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GENERALIZED DERIVATION AND DOUBLE OPERATOR INTEGRALS

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Abstract. Let H be a separable infinite dimensional complex Hilbert space, and let $\mathbb{B}(H)$ denote the algebra of all bounded linear operators on H. Let A, B be operators in $\mathbb{B}(H)$. We define the generalized derivation $\delta_{A,B} : \mathbb{B}(H) \mapsto \mathbb{B}(H)$ by $\delta_{A,B}(X) = AX - XB$. In this paper we consider the question posed by Turnsek [29], when $\overline{\operatorname{ran}(\delta_{A,B}\mid_{C_p})}^{c_p} = \overline{\operatorname{ran}(\delta_{A,B} \cap_{C_p})}^{c_p}$? We prove that this holds in the case where A and B satisfy the Fuglede–Putnam theorem. Finally, we apply the obtained results to double operator integrals.

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1. Introduction

Let H be a separable infinite dimensional complex Hilbert space, and let $\mathbb{B}(H)$ denote the algebra of all bounded linear operators on H. Let $T \in \mathbb{B}(H)$ be compact, and let $s_1(T) \geq s_2(T) \geq \cdots \geq 0$ denote the singular values of T, i.e., the eigenvalues of $|T| = (T^*T)^{\frac{1}{2}}$ arranged in their decreasing order. The operator T is said to belong to the Schatten p-class C_p if $\|T\|_p = [\sum_{i=1}^{\infty} s_j(T)^p]^{\frac{1}{p}} = [\operatorname{tr}|T|^p]^{\frac{1}{p}} < \infty$, $1 \leq p < \infty$, where tr denotes the trace functional. Hence $C_1(H)$ is the trace class, $C_2(H)$ is the Hilbert–Schmidt class, and C_∞ is the class of compact operators with $\|T\|_\infty = s_1(T) = \sup_{\|f\|=1} \|Tf\|$ denoting the usual operator norm. For the general theory of Schatten p-classes (see [27], [28]).

Let A, B be operators in $\mathbb{B}(H)$. We define the generalized derivations $\delta_{A,B} : \mathbb{B}(H) \mapsto \mathbb{B}(H)$ by $\delta_{A,B}(X) = AX - XB$ and $\delta^*_{A,B} : \mathbb{B}(H) \mapsto \mathbb{B}(H)$ by $\delta^*_{A,B}(X) = A^*X - XB^*$. It is clear that $\delta_{A,B}(C_p) \subseteq C_p$. However it may also happen that $\delta_{A,B}(X) \in C_p$ for some $X \in B(H) \setminus C_p$, hence $\operatorname{ran}(\delta_{A,B}\mid_{C_p}) \subseteq \operatorname{ran}(\delta_{A,B} \cap C_p)$ and then we also have $\overline{\operatorname{ran}(\delta_{A,B}\mid_{C_p})}^{C_p} \subseteq \overline{\operatorname{ran}(\delta_{A,B} \cap C_p)}^{C_p}$, where $\overline{(\cdot)}^{C_p}$ denotes the closure of the C_p norm. Turnsek [29] asked when the reverse inclusion is possible. He proved that

$$\overline{\operatorname{ran}(\delta_{A,B}\mid_{C_p})}^{C_p} = \overline{\operatorname{ran}(\delta_{A,B} \cap_{C_p})}^{C_p} \tag{1.1}$$

whenever A and B are normal operators. Or, equivalently, if $\delta_{A,B}(X) \in C_p$, then $\delta_{A,B}(X) = \operatorname{Lim}_n \delta_{A,B}(X_n)$, and $X_n \in C_p$. In this paper we prove that this

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holds in the case where A and B satisfy the Putnam–Fuglede theorem (i.e., if $AX = XB$ for $X \in C_p$, then $A^*X = XB^*$).

