Georgian Mathematical Journal
Volume 12 (2005), Number 4, 693–696

ON AN APPROXIMATION ORDER OF THE OPTIMAL STOPPING PROBLEM FOR n-DIMENSIONAL DIFFUSION PROCESSES

GIORGI LOMINASHVILI

Abstract. An approximation order of the optimal stopping problem for multidimensional diffusion processes by the corresponding semidiscretization is considered.

2000 Mathematics Subject Classification: 60G40.
Key words and phrases: Optimal stopping, n-dimensional diffusion process, approximation order.

1. Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $W_t = (W_t^1, \dots, W_t^n)$ be n-dimensional standard Brownian motion defined on this space. Denote by $F^W = (\mathcal{F}_t^W)_{t \geq 0}$ the natural filtration of W_t completed with respect to P.

On the time interval $T = [0, \infty)$ let us consider the stochastic differential equation

$$dX_s = b(X_s)ds + \sigma(X_s)dW_s, \quad X_0 = x \in \mathbb{R}^n, \tag{1}$$

where

A. $b(x) = (b^1(x), \dots, b^n(x))$ and $\sigma(x) = (\sigma_j^i(x))$, $i, j = 1, \dots, n$, are locally Lipschitz functions from $\mathbb{R}^n$ into $\mathbb{R}^n$ and the space of $n \times n$ real matrices, respectively, which satisfy the linear growth condition

$$\|b(x)\| \leq k(1 + \|x\|), \quad \|\sigma(x)\| \leq k(1 + \|x\|)$$

with a constant k; here $\|\cdot\|$ stands for the Euclidean norm (Frobenius norm for $\sigma(x)$).

As is well known (see [2], Ch. VIII, § 2, Theorem 4), under condition **A** the stochastic differential equation (1) has a unique strong solution $X_t = (X_t^1, \dots, X_t^n)$ which is a Markov diffusion process.

Let D be a bounded domain in the space $\mathbb{R}^n$, and $\overline{D}$ and ∂D be its closure and smooth boundary (say, of C^2), respectively. Assume that $g = g(x)$ is a Lipschitz function defined on $\overline{D}$ with the Lipschitz constant L, and $c = c(x)$ is a continuous function defined on $\overline{D}$.

Let us introduce the first moment at which the process X_t, $t \in T$, exits from the domain D:

$$\sigma(D) = \inf\big(t \geq 0 : \ X_t \notin D\big),$$

ISSN 1072-947X / \$8.00 / © Heldermann Verlag *www.heldermann.de*

for which $\sup_{x\in\overline{D}} E_x\sigma(D) < \infty$ and let us consider the optimal stopping problem in the domain $\overline{D}$:

$$S(x) = \sup_{\tau\in\mathfrak{M}} E_x\bigg[g(X_{\tau\wedge\sigma(D)})e^{-r(\tau\wedge\sigma(D))} + \int\limits_0^{\tau\wedge\sigma(D)} c(X_t)e^{-rt}\,dt\bigg], \tag{2}$$

where P_x is the probability measure corresponding to the initial condition $X_0 = x$, E_x is the sign of taking expectation w.r.t. P_x, $\mathfrak{M}$ is the class of all stopping moments w.r.t. the filtration $F^W = (\mathcal{F}_t^W)_{t\geq 0}$, and $r > 0$ is some constant.

Let $X_t^{\sigma(D)} = X_{t\wedge\sigma(D)}$ be the stopped process which is a solution of the following stochastic differential equation:

$$dX_s^{\sigma(D)} = \widetilde{b}(X_s^{\sigma(D)})\,ds + \widetilde{\sigma}(X_s^{\sigma(D)})dW_s,$$

where $\widetilde{b}(x) = b(x)\chi_{_D}(x)$, $\widetilde{\sigma}(x) = \sigma(x)\chi_{_D}(x)$ and $\chi_{_D}(x)$ is the characteristic function of D.

The optimal stopping problem (2) can be equivalently rewritten in terms of stopped diffusion processes as

$$S(x) = \sup_{\tau\in\mathfrak{M}} E_x\bigg[g(X_\tau^{\sigma(D)})e^{-\int_0^\tau \widetilde{r}(X_s^{\sigma(D)})\,ds} + \int\limits_0^\tau \widetilde{c}(X_t^{\sigma(D)})e^{-\int_0^t \widetilde{r}(X_s^{\sigma(D)})\,ds}\,dt\bigg], \tag{3}$$

where $\widetilde{r}(x) = r\cdot\chi_{_D}(x)$, $\widetilde{c}(x) = c(x)\cdot\chi_{_D}(x)$.

