

with unconditional convergence in any space $B_{p'}^{\sigma,a}$ with $\sigma > 0$. Hence, by (3.39),

$$(f, \psi) = \left(\sum_{\beta,j,m} 2^{jn} (f, k^\beta(2^{ja} \cdot - m)) \Phi_{jm}^\beta, \psi \right) = \left(\sum_{\beta,j,m} k_{jm}^\beta(f) \Phi_{jm}^\beta, \psi \right), \quad (3.48)$$

that is

$$f = \sum_{\beta,j,m} k_{jm}^\beta(f) \Phi_{jm}^\beta$$

in \mathcal{S}' . By (i) and our preceding remarks it follows that (2.13) converges unconditionally in \mathcal{S}' . \square

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