Let A and B be normal operators in $\mathbb{B}(H)$ and denote by E and F, respectively, their spectral measures. Furthermore, let f be a bounded Borel measurable function defined on $\sigma(A) \times \sigma(B)$. We define the bounded operator $f(A,B) : C_2(H) \to C_2(H)$ by

$$f(A,B)X = \int\limits_{\sigma(A)} \int\limits_{\sigma(B)} f(z,w)E(dz)XF(dw).$$

For analytic (in each variable) functions, defined in a neibourghood of the Cartesian product of the spectrum of two arbitrary bounded Hilbert space operators, the corresponding functional calculus on $\mathbb{B}(H)$, sometimes under different names, was considered in pretty good details by many authors (see [6] or [8] for details). For essentially bounded Borel functions defined on the Cartesian product of the spectrum of two normal operators, the corresponding functional calculus could not, in general, be defined on the whole $\mathbb{B}(H)$ (see [9] for a counter example), but merely on $C_2(H)$, which is, therefore, a natural framework for introducing double operator integrals. For their presentation, D. R. Jocic [13] used a slightly different approach than the one used in [4] and [5], where one can also find more details about integration in other Schatten classes. In this paper we follow the techniques introduced by Turnsek [29] to show that the above result remains true for a large classes of operators including normal operators in p-Schatten classes. The following formulas are the main properties of this functional calculus:

(i) $f(A,B)X = AX$ for $f(z,w) = z$, $f(A,B)X = XB$ for $f(z,w) = w$.
(ii) $(\alpha f + \beta g)(A,B)X = \alpha f(A,B)X + \beta g(A,B)X$.
(iii) $(fg)(A,B)X = f(A,B)(g(A,B)X)$.
(iv) $f(A,B)^* = \overline{f}(A,B)$.

From (iii) and (iv) it follows that $f(A,B)$ is a normal operator on $C_2(H)$, and also

(v) $\mathrm{tr}(f(A,B)X(g(A,B)Y)^*) = \mathrm{tr}(((f\overline{g})(A,B)X)Y^*)$.

2. On the Closure of the Range of a Generalized Derivation

Let $\mathcal{B}$ be a complex Banach space. We say that $b \in \mathcal{B}$ is orthogonal to $a \in \mathcal{B}$ if for all complexes λ there holds $\|a + \lambda b\| \geq \|a\|$. This definition has a natural geometric interpretation, namely, $b \perp a$ if and only if the complex line $\{a + \lambda b \mid \lambda \in C\}$ is disjoint with the open ball $K(0, \|a\|)$, i.e, if and only if this complex line is a tangent one. Note that if b is orthogonal to a, then a need not be orthogonal to b. If $\mathcal{B}$ is a Hilbert space, then from the above orthogonality condition it follows that $\langle a, b\rangle = 0$, i.e, orthogonality in the usual sense. Let $\mathcal{V}$ be a subspace of a Banach space $\mathcal{B}$ and define

$$\mathcal{V}_\perp = \{x \in \mathcal{B} : \|v + x\| \geq \|x\| \, for\, all \, v \in \mathcal{V}\}.$$

We begin by recalling the following well known lemma and theorem.

Lemma 2.1 ([30]). *Let $\mathcal{B}$ be a reflexive Banach space and $\mathcal{V}$ a closed subspace. If $\mathcal{V}_\perp = \{0\}$, then $\mathcal{V} = \mathcal{B}$.*

Theorem 2.1 ([14]). *Let A, B be two operators in $\mathbb{B}(H)$ such that $\ker \delta_{A,B} \subseteq \ker \delta_{A^*,B^*}$. Then $S \in \ker \delta_{A,B} \cap C_p$ if and only if*

$$\|S + \delta_{A,B}(X)\|_p \geq \|S\|_p$$

for all $X \in C_p$.

Theorem 2.2. *with the assumption of Theorem 2.1 the algebraic direct sum satisfies*

$$\overline{\operatorname{ran} \delta_{A,B}} \oplus \ker \delta_{A,B} = C_p.$$

Proof. Since the range and the kernel are orthogonal, the sum is closed. Now Assume that $\|\delta_{A,B}(X) + Y + Z\|_p \geq \|Z\|_p$ for all $X \in C_p$ and all $Y \in \ker \delta_{A,B}$. If we take $Y = 0$, then from Theorem 2.1 it follows that $Z \in \ker \delta_{A,B}$. Thus put $Y = -Z$ to get $\|\delta_{A,B}(X)\|_p \geq \|Z\|_p$ for all $X \in C_p$. But this implies $Z = 0$ and from Lemma 2.1 it follows that $\overline{\operatorname{ran} \delta_{A,B}} \oplus \ker \delta_{A,B} = C_p$. □

Remark 2.1. By applying Theorem 2.1 we get for $p > 1$

$$\operatorname{ran}(\delta_{A,B} \mid_{C_p})_\perp = (\operatorname{ran} \delta_{A,B} \cap C_p)_\perp = \ker(\delta_{A,B} \mid_{C_p}).$$

