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OSCILLATION AND NONOSCILLATION THEOREMS FOR SUPERLINEAR EMDEN–FOWLER EQUATIONS OF EVEN ORDER

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Abstract. We study the existence of oscillatory solutions of the even order superlinear ordinary differential equations

$$(E_1^n) \qquad y^{(n)} + p(t)|y|^\alpha \operatorname{sgn} y = 0,$$

$$(E_2^n) \qquad y^{(n)} - p(t)|y|^\alpha \operatorname{sgn} y = 0,$$

where n is an even integer ≥ 2, $\alpha > 1$ and $p(t) \in C[t_0, \infty)$, $t_0 > 0$ and $p(t) > 0$. When $n = 2$, our results reduce to those of Jasny and Kurzweil, and Erbe and Muldowney for (E_1^2). When $n = 4$, our result becomes that of Kura for equation (E_2^4). We present here new techniques suitable for the study of oscillation and nonoscillation of solutions of the general equations of even order (E_1^n) and (E_2^n).

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1. Introduction

We consider the Emden–Fowler equations of even order

$$(E_1^n) \qquad y^{(n)} + p(t)|y|^\alpha \operatorname{sgn} y = 0,$$

$$(E_2^n) \qquad y^{(n)} - p(t)|y|^\alpha \operatorname{sgn} y = 0,$$

where $n = 2m$, m is any positive integer, $\alpha > 1$ and $p(t)$ is a positive continuous function on $[t_0, \infty)$ for some $t_0 > 0$. Equations (E_1^n) and (E_2^n) are called superlinear if $\alpha > 1$. A nontrivial solution $y(t)$ of either (E_1^n) or (E_2^n) is called proper if it exists on some half-line $[T_y, \infty) \subseteq [t_0, \infty)$ and satisfies $\sup\{|y(t)| : t \geq T_y\} > 0$ where $T_y \geq t_0$. Because of the nonlinear nature, equations (E_1^n) and (E_2^n) in general have non-proper solutions, i.e., solutions that cannot be extended beyond certain point t_* where $t_0 < t_* < \infty$. These solutions are also called singular solutions. A solution $y(t)$ of (E_1^n) or (E_2^n) can be singular in two ways, namely (a) $\lim\limits_{t \to t_*} y(t) = 0$ and (b) $\limsup_{t \to t_*} |y(t)| = \infty$. Since equations (E_1^n) and (E_2^n) are superlinear, singular solutions of first kind, i.e., the case (a), do not exist (Kiguradze and Chanturia [25, p. 205, Theorem 11.5]). Singular solutions of second kind, i.e., the case (b), can and in fact exist for both (E_1^n) and (E_2^n) and are also referred to as solutions of finite escape time (o.f.e.t.) (Kiguradze and Kvinikadze [26], Chanturia [4], Bartusek [3]).

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A solution $y(t)$ of either (E_1^n) or (E_2^n) is called oscillatory if it has no last zeros, i.e., for every zero t_1 of $y(t)$ there exists $t_2 > t_1$ such that $y(t_2) = 0$. This definition allows to obtain oscillatory solutions of finite escape time. A proper solution $y(t)$ is called oscillatory if it has arbitrarily large zeros, i.e., to every $t_1 > t_0$ there corresponds $t_2 \geq t_1$ such that $y(t_2) = 0$. A solution $y(t)$ is called nonoscillatory if it is not oscillatory and a proper nonoscillatory solution can have only finitely many zeros. Equations (E_1^n), (E_2^n) are called nonoscillatory if every proper solution is nonoscillatory. On the other hand, they are called oscillatory if every solution (not necessarily proper) is oscillatory.

The theory of oscillation and nonoscillation of second order Emden–Fowler equation (E_1^2), i.e.,

$$y'' + p(t)|y|^\alpha \operatorname{sgn} y = 0, \tag{1.1}$$

has been extensively studied in the last half of the 20th century. When $\alpha > 1$, equation (1.1) may have both oscillatory and nonoscillatory solutions for a given function $p(t)$. Indeed, it is convenient to classify the study of oscillation theory of nonlinear equations (E_1^n) and (E_2^n) into four different classes:

(I) All solutions are oscillatory.
(II) There exist nonoscillatory solutions.
(III) There exist oscillatory solutions.
(IV) All solutions are nonoscillatory.