Assume that h, $0 < h < 1$, is the parameter that tends to zero. Denote by $\mathfrak{M}_h$ the class of stopping moments τ_h, where τ_h takes the values $0, h, \dots, nh, \dots$, and the set $\{\tau_h \leq nh\} \in \mathcal{F}_{nh}^W$.

The discrete analog of the problem (3) is written as follows (see [1]):

$$S_h(x) = \sup_{\tau_h\in\mathfrak{M}_h} E_x\bigg[g(X_{\tau_h}^{\sigma(D)})e^{-\int_0^{\tau_h} \widetilde{r}(X_s^{\sigma(D)})\,ds} + \int\limits_0^{\tau_h} \widetilde{c}(X_t^{\sigma(D)})e^{-\int_0^t \widetilde{r}(X_s^{\sigma(D)})\,ds}\,dt\bigg]. \tag{4}$$

Denote by τ^* the optimal stopping time of problem (3) (see [3], Ch. 3, Theorem 3). Besides, introduce the so-called pseudo-optimal stopping moment τ_h' defined as follows:

$$\tau_h' = \inf\big(nh:\ nh \geq \tau^*\big),$$

which is an admissible stoping time from $\mathfrak{M}_h$. In [1], requiring only continuity of $g(x)$, Bensoussan and Robins showed that $S_h(x)$ converges to $S(x)$ as $h \to 0$.

2. Formulation and Proof of the Basic Result

Theorem. *Let (X_t, P_x), $t\in T$, be an n-dimensional diffusion process given by the stochastic differential equation* (1) *with the coefficients $b(x)$ and $\sigma(x)$ satisfying condition* $\mathbf{A}$, *and $g(x)$ be the Lipschitz function defined on $\overline{D}$. Then for the functions $S(x)$ and $S_h(x)$ defined by* (3) *and* (4), *respectively, the following estimate is valid:*

$$\sup_{x\in\overline{D}} \big|S(x) - S_h(x)\big| \leq p\cdot h^{1/2},$$

where $p = C_1 + rH + L\sqrt{2nC_2^2 + 2n^2C_3^2}$, r *is introduced in relation* (2) *and* H, C_1, C_2, C_3 *are the constants such that for* $x \in \overline{D}$

$$|g(x)| \le H, \quad |c(x)| \le C_1, \quad |b^i(x)| \le C_2, \quad |\sigma^i_j(x)| \le C_3, \quad i, j = 1, \dots, n.$$

Proof. It is obvious from (4) that

$$S_h(x) \ge E_x\Big[g(X^{\sigma(D)}_{\tau'_h})e^{-\int_0^{\tau'_h}\widetilde{r}(X^{\sigma(D)}_s)\,ds} + \int_0^{\tau'_h}\widetilde{c}(X^{\sigma(D)}_t)e^{-\int_0^t\widetilde{r}(X^{\sigma(D)}_s)\,ds}\,dt\Big],$$

and therefore

$$0 \le S(x) - S_h(x) \le E_x\Big[g(X^{\sigma(D)}_{\tau^*})e^{-\int_0^{\tau^*}\widetilde{r}(X^{\sigma(D)}_s)\,ds} + \int_0^{\tau^*}\widetilde{c}(X^{\sigma(D)}_t)e^{-\int_0^t\widetilde{r}(X^{\sigma(D)}_s)\,ds}\,dt\Big]$$
$$- E_x\Big[g(X^{\sigma(D)}_{\tau'_h})e^{-\int_0^{\tau'_h}\widetilde{r}(X^{\sigma(D)}_s)\,ds} + \int_0^{\tau'_h}\widetilde{c}(X^{\sigma(D)}_t)e^{-\int_0^t\widetilde{r}(X^{\sigma(D)}_s)\,ds}\,dt\Big]. \qquad (5)$$

Clearly,

$$g(X^{\sigma(D)}_{\tau^*})e^{-\int_0^{\tau^*}\widetilde{r}(X^{\sigma(D)}_s)\,ds} - g(X^{\sigma(D)}_{\tau'_h})e^{-\int_0^{\tau'_h}\widetilde{r}(X^{\sigma(D)}_s)\,ds}$$
$$= \Big(g(X^{\sigma(D)}_{\tau^*}) - g(X^{\sigma(D)}_{\tau'_h})\Big)e^{-\int_0^{\tau'_h}\widetilde{r}(X^{\sigma(D)}_s)\,ds}$$
$$+ g(X^{\sigma(D)}_{\tau^*})e^{-\int_0^{\tau^*}\widetilde{r}(X^{\sigma(D)}_s)\,ds}\Big(1 - e^{-\int_{\tau^*}^{\tau'_h}\widetilde{r}(X^{\sigma(D)}_s)\,ds}\Big). \qquad (6)$$

