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NONEXISTENCE OF WEAK SOLUTIONS FOR THE p-DEGENERATE SUBELLIPTIC INEQUALITIES CONSTRUCTED BY GENERALIZED BAOUENDI–GRUSHIN VECTOR FIELDS

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Abstract. The purpose of this work is to study the nonexistence of weak solutions for p-degenerate subelliptic inequalities formed by the generalized Baouendi–Grushin vector fields. Also, we give the p-volume and p-area of the ball induced by the these vector fields.

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1. Introduction

Let

$$Z_i = \frac{\partial}{\partial x_i}, \quad Z_{n+j} = |x|^\alpha \frac{\partial}{\partial y_j}, \tag{1.1}$$

where $1 \le i \le n$; $1 \le j \le m$; $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$; $\alpha > 0$, are the generalized Baouendi–Grushin vector fields. We consider the p-degenerate subelliptic inequality

$$-\frac{d^\sigma}{\psi_{p\alpha}} \mathcal{L}_{p,\alpha} u \ge u^q, \quad u \ge 0, \quad u \not\equiv 0, \quad \text{on} \quad \mathbb{R}^{n+m}_*, \tag{1.2}$$

where $\sigma < p$, $1 < p$, $d = d(x,y)$ is a distance function (see (1.5) below), $\psi_{p\alpha} = \frac{|x|^{p\alpha}}{d^{p\alpha}}$, $q > 0$, $\mathbb{R}^{n+m}_* = \mathbb{R}^{n+m} \backslash \{(0,0)\}$, and for a differentiable function u,

$$\mathcal{L}_{p,\alpha} u = \operatorname{div}\left(A_\alpha |\nabla_L u|^{p-2} \nabla_L u\right), \tag{1.3}$$

with the $(n+m) \times (n+m)$ matrix

$$A_\alpha = \begin{pmatrix} I_n & 0 \\ 0 & |x|^\alpha I_m \end{pmatrix}$$

and the generalized gradient

$$\nabla_L = (Z_1, \dots, Z_n, Z_{n+1}, \dots, Z_{n+m}).$$

When $p = 2$, (1.3) is the generalized Baouendi–Grushin operator $\mathcal{L}_\alpha$ (see [4]).

The nonexistence of weak solutions for a singular elliptic inequality in $\mathbb{R}^n$ and a singular sub-Laplacian inequality in the Heisenberg group is investigated in [1], [7], [8], [9]. Recently, D'Ambrosio and Lucente have investigated a weak

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solution for a singular sub-elliptic inequality for $\mathcal{L}_\alpha$ in [2]. The purpose in this paper is to discuss weak solutions of (1.2).

We note that there exists a natural family of anisotropic dilations connected with (1.1), i.e.,

$$\delta_\lambda(x,y) = (\lambda x, \lambda^{\alpha+1} y), \quad \lambda > 0, \quad (x,y) \in \mathbb{R}^{n+m}. \tag{1.4}$$

The generator of the group $\{\delta_\lambda\}_{\lambda>0}$ is

$$X_\alpha = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} + (\alpha+1) \sum_{j=1}^{m} y_j \frac{\partial}{\partial y_j}.$$

$Q = n + (\alpha+1)m$ is the homogeneous dimension related to (1.1).

We introduce the distance function

$$d(x,y) = (|x|^{2(\alpha+1)} + (\alpha+1)^2 |y|^2)^{\frac{1}{2(\alpha+1)}}. \tag{1.5}$$

It should also be noted that

$$\nabla_L d = \frac{|x|^\alpha}{d^{2\alpha+1}} \left(|x|^\alpha x_1, |x|^\alpha x_2, \ldots, |x|^\alpha x_n, (\alpha+1)y_1, (\alpha+1)y_2, \ldots, (\alpha+1)y_m \right),$$

$$|\nabla_L d|^p = \frac{|x|^{p\alpha}}{d^{p\alpha}} = \psi_{p\alpha}$$

and

$$\mathcal{L}_{p,\alpha} d = \psi_{p\alpha} \frac{Q-1}{d}. \tag{1.6}$$

The open ball of radius R and centered at $(0,0) \in \mathbb{R}_*^{n+m}$ is denoted by

$$B_L(R) = B_L((0,0),R) = \{(x,y) \in \mathbb{R}^{n+m} | d(x,y) < R\}. \tag{1.7}$$

In [2], the polar coordinates for (1.1) are given (also see [6]). In Section 2, we define the p-volume and p-area of the ball $B_L(R)$ and, using the polar coordinates, calculate explicitly these quantities. Section 3 is devoted to the nonexistence of nontrivial weak solutions of (1.2) with $\sigma < p$ and $p - 1 < q \leq q_0 = \frac{(Q-\sigma)(p-1)}{Q-p}$. Our method in this section is based on a choice of test functions and consideration of the structure of operator (1.3). In the Appendix, we obtain the explicit fundamental solution of the operator $-\mathcal{L}_{p,\alpha}$ at $(x,y) = (0,0)$.

2. The Volume and the Area

We employ the following concepts. A function $u : \Omega \subset \mathbb{R}^{n+m} \to \mathbb{R}$ is said cylindrical if $u(x,y) = u(r,s)$(u depends only on $r = |x|$ and $s = |y|$), and in particular, u is said radial if $\underline{u(x,y) = u(d(x,y))}$, that is, u depends only on d.

Assume that $\Omega = B_L((0,0),R_2) \backslash \overline{B_L((0,0),R_1)}$, with $0 \leq R_1 < R_2 \leq +\infty$, and $u \in L^1(\Omega)$ is cylindrical. In order to calculate $\int_\Omega u\,dx\,dy$, we use the change of variables [2]

$$(x_1, \ldots, x_n, y_1, \ldots, y_m) \to (\rho, \theta, \theta_1, \ldots, \theta_{n-1}, \gamma_1, \ldots, \gamma_{m-1})$$

defined by

$$\begin{cases} x_1 = \rho(\sin\theta)^{\frac{1}{\alpha+1}}\cos\theta_1, \\ x_2 = \rho(\sin\theta)^{\frac{1}{\alpha+1}}\sin\theta_1\cos\theta_2, \\ x_3 = \rho(\sin\theta)^{\frac{1}{\alpha+1}}\sin\theta_1\sin\theta_2\cos\theta_3, \\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots \\ x_{n-1} = \rho(\sin\theta)^{\frac{1}{\alpha+1}}\sin\theta_1\sin\theta_2\ldots\sin\theta_{n-2}\cos\theta_{n-1}, \\ x_n = \rho(\sin\theta)^{\frac{1}{\alpha+1}}\sin\theta_1\sin\theta_2\ldots\sin\theta_{n-2}\sin\theta_{n-1}, \\ y_1 = \frac{1}{\alpha+1}\rho^{\alpha+1}\cos\theta\cos\gamma_1, \\ y_2 = \frac{1}{\alpha+1}\rho^{\alpha+1}\cos\theta\sin\gamma_1\cos\gamma_2, \\ y_3 = \frac{1}{\alpha+1}\rho^{\alpha+1}\cos\theta\sin\gamma_1\sin\gamma_2\cos\gamma_3, \\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots \\ y_{m-1} = \frac{1}{\alpha+1}\rho^{\alpha+1}\cos\theta\sin\gamma_1\sin\gamma_2\ldots\sin\gamma_{m-2}\cos\gamma_{m-1}, \\ y_m = \frac{1}{\alpha+1}\rho^{\alpha+1}\cos\theta\sin\gamma_1\sin\gamma_2\ldots\sin\gamma_{m-2}\sin\gamma_{m-1}, \end{cases} \tag{2.1}$$

