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COMMON FIXED POINT AND INVARIANT APPROXIMATION RESULTS ON NON-STARSHAPED DOMAINS

NAWAB HUSSAIN, DONAL O'REGAN, AND RAVI P. AGARWAL

Abstract. We extend the concept of R-subweakly commuting maps due to Shahzad [21] to the case of non-starshaped domains and obtain common fixed point results for this class of maps on non-starshaped domains in the setup of Fréchet spaces. As applications, we establish Brosowski–Meinardus type approximation theorems. Our results unify and extend the results of Al-Thagafi, Dotson, Habiniak, Jungck and Sessa, Sahab, Khan and Sessa and Shahzad.

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1. Introduction

Using fixed point theory, Brosowski [5], and Meinardus [13] established some interesting results on invariant approximation in the setting of normed spaces. Habiniak [8], Hicks and Humphries [9], Jungck and Sessa [11], Sahab et. al. [17], Sahney et. al. [18] and Singh [23] also obtained some results in approximation theory in the setting of normed spaces. Their work was extended, generalized and unified by many authors; for example, see [1], [3], [12], [19]–[22]. For any map $I : M \to X$ and $u \in X$, Al-Thagafi [1] introduced the following sets:

$$C_M^I(u) = \{x \in M : \ Ix \in P_M(u)\}, \ D_M^I(u) = P_M(u) \cap C_M^I(u),$$

where $P_M(u) = \{x \in M : \ d(x,u) = d(u,M) = \inf_{y\in M} d(u,y)\}$.

Theorem 1.1 ([1], Theorem 3.2)**.** *Let I and T be selfmaps of a normed space E with $u \in F(I) \cap F(T)$ and $M \subset E$ with $T(\partial M \cap M) \subset M$. Suppose that $D = D'_M(u)$ is closed and q-starshaped with $q \in F(I)$. Let T be I-nonexpansive on $D \cup \{u\}$ with $\mathrm{cl}(T(D))$ compact, I be continuous, linear, $ID = D$, and I commutes with T on D. Then $P_M(u) \cap F(T) \cap F(I) \neq \varnothing$.*

Recently, Hussain and Khan [10] have proved the following more general invariant approximation result for 1-subcommutative maps extending Theorem 1.1 to locally convex spaces.

Theorem 1.2 ([10], Theorem 3.3)**.** *Let T and I be selfmaps on a Hausdorff locally convex space (X, τ), M a subset of X such that $T(\partial M) \subseteq M$, and $u \in F(T) \cap F(I)$. Suppose that $D = D'_M(u)$ is nonempty and q-starshaped with $q \in F(I)$, T is I-nonexpansive on $D \cup \{u\}$, I is nonexpansive on $P_M(u) \cup \{u\}$*

and affine on D. If $I(D) = D$ and T and I are 1-subcommuting maps, then $P_M(u) \cap F(T) \cap F(I) \neq \varnothing$ provided one of the following conditions holds:

(i) *D is τ-sequentially compact;*
(ii) *T is a compact map;*
(iii) *D is weakly compact in (X, τ), I is weakly continuous and $I - T$ is demiclosed at 0;*
(iv) *D is weakly compact in an Opial space (X, τ) and I is weakly continuous.*

The aim of this paper is to prove the results extending the above-mentioned invariant approximation results. To do this, we introduce the concept of R-subweakly commuting maps defined on a non-starshaped domain. Then we establish general common fixed point theorems for R-subweakly commuting maps on non-starshaped domains in Fréchet spaces. We apply the theorems to derive some results on the existence of common fixed points from the set of best approximations. Examples are presented, which show that certain hypotheses of our results cannot be relaxed. Our results, on the one hand, unify the work of Brosowski [5], Dotson [6], [7], Habiniak [8], Khan et. al. [12], Meinardus [13] and, on the other hand, provide generalizations of the recent works of Al-Thagafi [1], Baskaran and Subrahmanyam [3], Jungck and Sessa [11], Sahab, Khan and Sessa [17], Sahney et. al. [18] and Shahzad [19]–[21].