Theorem 2.3. *Let A, B be operators in $\mathbb{B}(H)$ such that $\ker \delta_{A,B} \subseteq \ker \delta_{A^*,B^*}$, then*

$$\overline{\operatorname{ran}(\delta_{A,B} \mid_{C_p})}^{C_p} = \overline{\operatorname{ran}(\delta_{A,B} \cap C_p)}^{C_p}.$$

Proof. By applying Theorem 2.2 we get

$$\overline{\operatorname{ran} \delta_{A,B}} \oplus \ker \delta_{A,B} = C_p.$$

Now from Remark 2.1 we see that $\operatorname{ran}(\delta_{A,B} \mid_{C_p})_\perp = (\operatorname{ran} \delta_{A,B} \cap C_p)_\perp$. Since $\overline{\operatorname{ran}(\delta_{A,B} \mid_{C_p})}^{C_p} \subseteq \overline{(\operatorname{ran} \delta_{A,B} \cap C_p)}^{C_p}$. It is easy to see that if $\mathcal{X}$ and $\mathcal{Y}$ are closed subspaces of a Banach space $\mathcal{B}$, $\mathcal{X} \subseteq \mathcal{Y}$, $\mathcal{X}_\perp = \mathcal{Y}_\perp$ and $\mathcal{X} \oplus \mathcal{X}_\perp = \mathcal{Y} \oplus \mathcal{Y}_\perp$. Then $\mathcal{X} = \mathcal{Y}$. By this we complete the proof. □

Corollary 2.1. *Let A and B be operators in $\mathbb{B}(H)$. Then*

$$\overline{\operatorname{ran}(\delta_{A,B} \mid_{C_p(H)})}^{C_p(H)} = \overline{\operatorname{ran}(\delta_{A,B} \cap_{C_p(H)})}^{C_p(H)}$$

under either of the following hypotheses:
(1) *A and B are normal operators.*
(2) *A and B are such that $\|Ax\| \geq \|x\| \geq \|Bx\|$ for all $x \in H$.*
(3) *A is invertible and B is such that $\|A^{-1}\| \, \|B\| \leq 1$.*
(4) *$A = B$ and A is a cyclic subnormal operator.*
(5) *A and B are contractions.*

Proof. (1) It is well known that if the operators A, B are normal, then $\ker \delta_{A,B} \subseteq \ker \delta_{A^*,B^*}$.

(2) Under these restrictions the result of Y. Tong [31] guarantees that for all $T \in \ker(\delta_{A,B} \mid \mathcal{K}(H))$, $\overline{R(T)}$ reduces $A, \ker(T)^\perp$ reduces B, and $A \mid_{\overline{R(T)}}$ and

$B\mid_{\ker(T)^\perp}$ are unitary operators. Take $H_1 = H = \overline{\operatorname{ran} S} \oplus \overline{\operatorname{ran} S}^\perp$, $H_2 = H = \ker S \oplus \ker S^\perp$. According to the decomposition of H and for $A_1 : H_1 \to H_1$, $A_2 : H_2 \to H_2$, $S : H_2 \to H_1$, we can write

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad B^* = \begin{pmatrix} B_1^* & 0 \\ 0 & B_2^* \end{pmatrix}, \quad S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

From $AS = SB$ it follows that $A_1S = SB_1$ and since A_1, B_1 are unitary operators we obtain $A_1^*S = SB_1^*$ and the result holds by the above theorem.

The above inequality holds in particular if $A = B$ is isometric, in other words $\|Ax\| = \|x\|$ for all $x \in H$.

(3) In this case it suffices to take $A_1 = \|B\|^{-1} A$ and $B_1 = \|B\|^{-1} B$, then $\|A_1x\| \geq \|x\| \geq \|B_1x\|$ and the result holds by (2) for all $x \in H$.

(4) Since T commutes with A, it follows that T is subnormal [35]. But any compact subnormal operator is normal, hence T is normal. By applying the Fuglede–Putnam theorem we get $AT = TA$ implies $A^*T = TA^*$. □

3. Application

For generalized normal derivations, acting on $\mathbb{B}(H)$ Danko R.Jocic has shown in [13] that the following integral representation formula holds:

$$f(A)X - Xf(B) = \int\limits_{\sigma(A)} \int\limits_{\sigma(B)} \frac{f(z) - f(w)}{z - w} E(dz)(AX - XB)F(dw), \tag{3.1}$$

whenever $AX - XB$ is a Hilbert–Schmidt class operator and f is a Lipschitz class function on $\sigma(A) \cup \sigma(B)$. In this section we will prove that equality (3.1) still holds for the generalized derivation $AX - XB$, where A and B^* satisfy the Fuglede–Putnam theorem.