where $R_1 < \rho < R_2$, $\theta_1, \theta_2, \ldots, \theta_{n-2}, \gamma_1, \ldots, \gamma_{m-2} \in (0,\pi)$, $\gamma_{m-1} \in (0, 2\pi)$, and $\theta \in (a_1,\, a_2)$, $\theta_{n-1} \in (b_1,\, b_2)$. Let us note that a_1, a_2, b_1 and b_2 depend on n and m, that is

$$\begin{aligned} \theta \in (0,\pi),\ \ \theta_{n-1} \in (0,\pi), &\qquad \text{if}\quad n,\, m \geq 2; \\ \theta \in (0,\pi),\ \ \theta_{n-1} \in (0,2\pi), &\qquad \text{if}\quad m = 1,\, n \geq 2; \\ \theta \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2}), &\qquad \text{if}\quad n = 1,\, m \geq 2; \\ \theta \in (0, 2\pi), &\qquad \text{if}\quad n = m = 1. \end{aligned}$$

One easily sees that

$$r = |x| = \rho\,|\sin\theta|^{\frac{1}{\alpha+1}}, \qquad s = |y| = \frac{1}{\alpha+1}\rho^{\alpha+1}|\cos\theta| \tag{2.2}$$

and

$$\begin{aligned} d(x,y) &= d(r,s) \\ &= \left(|\rho(\sin\theta)^{\frac{1}{\alpha+1}}|^{2(\alpha+1)} + (\alpha+1)^2|\frac{1}{\alpha+1}\rho^{\alpha+1}\cos\theta|^2\right)^{\frac{1}{2(\alpha+1)}} = \rho. \end{aligned}$$

Let J be the Jacobian of the change of variables. A direct calculation gives

$$\begin{aligned} |\det J| &= \left|\frac{\partial(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)}{\partial(\rho, \theta, \theta_1, \ldots, \theta_{n-1}, \gamma_1, \ldots, \gamma_{m-1})}\right| \\ &= \left(\frac{1}{\alpha+1}\right)^m \rho^{Q-1}|\sin\theta|^{\frac{n}{\alpha+1}-1}|\cos\theta|^{m-1}\sin^{n-2}\theta_1\cdots\sin^2\theta_{n-3}\sin\theta_{n-2} \\ &\quad \cdot\sin^{m-2}\gamma_1\cdots\sin^2\gamma_{m-3}\sin\gamma_{m-2}. \end{aligned} \tag{2.3}$$

By

$$dx\,dy = |\det J| d\rho d\theta d\theta_1 \cdots d\theta_{n-1} d\gamma_1 \cdots d\gamma_{m-1} \tag{2.4}$$

we have, for $u \in L^1(\Omega)$,

$$\begin{aligned}\int\limits_{\Omega} u(x,y) = \tilde{\omega_n}\omega_m \int\limits_{a_1}^{a_2} d\theta \int\limits_{R_1}^{R_2} \left(\frac{1}{\alpha+1}\right)^m \rho^{Q-1} |\sin\theta|^{\frac{n}{\alpha+1}-1} |\cos\theta|^{m-1} u \\ \cdot \left(\rho|\sin\theta|^{\frac{1}{\alpha+1}}, \frac{\rho^{\alpha+1}}{\alpha+1}|\cos\theta|\right) d\rho, \qquad (2.5)\end{aligned}$$

where

$$\tilde{\omega}_n = \int\limits_0^{\pi} \sin^{n-2}\theta_1 d\theta_1 \int\limits_0^{\pi} \sin^{n-3}\theta_2 d\theta_2 \cdots \int\limits_0^{\pi} \sin\theta_{n-2} d\theta_{n-2} \int\limits_{b_1}^{b_2} d\theta_{n-1}$$

and

$$\omega_m = \int\limits_0^{\pi} \sin^{m-2}\gamma_1 d\gamma_1 \int\limits_0^{\pi} \sin^{m-3}\gamma_2 d\gamma_2 \cdots \int\limits_0^{\pi} \sin\gamma_{m-2} d\gamma_{m-2} \int\limits_0^{2\pi} d\gamma_{m-1}.$$

Furthermore, if u is of the form $u(x,y) = \psi_{p\alpha} v(d)$, then

$$\begin{aligned}\int\limits_{\Omega} \psi_{p\alpha} v(d) &= \tilde{\omega}_n \omega_m \int\limits_{a_1}^{a_2} d\theta \int\limits_{R_1}^{R_2} \left(\frac{1}{\alpha+1}\right)^m \rho^{Q-1} |\sin\theta|^{\frac{n}{\alpha+1}-1} |\cos\theta|^{m-1} \\ &\quad \cdot \frac{\rho^{p\alpha}|\sin\theta|^{\frac{p\alpha}{\alpha+1}}}{\rho^{p\alpha}} v(\rho) d\rho \\ &= s_{n,m} \int\limits_{R_1}^{R_2} \rho^{Q-1} v(\rho) d\rho, \qquad (2.6)\end{aligned}$$

where $s_{n,m} = \left(\frac{1}{\alpha+1}\right)^m \tilde{\omega}_n \omega_m \int_{a_1}^{a_2} |\sin\theta|^{\frac{n+p\alpha}{\alpha+1}-1} |\cos\theta|^{m-1} d\theta$. If $u \in C^2$ is radial, then

$$|\nabla_L u|^2 = \psi_{2\alpha} |u'|^2$$

and

$$\mathcal{L}_{p,\alpha} u = \psi_{p\alpha} |u'|^{p-2} \left((p-1)u'' + \frac{Q-1}{d} u'\right). \qquad (2.7)$$

Let $V_R(0)$ and $S_R(0)$ denote the p-volume and the p-area of the ball $B_L(R)$, respectively, which are defined by

$$V_R(0) = \int\limits_{B_L(R)} \psi_{p\alpha}\, dxdy, \quad S_R(0) = \frac{dV_R(0)}{dR}.$$

Federer's coarea formula [3] yields

$$S_R(0) = \int\limits_{\partial B_L(R)} \frac{\psi_{p\alpha}}{|\nabla_L d|}\, dH_{Q-1}.$$

When $p = 2$, the definitions coincide with those in [4]. In the setting of the Heisenberg group, similar definitions are given in [5].

We establish the following formulae.