2. Preliminaries

Let (X, d) be a metric linear space and M a nonempty subset of X. A metric d on a linear space X is called translation invariant if $d(x+z, y+z) = d(x, y)$ for all $x, y, z \in X$ or equivalently, $d(x, y) = d(x - y, 0)$. Let $I : M \to X$ be a mapping. A mapping $T : M \to X$ is called I-Lipschitz if, for any $x, y \in M$, there exists $k \geq 0$ such that $d(Tx, Ty) \leq kd(Ix, Iy)$. If $k < 1$ (respectively, $k = 1$), then T is called I-contraction (respectively, I-nonexpansive). Two selfmaps I and T on M are said to be R-weakly commuting [15], [20] if and only if $d(ITx, TIx) \leq Rd(Tx, Ix)$ for all $x \in M$ and some $R > 0$. Suppose M is q-starshaped with $q \in F(I)$ and is both T- and I-invariant. Then following Shahzad [21], we say that T and I are called R-subweakly commuting on M if there exists a real number $R > 0$ such that $d(ITx, TIx) \leq Rd((kTx + (1-k)q), Ix)$ for all $x \in M$, $k \in [0, 1]$. If $R = 1$, then the maps are called 1-subweakly commuting. Clearly, R-subweakly commuting maps are R-subcommuting. The class of R-subweakly commuting maps contains properly the class of commuting maps [21].

Let M be a subset of a Fréchet space (X, d) and $\mathcal{F} = \{f_a\}_{a \in M}$ be a family of functions from $[0, 1]$ into M such that $f_x(1) = x$ for each $x \in M$. The family $\mathcal{F}$ is said to be contractive [4], [7] if there exists a function $\phi : (0, 1) \to (0, 1)$ such that for all $x, y \in M$ and all $t \in (0, 1)$, we have $d(f_x(t), f_y(t)) \leq \phi(t)d(x, y)$. The family $\mathcal{F}$ is said to be jointly (weakly) continuous if $t \to t_0$ in $[0, 1]$ and $x \to x_0$ ($x \xrightarrow{\omega} x_0$) in M, then $f_x(t) \to f_{x_0}(t_0)$ ($f_x(t) \xrightarrow{\omega} f_{x_0}(t_0)$) in M; here $\to$ and $\xrightarrow{\omega}$ denote strong and weak convergence, respectively. We observe that if M is q-starshaped and $f_x(t) = (1-t)q + tx$ ($x \in M$; $t \in (0, 1)$), then $\mathcal{F} = \{f_x\}_{x \in M}$ is a contractive jointly continuous and a jointly weakly continuous family with

$\phi(t) = t$ provided d is translation invariant and satisfies $d(\alpha x, \alpha y) \le \alpha d(x, y)$, for each α with $0 < \alpha < 1$ and $x, y \in X$. Thus the class of subsets of X with the property of contractiveness and joint continuity contains the class of starshaped sets, which in turn contains the class of convex sets (see [2], [4], [7]).

Another extension of the concept of starshapedness was used by Naimpally et. al. in the result [14, Theorem 1.8(i)].

If for a selfmap T on M, there exist $q \in M$ and a fixed sequence of real numbers k_n $(0 < k_n < 1)$ converging to 1, such that $(1 - k_n)q + k_nTx \in M$ for each $x \in M$ and $n \in N$, then the set M is said to have property (N). A mapping I is said to have property (C) on a set M with property (N) if $I((1 - k_n)q + k_nTx) = (1 - k_n)Iq + k_nITx$ for each $x \in M$ and $n \in N$. Each T-invariant q-starshaped set has property (N) but not conversely in general (see Example 2.1). Each affine map on a q-starshaped set M satisfies condition (C). We extend the concept of R-subweakly commuting maps to a non-starshaped domain in the following way:

Let I and T be selfmaps of the set M and $\mathcal{F}$ be the family of functions as defined above. Then I and T are said to be R-subweakly commuting maps provided there exists a real number $R > 0$ such that

$$d(ITx, TIx)) \le Rd(f_{Tx}(k), Ix) \tag{1}$$

for each $k \in [0, 1]$ and $x \in M$.

Suppose that M is q-starshaped with $q \in F(I)$, $f_x(k) = (1-k)q+kx$, ($x \in M$; $k \in [0, 1]$), and M is both T- and I-invariant, then (1) reduces to the original concept of R-subweak commutativity of T and I.

The maps I and T are called R-subweakly commuting maps on a set M satisfying property (N) with $q \in F(I)$ provided there exists a real number $R > 0$ such that

$$d(ITx, TIx)) \le Rd((1 - k_n)q + k_nTx, Ix)$$

for each $x \in M$ and the sequence $\{k_n\}$ is as in the definition of property (N) of M.