Theorem 3.1. *Let A and B be operators in $\mathbb{B}(H)$ such that $\ker \delta_{A,B} \subseteq \ker \delta_{A^*,B^*}$. Then*

$$\|\delta_{A,B}(X) + S\|_2^2 = \|\delta_{A,B}(X)\|_2^2 + \|S\|_2^2$$

for all $X \in C_2$ and for all $S \in \ker \delta_{A,B} \cap C_2(H)$.

Proof. It is well known that the Hilbert–Schmidt class $C_2(H)$ is a Hilbert space under the inner product

$$\langle Y, Z \rangle = \operatorname{tr}(Z^*Y) = \operatorname{tr}(YZ^*).$$

Note here that for the Hilbert–Schmidt norm $\|\cdot\|_2$, the orthogonality result in Theorem 2.3 is to be understood in the usual Hilbert space sense. Thus if $\ker \delta_{A,B} \subseteq \ker \delta_{A^*,B^*}$, then

$$\|T + \delta_{A,B}(X)\|_2^2 = \|\delta_{A,B}(X)\|_2^2 + \|T\|_2^2,$$

for all $X \in C_2(H)$ and for all $T \in \ker \delta_{A,B} \cap C_2(H)$. This can be seen as an immediate consequence of the fact that

$$R(\delta_{A,B} \mid C_2(H))^\perp = \ker(\delta_{A,B} \mid C_2(H))^* = \ker(\delta_{A^*,B^*} \mid C_2(H)).$$

□

Now by the above theorem we can prove that (1.1) remains true in the case of the Hilbert–Schmidt class $C_2(H)$.

Theorem 3.2. *Let A and B be operators in $\mathbb{B}(H)$ such that* $\ker\delta_{A,B} \subseteq \ker\delta_{A^*,B^*}$. *Then*

$$\overline{\operatorname{ran}(\delta_{A,B}\mid_{C_2(H)})}^{C_2(H)} = \overline{\operatorname{ran}(\delta_{A,B}\cap_{C_2(H)})}^{C_2(H)}.$$

Proof. Since

$$\operatorname{ran}^{\perp}(\delta_{A,B}\mid_{C_2(H)}) = \ker(\delta_{A,B}\mid_{C_2(H)}),$$

if $S \in \operatorname{ran}^{\perp}(\delta_{A,B}\mid_{C_2(H)})$, then $S \in \ker(\delta_{A,B}\mid_{C_2(H)})$. Now Theorem 3.1 implies that $S \in (\operatorname{ran}\delta_{A,B} \cap C_2(H))^{\perp}$, that is,

$$\operatorname{ran}^{\perp}(\delta_{A,B}\mid_{C_2(H)}) \subseteq (\operatorname{ran}\delta_{A,B} \cap C_2(H))^{\perp}.$$

Since

$$(\operatorname{ran}\delta_{A,B} \cap C_2(H))^{\perp} \subseteq \operatorname{ran}^{\perp}(\delta_{A,B}\mid_{C_2(H)})$$

we have

$$\operatorname{ran}^{\perp}(\delta_{A,B}\mid_{C_2(H)}) = (\operatorname{ran}\delta_{A,B} \cap C_2(H))^{\perp}.$$

Consequently,

$$\overline{\operatorname{ran}(\delta_{A,B}\mid_{C_2(H)})}^{C_2(H)} = \overline{(\operatorname{ran}\delta_{A,B} \cap C_2(H))}^{C_2(H)}.$$

□

For any operator A in $\mathbb{B}(H)$ set, as usual, $|A| = (A^*A)^{\frac{1}{2}}$ and $[A^*, A] = A^*A - AA^* = |A|^2 - |A^*|^2$ (the self-commutator of A), and consider the following standard definitions: A is normal if $A^*A = AA^*$, hyponormal if $A^*A - AA^* \geq 0$, p-quasihyponormal if $A^*((A^*A)^p - (AA^*)^p)A \geq 0$ $(p > 0)$, (p,k)-quasihyponormal if $A^{*k}((A^*A)^p - (AA^*)^p)A^k \geq 0$ $(p > 0, k \in \mathbb{N})$, and normaloid if $\|A\| = r(A)$ (the spectral radius of A). Let (N), (HN), $Q(p)$, $(Q(p,k))$ and (NL) denote the classes consisting of normal, hyponormal, p-quasihyponormal, (p,k)-quasihyponormal, and normaloid operators. These classes are related by the proper inclusion