Theorem 2.1.

$$(1) \quad V_R(0) = \frac{1}{2Q} R^Q\, K(p\alpha, n, m)\omega_n\omega_m, \tag{2.8}$$

$$(2) \quad S_R(0) = \frac{1}{2} R^{Q-1}\, K(p\alpha, n, m)\omega_n\omega_m, \tag{2.9}$$

where

$$K(p\alpha, n, m) = \left(\frac{1}{\alpha+1}\right)^m \frac{\Gamma\left(\frac{n+p\alpha}{2(\alpha+1)}\right)\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{n+p\alpha}{2(\alpha+1)} + \frac{m}{2}\right)},$$

$$\omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} = \begin{cases} \frac{2\pi^k}{(k-1)!}, & \text{if} \quad n = 2k, \\ \frac{2(2\pi)^k}{(2k-1)!!}, & \text{if} \quad n = 2k+1, \end{cases}$$

$$\omega_m = \frac{2\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})} = \begin{cases} \frac{2\pi^t}{(t-1)!}, & \text{if} \quad m = 2t, \\ \frac{2(2\pi)^t}{(2t-1)!!}, & \text{if} \quad m = 2t+1. \end{cases}$$

Proof. (1) We will prove only the case n, $m \geq 2$ since other cases are similar. We obtain

$$\begin{aligned} V_R(0) &= \int\limits_{B_L(R)} \psi_{p\alpha}\, dxdy \\ &= \int\limits_0^\pi d\theta \int\limits_0^\pi d\theta_1 \int\limits_0^\pi d\theta_2 \dots \int\limits_0^\pi d\theta_{n-2} \int\limits_0^\pi d\theta_{n-1} \int\limits_0^\pi d\gamma_1 \dots \int\limits_0^{2\pi} d\gamma_{m-1} \\ &\quad \cdot \int\limits_0^R \left(\frac{1}{\alpha+1}\right)^m \left(\frac{\rho|\sin\theta|^{\frac{1}{\alpha+1}}}{\rho}\right)^{p\alpha} \rho^{Q-1}|\sin\theta|^{\frac{n}{\alpha+1}-1}|\cos\theta|^{m-1} \\ &\quad \cdot \sin^{n-2}\theta_1 \dots \sin^2\theta_{n-3}\sin\theta_{n-2}\sin^{m-2}\gamma_1 \dots \sin^2\gamma_{m-3}\sin\gamma_{m-2}d\rho \\ &= \frac{1}{2Q}\left(\frac{1}{\alpha+1}\right)^m R^Q\omega_n\omega_m I\left(\frac{n+p\alpha}{\alpha+1} - 1, m-1\right), \end{aligned} \tag{2.10}$$

where

$$\omega_n = \int\limits_0^\pi \sin^{n-2}\theta_1 d\theta_1 \int\limits_0^\pi \sin^{n-3}\theta_2 d\theta_2 \dots \int\limits_0^\pi \sin\theta_{n-2}d\theta_{n-2} \int\limits_0^{2\pi} d\theta_{n-1},$$

$$I\left(\frac{n+p\alpha}{\alpha+1} - 1, m-1\right) = \int\limits_0^\pi |\sin\theta|^{\frac{n+p\alpha}{\alpha+1}-1}|\cos\theta|^{m-1}d\theta.$$

Noting

$$\int_0^{\frac{\pi}{2}} \sin^n \theta d\theta = \begin{cases} \frac{\pi}{2}\frac{(2k-1)!!}{(2k)!!} & \text{if} \quad n = 2k, \\ \frac{(2k)!!}{(2k+1)!!} & \text{if} \quad n = 2k+1, \end{cases}$$

and

$$\int_0^{\pi} \sin^n \theta d\theta = 2\int_0^{\frac{\pi}{2}} \sin^n \theta d\theta,$$

we obtain that when $n = 2k$,

$$\begin{aligned} \omega_n &= 2^{n-2}\frac{\pi}{2}\frac{(2(k-1)-1)!!}{(2(k-1))!!}\frac{(2(k-2))!!}{(2(k-2)+1)!!}\cdots\frac{\pi}{2}\frac{1!!}{2!!}\cdot 1\cdot 2\pi \\ &= \frac{2\pi^k}{\Gamma(k)} \end{aligned} \tag{2.11}$$

and when $n = 2k+1$,

$$\begin{aligned} \omega_n &= 2^{n-2}\frac{(2(k-1))!!}{(2(k-1)+1)!!}\frac{\pi}{2}\frac{(2(k-1)-1)!!}{(2(k-1))!!}\cdots\frac{\pi}{2}\frac{1!!}{2!!}\cdot 1\cdot 2\pi \\ &= \frac{2\pi^{\frac{2k+1}{2}}}{\Gamma(\frac{2k+1}{2})}. \end{aligned} \tag{2.12}$$

Similarly, we have

$$\begin{aligned} \omega_m &= \frac{2\pi^t}{\Gamma(t)} & \text{if} \quad m = 2t, \\ \omega_m &= \frac{2\pi^{\frac{2t+1}{2}}}{\Gamma(\frac{2t+1}{2})} & \text{if} \quad m = 2t+1. \end{aligned}$$

Since

$$\begin{aligned} I\left(\frac{n+p\alpha}{\alpha+1}-1, m-1\right) &= 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{\frac{n+p\alpha}{\alpha+1}-1}(\cos\theta)^{m-1}d\theta \\ &= B\left(\frac{n+p\alpha}{2(\alpha+1)}, \frac{m}{2}\right) = \frac{\Gamma\left(\frac{n+p\alpha}{2(\alpha+1)}\right)\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{n+p\alpha}{2(\alpha+1)}+\frac{m}{2}\right)}, \end{aligned}$$

we obtain (2.8) from (2.10).

(2) Using (2.8) and $S_R(0) = \frac{dV_R(0)}{dR}$, we arrive at (2.9). □

Remark 2.2. When $\alpha = 0$, we see that (2.8) and (2.9) become, respectively, the volume and the area of the ball in the Euclidean space.

Let $\varphi_0 \in C_0^\infty(\Omega)$ satisfy the properties

$$0 \le \varphi_0 \le 1 \quad \text{and} \quad \varphi_0(y) = \begin{cases} 1 & \text{if} \quad 0 \le y \le 1, \\ 0 & \text{if} \quad y \ge 2. \end{cases} \tag{2.13}$$

The quantities

$$\int\limits_{\Omega} \frac{|\varphi_0'(\tau)|^q}{\varphi_0(\tau)^{q-1}} d\tau, \tag{2.14}$$

where $q > 1$, are said finite if there exists a suitable φ_0 with property (2.13) such that the integrals are finite. Such a function φ_0 satisfying the above hypotheses is called the admissible function (see [1] or [7], [8]).

3. The Nonexistence

In this section we study inequality (1.2) with $\sigma < p$. Letting $max\{1 - p, \frac{p-1-q}{p}\} < \beta < 0$, we give

Definition 3.1. Let $q > 0$. A function u is called a weak solution of (1.2) if $u \in L^q_{loc}(\mathbb{R}^{n+m}_*)$, $|\nabla_L u|^p u^{\beta-1} \in L^1_{loc}(\mathbb{R}^{n+m}_*)$, $\frac{u^{q+\beta}}{d^\sigma}\psi_{p\alpha} \in L^1_{loc}(\mathbb{R}^{n+m}_*)$ and

$$\int\limits_{\mathbb{R}^{n+m}} \frac{u^q}{d^\sigma}\psi_{p\alpha}\phi\, dxdy \le \int\limits_{\mathbb{R}^{n+m}} |\nabla_L u|^{p-2}\langle \nabla_L u, \nabla_L \phi\rangle\, dxdy \tag{3.1}$$

for any nonnegative function $\phi \in C_0^\infty(\mathbb{R}^{n+m}_*)$.