Example 2.1. Consider $X = \Re^2$ and $M = \{(0, y) : y \in [-1, 1]\} \cup \left\{\left(1 - \frac{1}{n+1}, 0\right) : n \in N\right\} \cup \{(1, 0)\}$ with the metric induced by the norm $\|(x, y)\| = |x| + |y|$. Define T on M as follows:

$$T(0, y) = (0, -y), \quad T\left(1 - \frac{1}{n+1}, 0\right) = \left(0, 1 - \frac{1}{n+1}\right), \quad T(1, 0) = (0, 1).$$

Clearly, M is not starshaped [14] but M has the property (N) for $q = (0, 0)$ and $k_n = 1 - \frac{1}{n+1}$, $\forall\, n \in N$.

Define $I(0, y) = I\left(1 - \frac{1}{n+1}, 0\right) = (0, 0)$, $\forall\, y \in [-1, 1]$ and $n \in N$ and $I(1, 0) = (1, 0)$. Then $|TIx - ITx| = 0$ or 1. Thus $|TIx - ITx| \le R|k_nTx - Ix|$ for all x in M, for each $R \ge 1$ and $q = (0, 0) \in F(I)$. Thus I and T are R-subweakly commuting but not commuting on M.

A Fréchet space X satisfies Opial's condition if for every sequence $\{x_n\}$ in X weakly convergent to $x \in X$, the inequality

$$\liminf d(x_n, x) < \liminf d(x_n, y) \text{ holds for all } y \neq x.$$

The map $T : M \to X$ is said to be (i) demiclosed at 0 if for every sequence $\{x_n\}$ in M converging weakly to x and $\{Tx_n\}$ converging strongly to 0, we have $Tx = 0$, (ii) condensing if T is continuous and for each nonempty bounded subset B of M with $\alpha(B) > 0$, $T(B)$ is bounded and $\alpha(T(B)) < \alpha(B)$, where $\alpha(B) = \{r > 0 :\ B \text{ can be covered by a finite number of sets of diameter} \leq r\}$, (iii) hemicompact if each sequence $\{x_n\}$ in M has a convergent subsequence whenever $d(x_n, Tx_n) \to 0$ as $n \to \infty$ (iv) demicompact if T is continuous and every bounded sequence $\{x_n\}$ in M, such that $\{Tx_n - x_n\}$ is convergent in X, has a convergent subsequence (v) completely continuous if $\{x_n\}$ converges weakly to x implies that $\{Tx_n\}$ converges to Tx.

3. Common Fixed Point Results

The following result is a consequence of Theorem 1 of Pant [15] which will be needed in the sequel.

Theorem 3.1 ([20, Theorem 2.1])**.** *Let (X, d) be a complete metric space and T, $I : X \to X$ be R-weakly commuting mappings such that $T(X) \subseteq I(X)$, and T is I-contraction. If either T or I is continuous, then $F(T) \cap F(I)$ is a singleton.*

Lemma 3.2 (cf. [10], Theorem 2.2)**.** *Let M be a nonempty weakly compact subset of a Fréchet space X satisfying Opial's condition. Let $I : M \to X$ be a weakly continuous map and $T : M \to X$ an I-nonexpansive map. Then $I - T$ is demiclosed.*

Using Theorem 3.1, we obtain a common fixed point generalization of Theorems 1 and 2 of Dotson [7], and Theorem 4 of Habiniak [8].

Theorem 3.3. *Let T, I be selfmaps on a subset M of a Fréchet space X. Suppose that M has a contractive jointly continuous family $\mathcal{F} = \{f_x\}_{x\in M}$ such that $I(f_x(\alpha)) = f_{I(x)}(\alpha)$ for all $x \in M$ and all $\alpha \in [0, 1]$. Assume that T is I-nonexpansive and $M = IM$. Suppose that T and I are R-subweakly commuting. If T or I is continuous, then $F(T) \cap F(I) \neq \varnothing$ provided one of the following conditions holds:*

(i) *M is closed and $\mathrm{cl}(T(M))$ is compact;*
(ii) *M is compact;*
(iii) *M is closed, $F(I)$ is bounded and T is a compact map;*
(iv) *M is bounded and closed and I is a demicompact map;*
(v) *M is weakly compact and T is completely continuous;*
(vi) *M is weakly compact, I and T are weakly continuous and the family $\mathcal{F} = \{f_x\}_{x\in M}$ is jointly weak continuous instead of jointly continuous.*