$$(N) \subset (HN) \subset (Q(p)) \subset (Q(p,k)) \subset (NL).$$

A is said to be p-hyponormal if $(A^*A)^p - (AA^*)^p \geq 0$ for some $0 < p \leq 1$. If $p = 1$, A is said to be hyponormal and if $p = \frac{1}{2}$, A is said to be semi-hyponormal. A is said to be log-hyponormal if A is invertible and satisfies the equality

$$\log(A^*A) \geq \log(AA^*).$$

It is known that invertible p-hyponormal operators are log-hyponormal operators but the converse is not true [32]. However it is very interesting that we may regard log-hyponormal operators as 0-hyponormal operators [32, 33]. The idea of a log-hyponormal operator is due to Ando [2] and the first paper in which log-hyponormality appeared is [10]. See [1, 32, 33, 34] for the properties of log-hyponormal operators.

Corollary 3.1. *Let A and B be operators in $\mathbb{B}(H)$. Then*

$$\overline{\operatorname{ran}(\delta_{A,B}\mid_{C_2(H)})}^{C_2(H)} = \overline{\operatorname{ran}(\delta_{A,B} \cap_{C_2(H)})}^{C_2(H)}$$

under either of the following hypotheses:

(1) *A and B^* are hyponormal operators.*

(2) *A is invertible and B is such that $\|A^{-1}\|\,\|B\| \leq 1$.*

(3) *A and B are such that $\|Ax\| \geq \|x\| \geq \|Bx\|$ for all $x \in H$.*

(4) *A is a (p,k)-quasihyponormal operator and B^* is an invertible (p,k)-quasihyponormal operator.*

(5) *A and B^* are* log*-hyponormal operators.*

Proof. Either of the above hypotheses satisfies the Fuglede–Putnam theorem (see [23],[24] [3], [7].) □

Now we are ready to apply Theorem 3.2 to double operator integrals.

Remark 3.1. The set $\mathcal{S} = \{X : AX - XB \in C_p\}$ contains C_p; if $X \in C_p$, then $X \in \mathcal{S}$ and, e.g., $I \in \mathcal{S}$ but $I \notin C_p$. If $A \in C_p$, the conclusions of Theorems 2.1, 2.3, 3.1, and 3.2, Corollaries 2.1 and 2.2 hold for all $X \in \mathbb{B}(H)$.

A restricting aspect of formula (3.1) is that X must be in C_2, which obviously does not include the identity operator I on any infinite dimensional Hilbert space H. So this essentially bounded functional calculus is fully applicable (at least not in this form) to perturbations $A - B$ or to the generalized normal derivations $AX - XB$ on $\mathbb{B}(H)$. From the above remark, we will show that this formula holds as well for all $X \in \mathbb{B}(H)$ having a Hilbert–Schmidt class derivation or (in general for all $X \in \mathbb{B}(H)$ such that $AX - XB \in C_p$).

Theorem 3.3. *Let A, B be operators in $\mathbb{B}(H)$ such that $AX - XB \in C_2$ and $\ker \delta_{A,B} \subseteq \ker \delta_{A^*,B^*}$. Then for every Lipschitz function f defined on $\sigma(A) \cup \sigma(B)$ we have*

$$f(A)X - Xf(B) = g(A,B)(AX - XB)$$

for all $X \in B(H)$, where $g(z,w) = \frac{f(z)-f(w)}{(z-w)}$ if $z \neq w$ and $g(z,w) = 0$ if $z = w$.

Proof. Assume that A and B are operators in $\mathbb{B}(H)$ such that $\ker \delta_{A,B} \subseteq \ker \delta_{A^*,B^*}$. By applying Theorem 3.2 it follows that there exists a sequence of operators (X_n) in $C_2(H)$ such that

$$AX_n - X_nB \to_n AX - XB$$

in $C_2(H)$ norm. Then we have

$$g(A,B)(AX_n - X_nB) = f(A)X_n - X_nf(B) \to_n g(A,B)(AX - XB). \quad (3.2)$$

If L is a Lipschitz constant of the function f, then applying ([12], Corollary 1) we get