The main results of this section are the following.

Theorem 3.2. *Let $\sigma < p$, $1 < p < Q$, $p - 1 < q \le \frac{(Q-\sigma)(p-1)}{Q-p}$. Then (1.2) has no nontrivial weak solution.*

Theorem 3.3. *If $\sigma < p$, $1 < Q \le p$, $p - 1 < q$, then (1.2) has no nontrivial weak solution.*

Proof of Theorem 3.2. Let u be a weak solution of (1.2) and $\varphi \in C_0^\infty(\mathbb{R}^{n+m}_*)$, $\varphi \ge 0$, be the admissible function, which we will specify later. Setting $\phi = u^\beta\varphi$ and using (3.1), we have

$$\begin{aligned}\int \frac{u^{q+\beta}}{d^\sigma}\psi_{p\alpha}\varphi dxdy &\le \int |\nabla_L u|^{p-2}\langle \nabla_L u, \nabla_L(u^\beta\varphi)\rangle dxdy\\ &= \beta\int |\nabla_L u|^p u^{\beta-1}\varphi dxdy + \int |\nabla_L u|^{p-2}u^\beta\langle\nabla_L u, \nabla_L\varphi\rangle dxdy\\ &\le \beta\int |\nabla_L u|^p u^{\beta-1}\varphi dxdy + \int |\nabla_L u|^{p-1}u^\beta|\nabla_L\varphi| dxdy,\end{aligned}$$

where and in the sequel we omit the domains $\mathbb{R}^{n+m}$ in the integrals. Letting $\epsilon > 0$, by Young's inequality we obtain

$$\begin{aligned}&\int \frac{u^{q+\beta}}{d^\sigma}\psi_{p\alpha}\varphi dxdy + |\beta|\int |\nabla_L u|^p u^{\beta-1}\varphi dxdy\\ &\le \int |\nabla_L u|^{p-1}u^\beta|\nabla_L\varphi| dxdy\\ &\le \frac{(p-1)\epsilon^{\frac{p}{p-1}}}{p}\int |\nabla_L u|^p u^{\beta-1}\varphi dxdy + \frac{1}{p\epsilon^p}\int u^{\beta+p-1}\frac{|\nabla_L\varphi|^p}{\varphi^{p-1}} dxdy.\end{aligned} \tag{3.2}$$

By putting $c_1(\epsilon) = |\beta| - \frac{(p-1)\epsilon^{\frac{p}{p-1}}}{p} > 0$ and $c_2(\epsilon) = \frac{1}{p\epsilon^p}$, (3.2) becomes

$$\int \frac{u^{q+\beta}}{d^\sigma}\psi_{p\alpha}\varphi dxdy + c_1(\epsilon)\int |\nabla_L u|^p u^{\beta-1}\varphi dxdy \\ \le c_2(\epsilon)\int u^{\beta+p-1}\frac{|\nabla_L\varphi|^p}{\varphi^{p-1}}dxdy. \tag{3.3}$$

Choose $a_1 > 1$ such that $\frac{1}{a_1} + \frac{1}{a_1'} = 1$ and let $c_3(\epsilon) = \frac{\epsilon^{pa_1}}{a_1}$, $c_4(\epsilon) = \frac{1}{a_1'\epsilon^{pa_1'}}$. Using Young's inequality, we have

$$\int u^{\beta+p-1}\frac{|\nabla_L\varphi|^p}{\varphi^{p-1}}\,dxdy \\ \le c_3(\epsilon)\int \frac{u^{a_1(\beta+p-1)}}{d^\sigma}\psi_{p\alpha}\varphi dxdy + c_4(\epsilon)\int \frac{|\nabla_L\varphi|^{pa_1'}}{\varphi^{pa_1'-1}}\left(\frac{d^\sigma}{\psi_{p\alpha}}\right)^{a_1'-1}dxdy. \tag{3.4}$$

In particular, putting $a_1 = \frac{q+\beta}{\beta+p-1} > 1$, $c_\epsilon = 1 - c_2(\epsilon)c_3(\epsilon) > 0$, $\tilde{c}_\epsilon = c_2(\epsilon)c_4(\epsilon)$ and combining (3.4) and (3.3), we obtain

$$c_\epsilon\int \frac{u^{q+\beta}}{d^\sigma}\psi_{p\alpha}\varphi dxdy + c_1(\epsilon)\int |\nabla_L u|^p u^{\beta-1}\varphi dxdy \\ \le \tilde{c}_\epsilon\int \frac{|\nabla_L\varphi|^{pa_1'}}{\varphi^{pa_1'-1}}\left(\frac{d^\sigma}{\psi_{p\alpha}}\right)^{a_1'-1}dxdy. \tag{3.5}$$

On the other hand, letting $a_2 > 1$, $\frac{1}{a_2} + \frac{1}{a_2'} = 1$, $\phi = \varphi$ in (3.1) and using Hölder's inequality, we have

$$\int \frac{u^q}{d^\sigma}\psi_{p\alpha}\varphi dxdy \le \int |\nabla_L u|^{p-1}|\nabla_L\varphi|\,dxdy \\ \le \left(\int |\nabla_L u|^p u^{\beta-1}\varphi dxdy\right)^{\frac{p-1}{p}}\left(\int u^{(1-\beta)(p-1)}\frac{|\nabla_L\varphi|^p}{\varphi^{p-1}}\,dxdy\right)^{\frac{1}{p}} \\ \le \left(\int |\nabla_L u|^p u^{\beta-1}\varphi dxdy\right)^{\frac{p-1}{p}}\left(\int \frac{u^{a_2(1-\beta)(p-1)}}{d^\sigma}\psi_{p\alpha}\varphi dxdy\right)^{\frac{1}{pa_2}} \\ \times\left(\int \frac{|\nabla_L\varphi|^{pa_2'}}{\varphi^{pa_2'-1}}\left(\frac{d^\sigma}{\psi_{p\alpha}}\right)^{a_2'-1}dxdy\right)^{\frac{1}{pa_2'}}. \tag{3.6}$$