Proof. For $n \in N$, let $\lambda_n = \frac{n}{n+1}$. Then $\lambda_n \in (0,1)$. Define $T_n x = f_{T(x)}(\lambda_n)$, $x \in M$. Each T_n is a well-defined selfmap of M. For any x, y in M, we get

$$d(T_n x, T_n y) = d(f_{Tx}(\lambda_n), f_{Ty}(\lambda_n)) \leq \phi(\lambda_n) d(Tx, Ty) \leq \phi(\lambda_n) d(Ix, Iy). \quad (2)$$

The continuity of T (if I is continuous, then T is also continuous) and (2) imply that each T_n is continuous and I-contraction on M. As $I(f_x(\alpha)) = f_{I(x)}(\alpha)$, it follows from the property of $\mathcal{F}$ and (1) that for each $x \in M$,

$$\begin{aligned} d(T_n Ix, IT_n x) &= d(f_{TI(x)}(\lambda_n), I(f_{T(x)}(\lambda_n))) \\ &= d(f_{TI(x)}(\lambda_n), (f_{IT(x)}(\lambda_n))) \\ &\leq \phi(\lambda_n) d(TIx, ITx) \\ &\leq \phi(\lambda_n) R d(f_{Tx}(\lambda_n), Ix) \\ &= \phi(\lambda_n) R d(T_n x, Ix). \end{aligned}$$

Thus T_n and I are $\phi(\lambda_n)R$-weakly commuting on M for each n and $T_n(M) \subseteq M = I(M)$. By Theorem 3.1, for each $n \geq 1$, there is a unique $x_n \in M$ such that $x_n = T_n x_n = I x_n$.

(i) By the compactness of $\mathrm{cl}(T(M))$, $\{Tx_n\}$ has a subsequence $\{Tx_{n_j}\}$ which converges to $z \in M$ as $j \to \infty$. The joint continuity of $\mathcal{F}$ gives that $x_{n_j} = T_{n_j} x_{n_j} = f_{T(x_{n_j})}(\lambda_{n_j}) \to f_z(1) = z$ as $j \to \infty$. As T is continuous so $Tx_{n_j} \to Tz$ as $j \to \infty$ and hence by the uniqueness of limit, $Tz = z$. As $TM \subseteq IM$, it follows that $z = Tz = Iy$ for some $y \in M$. Further, $d(Tx_{n_j}, Ty) \leq d(Ix_{n_j}, Iy) = d(x_{n_j}, z)$. Taking the limit as $j \to \infty$ yields $Tz = Ty$. Thus $z = Tz = Ty = Iy$. Since T and I are R-subweakly commuting, it follows that (note $f_{Ty}(1) = Ty$)

$$d(Tz, Iz) = d(TIy, ITy) \leq R d(Ty, Iy) = 0.$$

Hence $z \in F(T) \cap F(I)$.

(ii) It follows from (i) as T is continuous.

(iii) As T is compact and $\{x_n\}$ being in $F(I)$ is bounded so $\{Tx_n\}$ has a subsequence $\{Tx_{n_j}\}$ such that $Tx_{n_j} \to x \in M$ as $j \to \infty$. By the joint continuity of $\mathcal{F}$ we obtain $x_{n_j} = T_{n_j} x_{n_j} = f_{T(x_{n_j})}(\lambda_{n_j}) \to f_x(1) = x$. As T is continuous so $Tx_{n_j} \to Tx$ as $j \to \infty$ and hence $Tx = x$. The result now follows as in (i).

(iv) As in (i), there is a unique $x_n \in M$ such that $x_n = T_n x_n = I x_n$. Since $\{x_n\}$ is bounded and $\{x_n - Ix_n\}$ is a constant sequence converging strongly to 0 so by the demicompactness of I, $\{x_n\}$ has a subsequence $\{x_{n_j}\}$ converging strongly to $z \in M$ as $j \to \infty$. Since T is continuous, Tx_{n_j} converges strongly to Tz as $j \to \infty$. Also, $x_{n_j} = T_{n_j} x_{n_j} = f_{T(x_{n_j})}(\lambda_{n_j}) \to f_{T(z)}(1) = Tz$ as $j \to \infty$. By the uniqueness of limit, we get $z = Tz$. The result now follows as in (i).