If we use the elementary inequality $1 - e^{-z} < z$, $z > 0$, then (6) leads to the following inequality

$$g(X^{\sigma(D)}_{\tau^*})e^{-\int_0^{\tau^*}\widetilde{r}(X^{\sigma(D)}_s)\,ds} - g(X^{\sigma(D)}_{\tau'_h})e^{-\int_0^{\tau'_h}\widetilde{r}(X^{\sigma(D)}_s)\,ds}$$
$$\le \Big|g(X^{\sigma(D)}_{\tau^*}) - g(X^{\sigma(D)}_{\tau'_h})\Big| + \big|g(X^{\sigma(D)}_{\tau^*})\big|\int_{\tau^*}^{\tau'_h}\widetilde{r}(X^{\sigma(D)}_s)\,ds. \qquad (7)$$

Taking into account (7) in (5), we obtain

$$0 \le S(x) - S_h(x) \le E_x\Big[\Big|g(X^{\sigma(D)}_{\tau^*}) - g(X^{\sigma(D)}_{\tau'_h})\Big| + \big|g(X^{\sigma(D)}_{\tau^*})\big|\int_{\tau^*}^{\tau'_h}\widetilde{r}(X^{\sigma(D)}_s)\,ds +$$
$$+ \int_{\tau^*}^{\tau'_h}\big|\widetilde{c}(X^{\sigma(D)}_t)\big|e^{-\int_0^t\widetilde{r}(X^{\sigma(D)}_s)\,ds}\,dt\Big].$$

We have

$$\int\limits_{\tau^*}^{\tau_h'} \big|\widetilde{c}(X_t^{\sigma(D)})\big| e^{-\int\limits_0^t \widetilde{r}(X_s^{\sigma(D)})\,ds}\,dt \le C_1(\tau_h'-\tau^*) \le C_1 h,$$

$$\big|g(X_{\tau^*}^{\sigma(D)})\big| \int\limits_{\tau^*}^{\tau_h'} \widetilde{r}(X_s^{\sigma(D)})\,ds \le rH(\tau_h'-\tau^*) \le rHh.$$

It remains to estimate the value $E_x\Big|g(X_{\tau^*}^{\sigma(D)}) - g(X_{\tau_h'}^{\sigma(D)})\Big|$. Since $g(x)$ is the Lipschitz function defined on $\overline{D}$, we have

$$E_x\Big|g(X_{\tau^*}^{\sigma(D)}) - g(X_{\tau_h'}^{\sigma(D)})\Big| \le LE_x\big\|X_{\tau^*}^{\sigma(D)} - X_{\tau_h'}^{\sigma(D)}\big\| \le L\Big(E_x\big\|X_{\tau^*}^{\sigma(D)} - X_{\tau_h'}^{\sigma(D)}\big\|^2\Big)^{1/2},$$

$$E_x\big\|X_{\tau^*}^{\sigma(D)} - X_{\tau_h'}^{\sigma(D)}\big\|^2 = E_x\bigg\|\int\limits_{\tau^*}^{\tau_{h'}} \widetilde{b}(X_t^{\sigma(D)})\,dt + \int\limits_{\tau^*}^{\tau_{h'}} \widetilde{\sigma}(X_t^{\sigma(D)})\,dW_t\bigg\|^2$$

$$\le 2E_x \sum_{i=1}^n \bigg[\int\limits_{\tau^*}^{\tau_{h'}} \widetilde{b}^i(X_t^{\sigma(D)})\,dt\bigg]^2 + 2E_x \sum_{i=1}^n \bigg[\int\limits_{\tau^*}^{\tau_{h'}} \sum_{j=1}^n \big(\widetilde{\sigma}_j^i(X_t^{\sigma(D)})\big)^2\,dt\bigg]$$

$$\le E_x\Big[2nC_2^2(\tau_h'-\tau^*)^2 + 2n^2C_3^2(\tau_h'-\tau^*)\Big].$$

Finally,

$$\sup_{x\in\overline{D}} \big|S(x) - S_h(x)\big| \le C_1 h + rHh + \sqrt{h(2nC_2^2 + 2n^2C_3^2)}\cdot L \le p\cdot h^{1/2}. \qquad \square$$

References

1. A. Bensoussan and M. Robin, On the convergence of the discrete time dynamic programming equation for general semigroups. *SIAM J. Control Optim.* **20**(1982), No. 5, 722–746.
2. I. I. Gikhman and A. V. Skorokhod, Introduction to the theory of random processes. (Russian) 2nd ed. *Nauka, Moscow,* 1977.
3. A. N. Shiryaev, Statistical sequential analysis. Optimal stopping rules. (Russian) 2nd ed. *Nauka, Moscow,* 1976.

(Received 7.03.2005; revised 12.09.2005)

Author's address:

A. Tsereteli Kutaisi State University
59, Tamar Mepe St., Kutaisi 4600
Georgia