Statements (I) and (II) are contradictory to each other and so are statements (III) and (IV). It is known that the results concerning (I) and (II) are easier to prove. One generally assumes the existence of an arbitrarily positive solution of (E_1^n) or (E_2^n) and imposes conditions on $p(t)$, which leads to a contradiction, hence an oscillation theorem is formulated in the form of statement (I). To prove the existence of a nonoscillatory hence eventually positive solution, we have the powerful fixed point theorem in function spaces. Statements of types (III) and (IV) are more difficult to prove. In case of (III), one needs to identify a certain specific positive solution which, together with the condition on $p(t)$, is not sufficient for guaranteing that all solutions are oscillatory, will produce the desired conclusion. Finally, for results of type (IV), we need to exclude the existence of any oscillatory solutions. We know that it is more difficult to handle oscillatory functions than positive functions and thus results relating to (IV) are less available than the other three. Indeed, in the book of Kiguradze and Chanturia [25] already mentioned, which is the most competent book available on the subject under discussion, there is no result on nonoscillation for equations (E_1^n) and (E_2^n) when $n \geq 4$ except Kura's work [28] which was mentioned as a problem; see [25; p. 236, problem 15.3].

In the second order linear case, with equation (1.1) where $\alpha = 1$, we can refer to the Sturm Separation Theorem which states (I)$\Leftrightarrow$(III) and (II) $\Leftrightarrow$(IV). This reduces the study of (III) and (IV) to that of simpler cases of (I) and (II). When $n = 2$ and $\alpha > 1$, the classification problem of (E_1^2) in respect of oscillation is somewhat complete. For statements (I) and (II), we have

Theorem A (Atkinson [2]). *Equation* (1.1) *is oscillatory, i.e., all its solutions are oscillatory if and only if*

$$\int^{+\infty} tp(t)dt = +\infty. \tag{1.2}$$

For statement (III), we have

Theorem B$_1$ (Jasny [18], Kurzweil [29]). *If* $(p(t)t^{(\alpha+3)/2})' \geq 0$, *then equation* (1.1) *has oscillatory solutions.*

Theorem B$_2$ (Erbe and Muldowney [8]). *If* $(p(t)t^{(\alpha+3)/2})' \leq 0$ *and*

$$\lim_{t\to\infty} p(t)t^{(\alpha+3)/2} = k > 0,$$

then equation (1.1) *has oscillatory solutions.*

Clearly, if $p(t) = t^{-(\alpha+3)/2}$, then it satisfies the conditions in Theorems B$_1$ and B$_2$ but fails to satisfy (1.2).

Finally, for (IV) we have

Theorem C$_1$ (Kiguradze [21]). *If for some* $\varepsilon > 0$, $(p(t)t^{(\alpha+3)/2+\varepsilon})' \leq 0$, *then equation* (1.1) *is nonoscillatory.*

Theorem C$_2$ (Wong [43]). *If for some* $\varepsilon > 0$, $(p(t)t^{(\alpha+3)/2+\varepsilon})' \geq 0$ *and* $\lim_{t\to\infty} p(t)t^{(\alpha+3)/2+\varepsilon} = k < \infty$, *then equation* (1.1) *is nonoscillatory.*

Thus for the specific Emden–Fowler equation $y'' + t^\beta|y|^\alpha \operatorname{sgn} y = 0$, we have

(a) $\beta \geq -2 \iff$ oscillation
(b) $\beta \geq -\frac{\alpha+3}{2} \Longrightarrow$ existence of oscillatory solutions
(c) $\beta < -\frac{\alpha+3}{2} \Longrightarrow$ nonoscillation.

For higher order equations with $n \geq 4$, we have only the results of Kura [28] for (E_2^4), i.e.,

$$y^{(iv)} = p(t)|y|^\alpha \operatorname{sgn} y \tag{1.3}$$

concerning cases (III) and (IV), namely,

Theorem D$_1$ (Kura [28]). *If* $(p(t)t^{(3\alpha+5)/2})' \geq 0$, *then equation* (1.3) *has oscillatory solutions.*

Theorem D$_2$ (Kura [28]). *If for some* $\varepsilon > 0$, $(p(t)t^{(3\alpha+5)/2+\varepsilon})' \leq 0$ *and* $\lim_{t\to\infty} (p(t)t^{(3\alpha+5)/2+\varepsilon}) = k > 0$, *then equation* (1.3) *is nonoscillatory.*

Applying Theorems D$_1$ and D$_2$ to the special example (E_2^4): $y^{(iv)} = t^\beta|y|^\alpha \times \operatorname{sgn} y$, we conclude that for $\beta \geq -\frac{3\alpha+5}{2}$, equation (E_2^4) has oscillatory solutions, while for $\beta < -\frac{3\alpha+5}{2}$, equation (E_2^4) is nonoscillatory.