(v) The sequence $\{x_n\}$ has a subsequence $\{x_{n_j}\}$ converging weakly to $y \in M$ as $j \to \infty$. Since T is completely continuous, T converges strongly to Ty as $j \to \infty$. Also, $x_{n_j} = T_{n_j} x_{n_j} = f_{T(x_{n_j})}(\lambda_{n_j}) \to f_{T(y)}(1) = Ty$ as $j \to \infty$. Thus $Tx_{n_j} \to T^2 y$ as $j \to \infty$ and consequently $T^2 y = Ty$ implies that $Tw = w$, where $w = Ty$. As in (i), we obtain $w \in F(I)$.

(vi) The sequence $\{x_n\}$ has a subsequence $\{x_{n_j}\}$ converging weakly to $y \in M$ as $j \to \infty$ and as I is weakly continuous so $Iy = y$. As T is weakly continuous so $Tx_{n_j} \xrightarrow{w} Ty$ as $j \to \infty$ and hence by the joint weak continuity of the family, we have $x_{n_j} = f_{T(x_{n_j})}(\lambda_{n_j}) \xrightarrow{w} f_{Ty}(1) = Ty$ as $j \to \infty$. Also $x_{n_j} \xrightarrow{w} y$ as $j \to \infty$. Thus $Ty = y$. □

Remark 3.4. The above result provides a common fixed point generalization of Theorems 1 and 2 due to Anderson et. al. [2]. Theorem 3.3(i) extends recent the results of Shahzad ([19, Theorem 6; 20, Theorem 2.2], [21, Lemma 2.2]) to a non-starshaped domain in a Fréchet space. If I is an identity map on M and X is a Banach space, then Theorem 3.3(ii & vi) reduces to the corresponding results of Dotson [7].

For a set M with property (N), we have the following common fixed point result which contains Theorems 2.1, 2.2 [3], Theorems 1, 2 [6], Theorem 2.2 [10] and Theorem 6 [11].

Theorem 3.5. *Let T, I be selfmaps on a subset M of a Fréchet space (X, d) where d is translation invariant and $d(\alpha x, \alpha y) \leq \alpha d(x, y)$ for each α with $0 < \alpha < 1$ and $x, y \in X$. Assume that $q \in F(I)$, M has property (N), I satisfies condition* (C)*, T is I-nonexpansive and $TM \subseteq IM$. Suppose that T and I are R-subweakly commuting. If T or I is continuous, then $F(T) \cap F(I) \neq \varnothing$ provided one of the following conditions holds:*

(i) *M is closed and* $\mathrm{cl}(T(M))$ *is compact;*
(ii) *M is compact;*
(iii) *M is closed, $F(I)$ is bounded and T is a compact map;*
(iv) *M is bounded and closed and I is a demicompact map;*
(v) *M is weakly compact and T is completely continuous;*
(vi) *M is weakly compact and I and T are weakly continuous;*
(vii) *M is weakly compact, I is weakly continuous and $I - T$ is demiclosed at* 0,
(viii) *M is weakly compact, I is weakly continuous and X satisfies Opial's condition;*
(ix) *M is closed bounded and T is a hemicompact map;*
(x) *M is closed bounded and T is a condensing map.*

Proof. Define

$$T_n : M \to M \text{ by } T_n(x) = k_n Tx + (1 - k_n)q \tag{3}$$

for all $x \in M$ and fixed sequence of real numbers k_n $(0 < k_n < 1)$ converging to 1. Then each T_n is a well-defined continuous selfmap of M as M has property (N). For any x, y in M we get

$$\begin{aligned} d(T_n x, T_n y) &= d(k_n Tx + (1 - k_n)q, k_n Ty + (1 - k_n)q) \\ &\leq k_n d(Tx, Ty) \leq k_n d(Ix, Iy). \end{aligned} \tag{4}$$

As I satisfies condition (C) and $Iq = q$, it follows that for each $x \in M$,

$$\begin{aligned} d(T_n Ix, IT_n x) &= d(k_n TIx + (1 - k_n)q, k_n ITx + (1 - k_n)q) \\ &\leq k_n d(TIx, ITx) \leq k_n R d(k_n Tx + (1 - k_n)q, Ix) \\ &= k_n R d(T_n x, Ix). \end{aligned}$$

Thus T_n and I are k_nR-weakly commuting on M for each n and $T_n(M) \subseteq I(M)$. By Theorem 3.1, for each $n \geq 1$, there is a unique $x_n \in M$ such that $x_n = T_nx_n = Ix_n$.