Choosing a_2 such that $a_2(1-\beta)(p-1) = q+\beta$ and combining (3.5) and (3.6), we have

$$\int \frac{u^q}{d^\sigma}\psi_{p\alpha}\varphi dxdy \le \left(\frac{\tilde{c}_\epsilon}{c_1(\epsilon)}\right)^{\frac{p-1}{p}}\left(\int \frac{|\nabla_L\varphi|^{pa_1'}}{\varphi^{pa_1'-1}}\left(\frac{d^\sigma}{\psi_{p\alpha}}\right)^{a_1'-1}dxdy\right)^{\frac{p-1}{p}} \\ \times\left(\frac{\tilde{c}_\epsilon}{c_\epsilon}\right)^{\frac{1}{pa_2}}\left(\int \frac{|\nabla_L\varphi|^{pa_1'}}{\varphi^{pa_1'-1}}\left(\frac{d^\sigma}{\psi_{p\alpha}}\right)^{a_1'-1}dxdy\right)^{\frac{1}{pa_2}}$$

$$
\begin{aligned}
&\times \left(\int \frac{|\nabla_L \varphi|^{pa_2'}}{\varphi^{pa_2'-1}} \left(\frac{d^\sigma}{\psi_{p\alpha}} \right)^{a_2'-1} dxdy \right)^{\frac{1}{pa_2'}} \\
&= C_\epsilon \left(\int \frac{|\nabla_L \varphi|^{pa_1'}}{\varphi^{pa_1'-1}} \left(\frac{d^\sigma}{\psi_{p\alpha}} \right)^{a_1'-1} dxdy \right)^{\frac{p-1}{p}+\frac{1}{pa_2}} \\
&\times \left(\frac{|\nabla_L \varphi|^{pa_2'}}{\varphi^{pa_2'-1}} \left(\frac{d^\sigma}{\psi_{p\alpha}} \right)^{a_2'-1} dxdy \right)^{\frac{1}{pa_2'}}, \qquad (3.7)
\end{aligned}
$$

where $C_\epsilon = \left(\frac{\tilde{c}_\epsilon}{c_1(\epsilon)} \right)^{\frac{p-1}{p}} \left(\frac{\tilde{c}_\epsilon}{c_\epsilon} \right)^{\frac{1}{pa_2}}$.

If we require that φ be radial, that is, $\varphi = \varphi(d)$, then

$$
\nabla_L \varphi = \varphi_d \nabla_L d, \quad |\nabla_L d|^{pa_i'} = (|\nabla_L d|^p)^{a_i'} = \psi_{p\alpha}^{a_i'}, \quad i = 1, 2.
$$

Using the change of coordinates in (2.6), we see that

$$
\begin{aligned}
&\int\limits_{\operatorname{supp}(\nabla_L \varphi)} \frac{|\nabla_L \varphi|^{pa_i'}}{\varphi^{pa_i'-1}} \left(\frac{d^\sigma}{\psi_{p\alpha}} \right)^{a_i'-1} dxdy \\
&= \int\limits_{\mathbb{R}^{n+m}} \frac{|\varphi_d(d) \nabla_L d|^{pa_i'}}{\varphi(d)^{pa_i'-1}} \left(\frac{d^\sigma}{\psi_{p\alpha}} \right)^{a_i'-1} dxdy \\
&= \int\limits_{\mathbb{R}^{n+m}} \psi_{p\alpha} \frac{|\varphi_d(d)|^{pa_i'}}{\varphi(d)^{pa_i'-1}} d^{\sigma(a_i'-1)} dxdy \\
&= s_{n,m} \int\limits_0^\infty \rho^{Q-1+\sigma(a_i'-1)} \frac{|\varphi_\rho(\rho)|^{pa_i'}}{\varphi(\rho)^{pa_i'-1}} d\rho. \qquad (3.8)
\end{aligned}
$$

Denote $\rho = R\tau$ and $\varphi(\rho) = \varphi_0 \left(\frac{\rho}{R} \right)$, where φ_0 is an admissible function satisfying (2.13) and (2.14). Hence

$$
\begin{aligned}
&\int\limits_{\operatorname{supp}(\nabla_L \varphi)} \frac{|\nabla_L \varphi|^{pa_i'}}{\varphi^{pa_i'-1}} \left(\frac{d^\sigma}{\psi_{p\alpha}} \right)^{a_i'-1} dxdy \\
&= s_{n,m} \int\limits_1^2 (R\tau)^{Q-1+\sigma(a_i'-1)} \frac{|\frac{1}{R} \varphi_0' \tau|^{pa_i'}}{\varphi_0^{pa_i'-1}(\tau)} R d\tau \\
&= s_{n,m} M_i R^{Q+\sigma(a_i'-1)-pa_i'}, \qquad (3.9)
\end{aligned}
$$

where

$$
M_i = \int\limits_1^2 \tau^{Q-1+\sigma(a_i'-1)} \frac{|\varphi_0'(\tau)|^{pa_i'}}{\varphi_0^{pa_i'-1}(\tau)} d\tau \leq 2^{Q-1+|\sigma|(a_i'-1)} \int\limits_1^2 \frac{|\varphi_0'(\tau)|^{pa_i'}}{\varphi_0^{pa_i'-1}(\tau)} d\tau.
$$

Since φ_0 is admissible, we infer that M_i is finite and independent of R.

Finally, choosing $\varphi(d) = \varphi(\rho) = \varphi_0\left(\frac{d}{R}\right)$. By (3.7) and (3.9), we obtain

$$\begin{aligned}
&\int\limits_{B_L(R)} \frac{u^q}{d^\sigma}\psi_{p\alpha}dxdy = \int\limits_{B_L(R)} \frac{u^q}{d^\sigma}\psi_{p\alpha}\varphi dxdy \le \int\limits_{B_L(2R)} \frac{u^q}{d^\sigma}\psi_{p\alpha}\varphi dxdy \\
&\le C_\epsilon\Bigg(\int\limits_{\mathrm{supp}(\nabla_L\varphi)} \frac{|\nabla_L\varphi|^{pa_1'}}{\varphi^{pa_1'-1}}\left(\frac{d^\sigma}{\psi_{p\alpha}}\right)^{a_1'-1} dxdy\Bigg)^{\frac{p-1}{p}+\frac{1}{pa_2}} \\
&\quad \times \Bigg(\int\limits_{\mathrm{supp}(\nabla_L\varphi)} \frac{|\nabla_L\varphi|^{pa_2'}}{\varphi^{pa_2'-1}}\left(\frac{d^\sigma}{\psi_{p\alpha}}\right)^{a_2'-1} dxdy\Bigg)^{\frac{1}{pa_2'}} \\
&= C_\epsilon\left(s_{n,m}M_1R^{Q+\sigma(a_1'-1)-pa_1'}\right)^{\frac{p-1}{p}+\frac{1}{pa_2}}\left(s_{n,m}M_2R^{Q+\sigma(a_2'-1)-pa_2'}\right)^{\frac{1}{pa_2'}} \\
&= C_1R^{\gamma_0},
\end{aligned} \tag{3.10}$$

where we have used $\frac{p-1}{p}+\frac{1}{pa_2}+\frac{1}{pa_2'} = 1$, $C_1 = s_{n,m}C_\epsilon M_1^{\frac{p-1}{p}+\frac{1}{pa_2}}M_2^{\frac{1}{pa_2'}}$, $a_1' = \frac{q+\beta}{q-p+1}$, $a_2' = \frac{q+\beta}{q-p+1+p\beta}$ and $\gamma_0 = Q-\sigma+\frac{(\sigma-p)q}{q-p+1}$.