$$\begin{aligned} &\|(f(A)X_n - X_nf(B)) - (f(A)X - Xf(B))\|_2 \\ &= \|f(A)(X_n - X) - (X_n - X)f(B)\|_2 \leq L\|A(X_n - X) - (X_n - X)B\|_2, \end{aligned}$$

which proves

$$f(A)X_n - Xf(B)_n \to_n f(A)X - Xf(B).$$

Now by this and (3.1) we complete the proof. □

Remark 3.2. It follows from the property (v) that if f and h are two Lipschitz functions, than an immediate consequence of the above theorem is

$$\begin{aligned}&\operatorname{tr}(f(A)X - Xf(B))(h(A)Y - Yh(B))^*)\\&\qquad = \operatorname{tr}(g(A,B)(AX - XB)(s(A,B)(AYB - Y))^*),\end{aligned}$$

which according to the basic properties of double operator integrals, gives

Corollary 3.2. *Under the conditions of Theorem* 3.3*, for all Lipschitz functions f and g defined on $\sigma(A) \cup \sigma(B)$ there holds*

$$\begin{aligned}&\operatorname{tr}(f(A)X - Xf(B))(h(A)Y - Yh(B))^*)\\&= \int_{\sigma(A)}\int_{\sigma(B)} g(z,w)\overline{s(z,w)}\operatorname{tr}(E(dz)(AX - XB)F(dw)(AY - YB)^*).\end{aligned}$$

Consequently,

$$\operatorname{tr}(f(A)X - Xf(B))(h(A)X - Xh(B))^* \in \|AXB - X\|_2^2\overline{Co}(g\overline{s})(\sigma(A), \sigma(B)),$$

where $\overline{Co}$ is the closed convex hull of a given set.

For $f = h = \overline{z}$, an immediate consequence of the above corollary is that

$$\|A^*X - XB^*\|_2^2 = \|AX - XB\|_2^2. \tag{3.3}$$

Note that if $AX - XB = 0$, then $A^*X = XB^*$ by (3.3). This is a remarkable version of the classical Fuglede–Putnam theorem (see [11], [26] and [3]).

By the same arguments as above we prove the following theorem.

Theorem 3.4. *Let A, B be operators in $\mathbb{B}(H)$ such that $AX - XB \in C_p$ $(p > 1)$ and $\ker\delta_{A,B} \subseteq \ker\delta_{A^*,B^*}$. Then for every Lipschitz function f defined on $\sigma(A) \cup \sigma(B)$ we have*

$$f(A)X - Xf(B) = g(A,B)(AX - XB)$$

for all $X \in \mathbb{B}(H)$, where $g(z,w) = \frac{f(z)-f(w)}{(z-w)}$ if $z \neq w$ and $g(z,w) = 0$ if $z = w$.

Corollary 3.3. *Let A, B be operators in $\mathbb{B}(H)$ such that $AX - XB \in C_p$. Then for every Lipschitz function f defined on $\sigma(A) \cup \sigma B$ we have*

$$f(A)X - Xf(B) = \int_{\sigma(A)}\int_{\sigma(B)} \frac{f(z) - f(w)}{z - w} E(dz)(AX - XB)F(dw),$$

for all $X \in \mathbb{B}(H)$ under either of the following hypotheses:

(1) *A and B are normal operators.*
(2) *A and B are such that $\|Ax\| \geq \|x\| \geq \|Bx\|$ for all $x \in H$.*
(3) *A is invertible and B is such that $\|A^{-1}\|\,\|B\| \leq 1$.*
(4) *If $A = B$ and A is a cyclic subnormal operator.*
(5) *A and B are contractions.*

Corollary 3.4. *Let A, B be operators in $\mathbb{B}(H)$ such that $AX - XB \in C_2$. Then for every Lipschitz function f defined on $\sigma(A) \cup \sigma B$ we have*

$$f(A)X - Xf(B) = \int\limits_{\sigma(A)} \int\limits_{\sigma(B)} \frac{f(z) - f(w)}{z - w} E(dz)(AX - XB)F(dw),$$

for all $X \in \mathbb{B}(H)$ under either of the following hypotheses:

(1) *A and B are hyponormal operators.*

(2) *A is invertible and B is such that $\|A^{-1}\| \, \|B\| \leq 1$.*

(3) *A and B are such that $\|Ax\| \geq \|x\| \geq \|Bx\|$ for all $x \in H$.*

(4) *A and B^* are* log*-hyponormal operators.*

(5) *A is a (p, k)-quasihyponormal operator and B^* is an invertible (p, k)-quasihyponormal operator.*

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