(i) By the compactness of $\mathrm{cl}(T(M))$, $\{Tx_n\}$ has a subsequence $\{Tx_{n_j}\}$ which converges to $z \in M$ as $j \to \infty$. Since $k_{n_j} \to 1$, $x_{n_j} = T_{n_j}x_{n_j} = k_{n_j}Tx_{n_j} + (1 - k_{n_j})q \to z$ as $j \to \infty$. As T is continuous, $Tx_{n_j} \to Tz$ as $j \to \infty$ and hence $Tz = z$. As $TM \subseteq IM$, it follows that $z = Tz = Iy$ for some $y \in M$. Further,

$$d(Tx_{n_j}, Ty) \leq d(Ix_{n_j}, Iy) = d(x_{n_j}, z).$$

Taking the limit as $j \to \infty$ yields $Tz = Ty$. Thus $z = Tz = Ty = Iy$. Since T and I are R-subweakly commuting, it follows that

$$d(Tz, Iz) = d(TIy, ITy) \leq Rd(Ty, Iy) = 0.$$

Hence $z \in F(T) \cap F(I)$.

(ii)–(iv) These proofs follow the pattern of Theorem 3.3 (ii)–(iv).

(v) As in the proof of Theorem 3.3(v), we can find a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ in M converging weakly to $y \in M$ as $j \to \infty$. Since T is completely continuous, Tx_{n_j} converges strongly to Ty as $j \to \infty$. Since $k_{n_j} \to 1$, $x_{n_j} = T_{n_j}x_{n_j} = k_{n_j}Tx_{n_j} + (1 - k_{n_j})q \to Ty$ as $j \to \infty$. Thus $Tx_{n_j} \to T^2y$ as $j \to \infty$ and consequently $T^2y = Ty$ implies that $Tw = w$, where $w = Ty$ and the result follows as in (i).

(vi) The sequence $\{x_n\}$ has a subsequence $\{x_{n_j}\}$ converging weakly to $y \in M$ as $j \to \infty$ and as I is weakly continuous, $Iy = y$. As T is weakly continuous, $Tx_{n_j} \xrightarrow{w} Ty$ as $j \to \infty$. Since $k_{n_j} \to 1$, $x_{n_j} = T_{n_j}x_{n_j} = k_{n_j}Tx_{n_j} + (1 - k_{n_j})q \xrightarrow{w} Ty$ as $j \to \infty$. Also $x_{n_j} \xrightarrow{w} y$ as $j \to \infty$. Thus $Ty = y$.

(vii) The sequence $\{x_n\}$ has a subsequence $\{x_{n_j}\}$ converging weakly to $y \in M$ as $j \to \infty$ and as I is weakly continuous, $Iy = y$. As M is bounded,

$$\begin{aligned}(I - T)(x_{n_j}) = x_{n_j} - Tx_{n_j} &= -(1/k_{n_j})(T_{n_j}x_{n_j} - (1 - k_{n_j})q)\\ &= (1 - k_{n_j})(x_{n_j} - q) \to 0 \text{ as } j \to \infty. \end{aligned} \tag{5}$$

Since $I - T$ is demiclosed at 0, $(I - T)y = 0$ and hence $Ty = Iy = y$ as required.

(viii) By Lemma 3.2, $I - T$ is demiclosed at 0. The result follows from (vii).

(ix) The sequence $\{x_{n_j}\}$ is bounded and by (5) $d(x_{n_j}, Tx_{n_j}) \to 0$ as $j \to \infty$. Since T is hemicompact, $\{x_{n_j}\}$ converges to some $y \in M$ as $j \to \infty$. By the continuity of T, Tx_{n_j} converges to Ty and so by (3) x_{n_j} converges to Ty as $j \to \infty$. Hence $y = Ty$ and the result follows as in (i).

(x) Every condensing map on a closed bounded subset of a complete metric space is hemicompact by Lemma 2.1 [25]. Hence the result follows from (ix). $\square$

Example 3.6. Let $X = \Re$ and $M = \left\{0, 1, 1 - \frac{1}{n+1} : n \in N\right\}$ be endowed with the usual metric.