In our recent paper [37], we have extended Kura's Theorem D$_1$ to equation (E_1^4) as follows:

Theorem E$_1$(Ou and Wong [37]). *If* $(p(t)t^{\frac{3\alpha+5}{2}})' \geq 0$, *then equation* (E_1^4) *has oscillatory solutions.*

Theorem E$_2$ (Ou and Wong [37]). *If* $(p(t)t^{\frac{3\alpha+5}{2}})' \leq 0$ *and* $\lim_{t\to\infty} p(t)t^{\frac{3\alpha+5}{2}} = k > 0$, *then equation* (E_1^4) *has oscillatory solutions.*

Likewise, here we also give analogues of Theorem D$_2$ for equation (E_1^4) although we are unable to establish a nonoscillation result in the same generality as theorem D$_2$ for equation (E_2^4).

The techniques used both for the second order equation (1.1) and for the fourth order equations (E_1^4) and (E_2^4) can be further developed to obtain similar results for the general even order equations (E_1^n) and (E_2^n). It is the purpose of this paper to present the results of this investigation.

For nth order equation (E_1^n), statements (I) and (II) can be easily disposed of the following extension of Theorem A.

Theorem F (Kiguradze [22], Licko and Svec [32]). *Equation* (E_1^n) *is oscillatory if and only if*

$$\int^{\infty} t^{n-1}p(t)dt = \infty. \tag{1.4}$$

Kiguradze [21] proved a more general result for the superlinear equation $y^{(n)} + p(t)f(y)=0$ where $f(y)$ is odd, $f(y) = -f(-y)$, $y > 0$ and satisfies $\int^{\infty} \frac{dy}{f(y)} < \infty$.

On the other hand, it is well known that equation (E_2^n) always has nonoscillatory solutions. To find results concerning statements (III) and (IV) for nth order equations, we consider the function $\varphi_n(t) = p(t)t^{(\frac{n-1}{2})\alpha+\frac{n+1}{2}}$. We see from Theorems B$_1$, B$_2$, D$_1$, D$_2$, E$_1$, E$_2$ that the existence of oscillatory solutions is related to the monotonicity properties of $\varphi_2(t) = p(t)t^{\frac{a+3}{2}}$ and $\varphi_4(t) = p(t)t^{\frac{3a+5}{2}}$ respectively. We shall show that in a similar fashion the monotonicity of the function $\varphi_n(t)$ yields results related to statements (III) and (IV) for the general even order equations (E_1^n) and (E_2^n).

The basic technique in studying oscillation problems (III) and (IV) for the second order equation (1.3) is to use Sturm's Comparison Theorem and compare it with a transformed equation based on the Euler equation $y'' + kt^{-2}y = 0$. Results on the fourth order equations are again based upon the known facts related to the fourth order Euler equation $y^{(iv)} + kt^{-4}y = 0$. For the nth order equation, we consider the Euler equation (N): $y^{(n)} + kt^{-n}y = 0$. Much of the information on solutions of (N) is contained in the characteristic polynomial $\Gamma_n(\lambda) = \lambda(\lambda-1)\cdots(\lambda-n+1)$. In Section 2, we study the nth order Euler equation (N) and derive a number of properties concerning $\Gamma_n(\lambda)$ which will be used throughout this paper. In Section 3, we prove the results on the existence of proper oscillatory solutions of equation (E_1^n) and do the same for equation (E_2^n) in Section 4. Section 5 is devoted to the study of nonoscillatory solutions of (E_1^n) and (E_2^n), and we extend our earlier results on the fourth order equation given in [37]. In the concluding section, we illustrate our results by examples and state several open problems for further research. We refer to the survey articles by Wong [41], Kartsatos [20] and Kiguradze [23] for further information concerning oscillation problems of second order and nth order Emden–Fowler equations.

2. Auxiliary Lemmas and an nth Order Euler Equation

In this section, we study the nth order Euler equation in terms of its transformed equation and the characteristic polynomial. We also give other results concerning solutions of equations (E_1^n) and (E_2^n) which are required in the next sections.