If $q < \frac{(p-1)(Q-\sigma)}{Q-p}$, then $\gamma_0 < 0$. Letting $R \to +\infty$, we have from (3.10)

$$\lim_{R\to\infty}\int\limits_{B_L(R)} \frac{u^q}{d^\sigma}\psi_{p\alpha}dxdy = \int\limits_{\mathbb{R}^{n+m}} \frac{u^q}{d^\sigma}\psi_{p\alpha}dxdy \le 0.$$

If $q = \frac{(p-1)(Q-\sigma)}{Q-p}$, then $\gamma_0 = 0$. Letting $R \to +\infty$ in (3.10), we have

$$\int\limits_{\mathbb{R}^{n+m}} \frac{u^q}{d^\sigma}\psi_{p\alpha}dxdy < +\infty. \tag{3.11}$$

Choose $b_1 > 1$ such that $\frac{1}{b_1}+\frac{1}{b_1'} = 1$. By the choice of φ and Hölder's inequality, we have from (3.1)

$$\begin{aligned}
&\int\limits_{B_L(R)} \frac{u^q}{d^\sigma}\psi_{p\alpha}dxdy = \int\limits_{B_L(R)} \frac{u^q}{d^\sigma}\psi_{p\alpha}\varphi dxdy \le \int\limits_{B_L(2R)} \frac{u^q}{d^\sigma}\psi_{p\alpha}\varphi dxdy \\
&\le \int\limits_{\mathrm{supp}(\nabla_L\varphi)} |\nabla_Lu|^{p-1}|\nabla_L\varphi|dxdy \\
&\le \Bigg(\int\limits_{\mathrm{supp}(\nabla_L\varphi)} |\nabla_Lu|^pu^{\beta-1}\varphi dxdy\Bigg)^{\frac{p-1}{p}}\Bigg(\int\limits_{\mathrm{supp}(\nabla_L\varphi)} u^{(1-\beta)(p-1)}\frac{|\nabla_L\varphi|^p}{\varphi^{p-1}}dxdy\Bigg)^{\frac{1}{p}} \\
&\le \Bigg(\int\limits_{\mathrm{supp}(\nabla_L\varphi)} |\nabla_Lu|^pu^{\beta-1}\varphi dxdy\Bigg)^{\frac{p-1}{p}}\left(\int\limits_{\mathrm{supp}(\nabla_L\varphi)} \frac{u^{b_1(1-\beta)(p-1)}}{d^\sigma}\psi_{p\alpha}\varphi dxdy\right)^{\frac{1}{pb_1}}
\end{aligned}$$

$$\cdot\Bigg(\int\limits_{\operatorname{supp}(\nabla_L\varphi)} \frac{|\nabla_L\varphi|^{pb_1'}}{\varphi^{pb_1'-1}}\left(\frac{d^\sigma}{\psi_{p\alpha}}\right)^{b_1'-1} dxdy\Bigg)^{\frac{1}{pb_1'}}. \tag{3.12}$$

We specify $b_1 = \frac{q}{(1-\beta)(p-1)} = \frac{Q-\sigma}{(Q-p)(1-\beta)} > 1$ and have by (3.5) and (3.12),

$$\begin{aligned}
&\int\limits_{B_L(R)} \frac{u^q}{d^\sigma}\psi_{p\alpha}dxdy \\
&\le c_5\Bigg(\int\limits_{\operatorname{supp}(\nabla_L\varphi)} \frac{|\nabla_L\varphi|^{pa_1'}}{\varphi^{pa_1'-1}}\left(\frac{d^\sigma}{\psi_{p\alpha}}\right)^{a_1'-1} dxdy\Bigg)^{\frac{p-1}{p}} \\
&\quad\cdot\Bigg(\int\limits_{\operatorname{supp}(\nabla_L\varphi)} \frac{|\nabla_L\varphi|^{pb_1'}}{\varphi^{pb_1'-1}}\left(\frac{d^\sigma}{\psi_{p\alpha}}\right)^{b_1'-1} dxdy\Bigg)^{\frac{1}{pb_1'}} \cdot \Bigg(\int\limits_{\operatorname{supp}(\nabla_L\varphi)} u^q\frac{\psi_{p\alpha}\varphi}{d^\sigma}dxdy\Bigg)^{\frac{1}{pb_1}} \\
&\le c_5 I_1\cdot I_2,
\end{aligned} \tag{3.13}$$

where c_5 is a positive constant,

$$\begin{aligned}
I_1 &= \Bigg(\int\limits_{\operatorname{supp}(\nabla_L\varphi)} \frac{|\nabla_L\varphi|^{pa_1'}}{\varphi^{pa_1'-1}}\left(\frac{d^\sigma}{\psi_{p\alpha}}\right)^{a_1'-1} dxdy\Bigg)^{\frac{p-1}{p}} \\
&\quad\cdot\Bigg(\int\limits_{\operatorname{supp}(\nabla_L\varphi)} \frac{|\nabla_L\varphi|^{pb_1'}}{\varphi^{pb_1'-1}}\left(\frac{d^\sigma}{\psi_{p\alpha}}\right)^{b_1'-1} dxdy\Bigg)^{\frac{1}{pb_1'}}, \\
I_2 &= \Bigg(\int\limits_{\operatorname{supp}(\nabla_L\varphi)} u^q\frac{\psi_{p\alpha}}{d^\sigma}dxdy\Bigg)^{\frac{1}{pb_1}}.
\end{aligned}$$

Setting $\varphi = \varphi_0(\frac{d}{R})$ with $d = \rho = R\tau$ and noting $\operatorname{supp}(\nabla_L\varphi_0(d)) = \{(x,y)\in \mathbb{R}^{n+m} | R < d < 2R\}$, we have

$$\begin{aligned}
I_1 &= \left(s_{n,m}M_1R^{Q+\sigma(a_1'-1)-pa_1'}\right)^{\frac{p-1}{p}}\left(s_{n,m}M_1'R^{Q+\sigma(b_1'-1)-pb_1'}\right)^{\frac{1}{pb_1'}} \\
&= s_{n,m}^{\frac{p-1}{p}+\frac{1}{pb_1'}} M_1^{\frac{p-1}{p}} (M_1')^{\frac{1}{pb_1'}} R^{\gamma_1},
\end{aligned} \tag{3.14}$$

where $M_1 = \int_1^2 \tau^{Q-1+\sigma(a_1'-1)}\frac{|\varphi_0'(\tau)|^{pa_1'}}{\varphi_0^{pa_1'-1}(\tau)}d\tau$, $M_1' = \int_1^2 \tau^{Q-1+\sigma(b_1'-1)}\frac{|\varphi_0'(\tau)|^{pb_1'}}{\varphi_0^{pb_1'-1}(\tau)}d\tau$, $a_1' = \frac{(Q-\sigma)(p-1)+\beta(Q-p)}{(p-\sigma)(p-1)}$, $b_1' = \frac{Q-\sigma}{p-\sigma+(Q-p)\beta}$, $\gamma_1 = [Q+\sigma(a_1'-1)-pa_1']\frac{p-1}{p} + [Q+\sigma(b_1'-1)-pb_1']\frac{1}{pb_1'} = 0$. From (2.14) we see that I_1 is finite and independent of R. Furthermore, it follows that

$$\lim_{R\to\infty}\int\limits_{R<d<2R} u^q\frac{\psi_{p\alpha}}{d^\sigma}dxdy = 0. \tag{3.15}$$

So we have $I_2 \to 0$ as $R\to\infty$.