(1) Define $T(x) = 0$ for each $x \in M$. Clearly, M has property (N) for $q = 0$ and $k_n = 1 - \frac{1}{n+1}$. Let I be defined by $I(1) = 0 = I(0)$ and $I\left(1 - \frac{1}{n+1}\right) = 1$, $\forall n \in N$. All the conditions of Theorem 3.5(ii) are satisfied and consequently T and I have a common fixed point. However,

it is interesting to note that the results of Al-Thagafi [1], Baskaran et. al. [3] and Shahzad [19]–[22] cannot be applied, since the map I is discontinuous and the underlying domain is not starshaped.

(2) Define $T(1) = 0$ and $T(0) = T\left(1 - \frac{1}{n+1}\right) = 1$, $\forall n \in N$. Clearly, M has property (N) for $q = 0$ and $k_n = 1 - \frac{1}{n+1}$, $\forall n \in N$. Let I be the identity map on M. All of the conditions of Theorem 3.5(ii) are satisfied except the I-nonexpansiveness of T (as T is not continuous). Note $F(T) \cap F(I) = \varnothing$.

Example 3.7. (a) Let $X = \Re^2$ be endowed with the norm $\|\cdot\|$ defined by $\|(a, b)\| = |a| + |b|$, $(a, b) \in \Re^2$.

(1) Let $M = A \cup B$, where $A = \{(a, b) \in X : 0 \le a \le 1,\ 0 \le b \le 4\}$ and $B = \{(a, b) \in X : 2 \le a \le 3,\ 0 \le b \le 4\}$. Define $T, I : M \to M$ by

$$T(a,b) = \begin{cases} (2,b) & \text{if } (a,b) \in A \\ (1,b) & \text{if } (a,b) \in B \end{cases}$$

and $I(x) = x$, $\forall x = (a, b) \in M$. All of the conditions of Theorem 3.5 (ii) are satisfied except the condition that M has property (N). That is, $(1 - k_n)q + k_nT(M) \not\subseteq M$ for any choice of $q \in M$ and $\{k_n\}$. Note that $F(T) \cap F(I) = \varnothing$.

(2) If $M = \{(a, b) \in X : 0 \le a < \infty,\ 0 \le b \le 1\}$ and $T, I : M \to M$ is defined by

$$T(a, b) = (a + 1, b) \text{ and } I(x) = x,\ \forall x = (a, b) \in M.$$

All of the conditions of Theorem 3.5 (i)–(ii) are satisfied except the condition that M or $\mathrm{cl}(T(M))$ is compact. Note that $F(T) \cap F(I) = \varnothing$. Notice that M is convex and T-invariant and so it has property (N) for any choice of q and $\{k_n\}$.

(b) Let $X = \Re$ and $M = \{x : 0 \le x \le 1,\ x \in Q\}$ be endowed with the usual metric. Define $T(x) = 0$ for each $x \in M$. Clearly, M is not starshaped but M has property (N) for $q = 0$ and $k_n = 1 - \frac{1}{n+1}$. Let f be defined on M by $f(x) = 1 - x$, $\forall x \in M$. All of the conditions of Theorem 3.5 (i) are satisfied except the condition that M is closed. Note that $F(T) \cap F(I) = \varnothing$.

4. Best Approximation Results

We establish generalizations of Theorems 1.1–1.2, Theorem 7 in [11], Theorem 2.6 in [12], Theorem 6 in [19] and Theorem 2 from [22] in the sense that the set of best approximations need not be starshaped and the maps are noncommuting, in the results to follow.

Theorem 4.1. *Let T and I be selfmaps on a Fréchet space X and M a subset of X such that $T(M) \subseteq M$, $u \in F(T) \cap F(I)$. Suppose that $D = D_M^I(u)$ is nonempty, $D = ID$, T is I-nonexpansive on $D \cup \{u\}$ and I is nonexpansive on $P_M(u) \cup \{u\}$. Then D is T-invariant. Further if D has a contractive jointly continuous family of functions $\mathcal{F} = \{f_x\}_{x \in D}$ such that $I(f_x(\alpha)) = f_{I(x)}(\alpha)$ for* *all $x \in D$ and all $\alpha \in [0,1]$ and T and I are R-subweakly commuting on D, then $F(I) \cap F(T) \cap P_M(u) \neq \varnothing$ provided one of the following conditions holds:*

(i) *D is closed and $\mathrm{cl}(T(D))$ is compact;*
(ii) *D is compact;*
(iii) *D is closed, $F(I)$ is bounded and T is a compact map;*
(iv) *D is bounded and closed and I is a demicompact map;*
(v) *D is separable weakly compact and T is completely continuous;*
(vi) *D is separable weakly compact, I and T are weakly continuous and the family $\mathcal{F} = \{f_x\}_{x\in D}$ is jointly weakly continuous instead of jointly continuous.*