As can be seen in most papers on the subject of oscillation and nonoscillation of the second order Emden–Fowler equation, the transformation

$$w(x) = t^{-\lambda} y(t), \quad x = \log t, \tag{2.1}$$

where λ is a real constant, plays a key role. Under this "oscillation preserving" transformation, equations (E_1^n) and (E_2^n) are respectively changed into

$$\sum_{k=0}^{n} \frac{1}{k!} \Gamma_n^{(k)}(\lambda) \frac{d^k w}{dx^k} + f(x)|w|^\alpha \operatorname{sgn} w = 0 \tag{2.2}$$

and

$$\sum_{k=0}^{n} \frac{1}{k!} \Gamma_n^{(k)}(\lambda) \frac{d^k w}{dx^k} - f(x)|w|^\alpha \operatorname{sgn} w = 0 \tag{2.3}$$

where

$$\begin{gathered} \Gamma_n(\lambda) = \Gamma_n^{(0)}(\lambda) = \prod_{j=0}^{n-1} (\lambda - j), \\ \Gamma_n^{(k)}(\lambda) = \frac{d^k \Gamma_n(\lambda)}{d\lambda^k}, \quad f(x) = p(t) t^{n+\lambda(\alpha-1)}. \end{gathered} \tag{2.4}$$

To see that equations (2.2) and (2.3) are equivalent to (E_1^n) and (E_2^n), we only need to show that

$$\frac{d^n y(t)}{dt^n} = t^{\lambda - n} \sum_{k=0}^{n} \frac{1}{k!} \Gamma_n^{(k)}(\lambda) \frac{d^k w}{dx^k}. \tag{2.5}$$

It is easy to verify that (2.5) is true for $n = 1, 2$. Using the identity

$$(\lambda - n)\Gamma_n^{(k)}(\lambda) + k\Gamma_n^{(k-1)}(\lambda) = \Gamma_{n+1}^{(k)}(\lambda), \tag{2.6}$$

we can then derive (2.5) by induction.

In view of the importance of the coefficient functions $\Gamma_n^{(k)}(\lambda)$ in the next sections, here we would like to study the properties of the polynomial $\Gamma_n(\lambda)$ and its derivatives $\Gamma_n^{(k)}(\lambda)$. First of all, we observe that $\Gamma_n^{(k)}(\lambda)$ has $n-k$ distinct zeros, and we denote them by r_k^i, where $\Gamma_n^{(k)}(r_k^i) = 0, i = 0, 1, \ldots, n-k$.

Lemma 1. *Let $n = 2m$ and $\Gamma_n(\lambda) = \Gamma_n^{(0)}(\lambda) = \prod_{j=0}^{n-1}(\lambda - j)$. Then*

(a) *$\Gamma_n(\lambda)$ is symmetrical about $m - \frac{1}{2}$, i.e., $\Gamma_n^{(0)}(m - \frac{1}{2} + \lambda) = \Gamma_n^{(0)}(m - \frac{1}{2} - \lambda)$.*

(b) *$\Gamma_n^{(2k-1)}(m - \frac{1}{2}) = 0$ for $1 \le k \le m$.*

(c) $\Gamma_n^{(2k)}(m-\frac{1}{2}) \neq 0$ *for* $1 \le k \le m$. $\Gamma_n(m-\frac{1}{2}) = (-1)^m \prod_{k=1}^{m} (\frac{2k-1}{2})^2$.

(d) *when* $\lambda > m-1$, $|\Gamma_n(\lambda+1)| > |\Gamma_n(\lambda)|$, *which means that* $|\Gamma_n(\lambda)|$ *attains the smallest local maximum at* $\lambda = m - \frac{1}{2}$.

Proof. (a) By direct computation, one can find

$$\Gamma_n^{(0)}(m - \frac{1}{2} + \lambda) = \prod_{i=1}^{m} \left(\lambda^2 - (i - \frac{1}{2})^2 \right) \tag{2.7}$$

which implies $\Gamma_n^{(0)}(m-\frac{1}{2}+\lambda) = \Gamma_n^{(0)}(m-\frac{1}{2}-\lambda)$.

(b) Differentiating (2.7) odd number times at $\lambda = m-\frac{1}{2}$ yields $\Gamma_n^{(2k-1)}(m-\frac{1}{2}) = 0$ for $1 \le k \le m$.

(c) From (b) it follows that $\lambda = m-\frac{1}{2}$ is a root of $\Gamma_n^{(2k-1)}(m-\frac{1}{2}) = 0$ and, by Rolle's mean-value theorem and the fact that $\Gamma_n^{(k)}(\lambda)$ has exactly $n-k$ distinct zeros, we find that the multiplicity of this root is 1, hence $\Gamma_n^{(2k)}(m-\frac{1}{2}) \neq 0$ for $1 \le k \le m$. We get $\Gamma_n(m-\frac{1}{2}) = (-1)^m \prod_{k=1}^{m} (\frac{2k-1}{2})^2$ by setting $\lambda = 0$ in (2.7).