From(3.11),(3.14) and (3.15) we deduce that

$$\lim_{R\to\infty}\int\limits_{B_L(R)}\frac{u^q}{d^\sigma}\psi_{p\alpha}dxdy=\int\limits_{\mathbb{R}^{n+m}}\frac{u^q}{d^\sigma}\psi_{p\alpha}dxdy\le 0.$$

This contradicts our assumption that $u>0$, and then $u=0$. Theorem 3.2 is proved. □

The proof of Theorem 3.3 is similar to that of Theorem 3.2. We only need to note that $\gamma_0=Q-\sigma+\frac{(\sigma-p)q}{q-p+1}=Q-p-\frac{(p-\sigma)(p-1)}{q-p+1}<0$.

Remark 3.4. By considering the inequality

$$-\frac{d^\sigma}{\psi_{p\alpha}}\mathcal{L}_{p,\alpha}u\ge|u|^q,\quad u\le 0,\quad u\not\equiv 0,\quad \text{on}\quad \mathbb{R}^{n+m}_*, \tag{3.16}$$

we can give a similar definition of a weak solution and formulate the following result:

under the hypotheses of Theorem 3.2, (3.16) has no nontrivial weak solutions.

In the proof, we take $0<\beta<1$ and repeat the preceding process.

APPENDIX A. A FUNDAMENTAL SOLUTION OF $\mathcal{L}_{p,\alpha}$

A fundamental solution of the operator $\mathcal{L}_\alpha$ at $(x,y)=(0,0)$ is obtained in [4]. Here we will derive a fundamental solution of $\mathcal{L}_{p,\alpha}$.

Theorem A.1. *Let $C_{p,Q}>0$ be defined by the formula*

$$C_{p,Q}=\frac{2}{\left(\frac{Q-p}{p-1}\right)^{p-1}K(p\alpha,n,m)\omega_n\omega_m}. \tag{A.1}$$

Then

$$\Gamma(x,y)=\frac{C_{p,Q}}{d(x,y)^{\frac{Q-p}{p-1}}}$$

is a fundamental solution of $-\mathcal{L}_{p,\alpha}$ with singularity at the origin.

Proof. Let

$$d_\epsilon=\left(|x|^{2(\alpha+1)}+(\alpha+1)^2|y|^2+\epsilon^{2(\alpha+1)}\right)^{\frac{1}{2(\alpha+1)}}.$$

Clearly, $d_\epsilon\in C^2(\mathbb{R}^{n+m})$ and

$$\nabla_L d_\epsilon=\frac{|x|^\alpha}{d_\epsilon^{2\alpha+1}}\begin{pmatrix}|x|^\alpha x\\(\alpha+1)t\end{pmatrix},$$

$$|\nabla_L d_\epsilon|^{p-2}=\frac{|x|^{\alpha(p-2)}d^{(\alpha+1)(p-2)}}{d_\epsilon^{(2\alpha+1)(p-2)}},$$

$$A_\alpha|\nabla_L d_\epsilon|^{p-2}\nabla_L d_\epsilon=\frac{|x|^{p\alpha}d^{(\alpha+1)(p-2)}}{d_\epsilon^{(2\alpha+1)(p-2)}}X_\alpha=\frac{\psi_{p\alpha,\epsilon}d^{(\alpha+1)(p-2)}}{d_\epsilon^{(\alpha+1)(p-2)+1}}X_\alpha, \tag{A.2}$$

where $\psi_{p\alpha,\epsilon} = \frac{|x|^{p\alpha}}{d_\epsilon^{p\alpha}}$. (A.2) yields

$$X_\alpha \psi_{p\alpha,\epsilon} = p\alpha \psi_{p\alpha,\epsilon} \frac{\epsilon^{2(\alpha+1)}}{d_\epsilon^{2(\alpha+1)}} \tag{A.3}$$

and

$$\begin{aligned} &X_\alpha \left(\frac{d^{(\alpha+1)(p-2)}}{d_\epsilon^{(\alpha+1)(p-2)+1}} \right) \\ &= \frac{d^{(\alpha+1)(p-2)}}{d_\epsilon^{(\alpha+1)(p-2)+1}} \left\{ (\alpha+1)(p-2) - [(\alpha+1)(p-2)+1] \frac{d^{2(\alpha+1)}}{d_\epsilon^{2(\alpha+1)}} \right\}. \end{aligned} \tag{A.4}$$

By(A.2), (A.3), (A.4), we get

$$\begin{aligned} \mathcal{L}_{p,\alpha} d_\epsilon &= \operatorname{div}\left(A_\alpha |\nabla_L d_\epsilon|^{p-2} \nabla_L d_\epsilon \right) = \operatorname{div}\left(\frac{\psi_{p\alpha,\epsilon} d^{(\alpha+1)(p-2)}}{d_\epsilon^{(\alpha+1)(p-2)+1}} X_\alpha \right) \\ &= \frac{\psi_{p\alpha,\epsilon} d^{(\alpha+1)(p-2)}}{d_\epsilon^{(\alpha+1)(p-2)+1}} \left\{ Q + (\alpha+1)(p-2) \right. \\ &\qquad \left. + p\alpha \frac{\epsilon^{2(\alpha+1)}}{d_\epsilon^{2(\alpha+1)}} - [(\alpha+1)(p-2)+1] \frac{d^{2(\alpha+1)}}{d_\epsilon^{2(\alpha+1)}} \right\}. \end{aligned} \tag{A.5}$$

Let $f \in C^2(0,\infty)$ and define $u(x,y) = f(d_\epsilon) = d_\epsilon^{\frac{p-Q}{p-1}}$, then

$$f'(d_\epsilon) = \frac{p-Q}{p-1} d_\epsilon^{\frac{p-Q}{p-1}-1}, \quad f''(d_\epsilon) = \frac{1-Q}{p-1} \frac{p-Q}{p-1} d_\epsilon^{\frac{p-Q}{p-1}-2}.$$

By (2.7) and (A.5), we obtain

$$\begin{aligned} &\mathcal{L}_{p,\alpha}\left(f(d_\epsilon)\right) = |f'(d_\epsilon)|^{p-2} \left[(p-1) f''(d_\epsilon) |\nabla_L d_\epsilon|^p + f'(d_\epsilon) \mathcal{L}_{p,\alpha} d_\epsilon \right] \\ &= \left(\frac{Q-p}{p-1} \right)^{p-1} d_\epsilon^{-Q} \left[(Q-1) |\nabla_L d_\epsilon|^p - d_\epsilon \mathcal{L}_{p,\alpha} d_\epsilon \right] \\ &= -\left[Q + (\alpha+1)(p-2) + p\alpha \right] \left(\frac{Q-p}{p-1} \right)^{p-1} \frac{|x|^{p\alpha} d^{(\alpha+1)(p-2)} \epsilon^{2(\alpha+1)}}{d_\epsilon^{Q+p\alpha+(\alpha+1)p}}. \end{aligned} \tag{A.6}$$