Proof. Let $y \in D$. Then $Iy \in D$ since $I(D) = D$. By the definition of D, $y \in M$ and since $T(M) \subseteq M$, it follows that $Ty \in M$. As T is I-nonexpansive on $D \cup \{u\}$, we have

$$d(Ty, u) = d(Ty, Tu) \leq d(Iy, u). \tag{6}$$

Since $Ty \in M$ and $Iy \in P_M(u)$, (6) implies that $Ty \in P_M(u)$. As I is nonexpansive on $P_M(u) \cup \{u\}$, we obtain

$$d(ITy, u) = d(ITy, Iu) \leq d(Ty, u) = d(Ty, Tu) \leq d(Iy, Iu) = d(Iy, u).$$

Thus $ITy \in P_M(u)$. This implies that $Ty \in C_M^I(u)$ and hence $Ty \in D$. Thus D is T-invariant. Now all the conditions of Theorem 3.3 are satisfied. Thus $F(I) \cap F(T) \cap P_M(u) \neq \varnothing$ under each one of the conditions (i)–(vi). □

Theorem 4.2. *Let T and I be selfmaps on a Fréchet space (X, d) where d is translation invariant and $d(\alpha x, \alpha y) \leq \alpha d(x, y)$, for each α with $0 < \alpha < 1$ and $x, y \in X$, and M be a subset of X such that $T(\partial M \cap M) \subseteq M$, $u \in F(T) \cap F(I)$. Suppose that $D = D_M^I(u)$ is nonempty, $D = ID$, T is I-nonexpansive on $D \cup \{u\}$ and I is nonexpansive on $P_M(u) \cup \{u\}$. Then D is T-invariant. Further if D has property* (N) *with $Iq = q$, I satisfies the condition* (C) *and T and I are R-subweakly commuting on D, then $F(I) \cap F(T) \cap P_M(u) \neq \varnothing$ provided one of the conditions* (i)–(x) *in Theorem* 3.5 *is satisfied by T, I with D in place of M.*

Proof. Let $y \in D$. Then $Iy \in D$ since $I(D) = D$. Note that for any $k \in (0, 1)$,

$$d(ku + (1-k)y, u) = d(ku + (1-k)y - u, 0) = (1-k)d(y, u) < \mathrm{dist}(u, M).$$

It follows that the line segment $\{ku + (1-k)y : 0 < k < 1\}$ and the set M are disjoint. Thus y is not in the interior of M and so $y \in \partial M \cap M$. Since $T(\partial M \cap M) \subset M$, Ty must be in M. Now the rest of the proof is similar to that of Theorem 4.1. Thus D is T-invariant. Now Theorem 3.5 guarantees that $F(I) \cap F(T) \cap P_M(u) \neq \varnothing$ under each one of the conditions (i)–(x). □

Remark 4.3. If $I(P_M(u)) \subseteq P_M(u)$, then $P_M(u) \subseteq C_M^I(u)$. Hence $D_M^I(u) = P_M(u)$. Also, if $I(C_M^I(u)) \subseteq C_M^I(u)$, then $I(D_M^I(u)) \subseteq I(C_M^I(u)) \subseteq D_M^I(u)$. Thus Theorems 4.1 and 4.2 hold for $D = P_M(u)$ as well as for $D = C_M^I(u)$. Hence Theorems 4.1-4.2 generalize several known results including those of Al-Thagafi [1], Brosowski [5], Hicks and Humphries [9], Sahab, Khan and Sessa [17], Sahney et. al. [18], Shahzad [19], [20] and Singh [23].

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Authors' addresses:

N. Hussain
Department of Mathematics, Faculty of Science
King Abdul Aziz University
P.O. Box 80203, Jeddah, 21589
Saudi Arabia
(On leave from Bahauddin Zakariya University, Multan, Pakistan)
E-mail: mnawab2000@yahoo.com

D. O'Regan
Department of Mathematics
National University of Ireland, Galway
Ireland
E-mail: donal.oregan@nuigalway.ie

R. P. Agarwal
Department of Mathematical Sciences
Florida Institute of Technology
Melbourne, Florida 32901–6975
U.S.A.
E-mail: agarwal@fit.edu