Denote

$$K(x,y) = -[Q + (\alpha+1)(p-2) + p\alpha] \left(\frac{Q-p}{p-1} \right)^{p-1} \frac{|x|^{p\alpha} d^{(\alpha+1)(p-2)}}{\left(1 + d^{2(\alpha+1)}\right)^{\frac{Q+p\alpha+(\alpha+1)p}{2(\alpha+1)}}}.$$

Using (A.2) and (A.6) we have

$$\mathcal{L}_{p,\alpha}\left(d_\epsilon^{\frac{p-Q}{p-1}} \right) = \epsilon^{-Q} K\left(\delta_{\frac{1}{\epsilon}}(x,y) \right),$$

where $\delta_{\frac{1}{\epsilon}}$ is defined by (1.4), and for any $u \in C_0^\infty$,

$$\left(\mathcal{L}_{p,\alpha} d^{\frac{p-Q}{p-1}}, u \right) = \lim_{\epsilon \to 0} \int\limits_{\mathbb{R}^{n+m}} \mathcal{L}_{p,\alpha}\left(d_\epsilon^{\frac{p-Q}{p-1}} \right) u(x,y)\,dxdy$$

$$
\begin{aligned}
&= \lim_{\epsilon\to 0} \epsilon^{-Q} \int\limits_{\mathbb{R}^{n+m}} K\left(\delta_{\frac{1}{\epsilon}}(x,y)\right) u(x,y)dxdy \\
&= \lim_{\epsilon\to 0} \int\limits_{\mathbb{R}^{n+m}} K\left(x,y\right) u(\epsilon x, \epsilon^{\alpha+1}y)dxdy \\
&= u(0,0) \int\limits_{\mathbb{R}^{n+m}} K\left(x,y\right) dxdy. \qquad \text{(A.7)}
\end{aligned}
$$

Note that

$$
\begin{aligned}
C_{p,Q}^{-1} &= - \int\limits_{\mathbb{R}^{n+m}} K\left(x,y\right) dxdy \\
&= [Q+(\alpha+1)(p-2)+p\alpha]\left(\frac{Q-p}{p-1}\right)^{p-1} \int\limits_{\mathbb{R}^{n+m}} \frac{|x|^{p\alpha} d^{(\alpha+1)(p-2)}}{\left(1+d^{2(\alpha+1)}\right)^{\frac{Q+p\alpha+(\alpha+1)p}{2(\alpha+1)}}} dxdy.
\end{aligned}
$$

From (A.7), (A.6) and (A.5), we see that $-\mathcal{L}_{p,\alpha}(\Gamma)=\delta_0$ in $\mathcal{D}'(\mathbb{R}^{n+m})$.

Now, we use (2.1) and (2.8) and let $R=1$. An easy calculation yields

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^{n+m}} \frac{|x|^{p\alpha} d^{(\alpha+1)(p-2)}}{\left[1+d^{2(\alpha+1)}\right]^{\frac{Q+p\alpha+p(\alpha+1)}{2(\alpha+1)}}} dxdy \\
&= \int_0^{\pi} d\theta_1 \int_0^{\pi} d\theta_2 \ldots \int_0^{\pi} d\theta_{n-2} \int_0^{\pi} d\theta_{n-1} \int_0^{\pi} d\gamma_1 \ldots \int_0^{2\pi} d\gamma_{m-1} \\
&\quad \cdot \int_0^{\infty} \left(\frac{1}{\alpha+1}\right)^m \frac{|\rho(\sin\theta)^{\frac{1}{\alpha+1}}|^{p\alpha}}{\left[1+\rho^{2(\alpha+1)}\right]^{\frac{Q+p\alpha+p(\alpha+1)}{2(\alpha+1)}}} \rho^{Q-1} |\sin\theta|^{\frac{n}{\alpha+1}-1} |\cos\theta|^{m-1} \\
&\quad \cdot \sin^{n-2}\theta_1 \ldots \sin^2\theta_{n-3} \sin\theta_{n-2} \sin^{m-2}\gamma_1 \ldots \sin^2\gamma_{m-3} \sin\gamma_{m-2}\, d\rho \\
&= s_{n,m,} \int_0^{+\infty} \frac{\rho^{(\alpha+1)(p-2)+p\alpha+Q-1}}{\left[1+\rho^{2(\alpha+1)}\right]^{\frac{Q+p\alpha+p(\alpha+1)}{2(\alpha+1)}}} d\rho \\
&= s_{n,m} \int_0^{+\infty} \frac{\rho^{2(\alpha+1)\frac{Q+p\alpha+p(\alpha+1)}{2(\alpha+1)}} \rho^{-(2\alpha+3)}}{\left[1+\rho^{2(\alpha+1)}\right]^{\frac{Q+p\alpha+p(\alpha+1)}{2(\alpha+1)}}} d\rho.
\end{aligned}
$$

Letting $s=\rho^{-2(\alpha+1)}$, we have

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^{n+m}} \frac{|x|^{p\alpha} d^{(\alpha+1)(p-2)}}{\left[1+d^{2(\alpha+1)}\right]^{\frac{Q+p\alpha+p(\alpha+1)}{2(\alpha+1)}}} dxdy \\
&= s_{n,m} \frac{1}{2(\alpha+1)} \int_0^{+\infty} \left(\frac{1}{1+s}\right)^{\frac{Q+p\alpha+p(\alpha+1)}{2(\alpha+1)}} ds
\end{aligned}
$$

$$= \frac{\frac{1}{2}K(p\alpha, n, m)\omega_n\omega_m}{Q + p\alpha + (p-2)(\alpha+1)},$$

where

$$s_{n,m} = \left(\frac{1}{\alpha+1}\right)^m \tilde{\omega}_n\omega_m \int\limits_0^\pi |\sin\theta|^{\frac{n+p\alpha}{\alpha+1}-1}|\cos\theta|^{m-1}d\theta$$

$$= \frac{Q}{2}\left(\frac{1}{\alpha+1}\right)^m \omega_n\omega_m \int\limits_0^1 \rho^{Q-1}d\rho \int\limits_0^\pi |\sin\theta|^{\frac{n+p\alpha}{\alpha+1}-1}|\cos\theta|^{m-1}d\theta$$

$$= QV_1(0,0) = \frac{1}{2}K(p\alpha, n, m)\omega_n\omega_m.$$

Hence

$$C_{p,Q} = \frac{2}{\left(\frac{Q-p}{p-1}\right)^{p-1} K(p\alpha, n, m)\omega_n\omega_m}. \qquad \square$$

Remark A.2. Since $\mathcal{L}_{p,\alpha}$ is translation invariant in $y \in \mathbb{R}^m$, we easily infer from Theorem A.1 that $\Gamma(x, y - y_0) = C_{p,Q}d(x, y - y_0)^{\frac{p-Q}{p-1}}$ is a fundamental solution of $-\mathcal{L}_{p,\alpha}$ with singularity at $(0, y_0)$, with $0 \in \mathbb{R}^n$, $y_0 \in \mathbb{R}^m$.

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