(d) If $\lambda > m-1$, then

$$(\Gamma_n(\lambda+1))^2 - (\Gamma_n(\lambda))^2 = \lambda^2(\lambda-1)^2 \cdots (\lambda-n+2)^2(2(\lambda+1)n - n^2) > 0,$$

so $|\Gamma_n(\lambda)|$ attains the smallest local maximum at $\lambda = m-\frac{1}{2}$. □

Lemma 2. *Let* $n = 2m$ *and* $\Gamma_n(\lambda) = \prod_{j=0}^{n-1}(\lambda - j)$. *Then*

(a) $\Gamma_{2m}(m-\frac{1}{2}+\lambda)$ *is an even function of* λ. *Furthermore,* $\Gamma_{2m}^{(k)}(m-\frac{1}{2}+\lambda)$ *is an odd function when* k *is odd and is an even function when* k *is even.*

(b) $\Gamma_{2m+1}(m+\lambda)$ *is an odd function of* λ. *Moreover,* $\Gamma_{2m+1}^{(k)}(m+\lambda)$ *is an even function when* k *is odd and is an odd function when* k *is even.*

Proof. (a) From (2.7) it follows that $\Gamma_{2m}(m-\frac{1}{2}+\lambda)$ is an even function. Since the derivative of an even function is an odd function and vice versa, it follows that $\Gamma_{2m}^{(k)}(m-\frac{1}{2}+\lambda)$ is odd or even whenever k is odd or even.

(b) Like in the case of (2.7), for $n = 2m+1$ we have

$$\Gamma_{2m+1}(m+\lambda) = \lambda \prod_{i=1}^{m} (\lambda^2 - i^2) \tag{2.8}$$

which is an odd function. This completes the proof of the lemma. □

Lemma 3. *Let* $n = 2m$. *Then*

(a) $\Gamma_n(m-\frac{1}{2})\Gamma_n''(m-\frac{1}{2}) < 0$.

(b) $\Gamma_n(m+\frac{1}{2})\Gamma_n'(m+\frac{1}{2}) > 0$.

Proof. Define $G_n(\lambda) = \Gamma_n(m-\frac{1}{2}+\lambda)$. Note that

$$G_n(\lambda) = \Gamma_n\left(m - \frac{1}{2} + \lambda\right) = \prod_{k=1}^{m} \left\{ \lambda^2 - \left(k - \frac{1}{2}\right)^2 \right\}, \tag{2.9}$$

and, after differentiating (2.9), we obtain

$$G'_n(\lambda) = G_n(\lambda) \sum_{k=1}^{m} \frac{2\lambda}{\lambda^2 - (k - \frac{1}{2})^2}. \tag{2.10}$$

Note that from (2.9) and (2.10) we have $G_n(0) = (-1)^m \prod_{k=1}^{m} (k - \frac{1}{2})^2$ and $G'_n(0) = 0$. A further differentiation yields

$$G''_n(0) = 2G_n(0) \sum_{k=1}^{m} \frac{1}{(k - \frac{1}{2})^2},$$

which proves (a).

To prove (b), we shall show that $G'_n(1)G_n(1) > 0$. Using (2.10) again, we find

$$\begin{aligned} G'_n(1) &= 2G_n(1) \left\{ \frac{2^2}{3 \cdot 1} - \frac{2^2}{5 \cdot 1} - \frac{2^2}{7 \cdot 3} - \cdots - \frac{2^2}{(2m-1)(2m+3)} \right\} \\ &= 2G_n(1) \left\{ \frac{4}{3} - \left(1 - \frac{1}{5}\right) - \left(\frac{1}{3} - \frac{1}{7}\right) - \cdots - \left(\frac{1}{2m-1} - \frac{1}{2m+3}\right) \right\} \\ &= 2G_n(1) \left\{ \frac{1}{2m+1} + \frac{1}{2m+3} \right\}. \end{aligned}$$

Hence it follows from (2.9) that

$$\Gamma_n\left(m + \frac{1}{2}\right) \Gamma'_n\left(m + \frac{1}{2}\right) = 2\Gamma_n^2\left(m + \frac{1}{2}\right) \left\{ \frac{1}{2m+1} + \frac{1}{2m+3} \right\} > 0,$$

which proves (b). □

Lemma 4. *Define $\varepsilon_0 > 0$ as follows:*

$$\varepsilon_0 = \min\left\{ \left| r_k^i - \left(m - \frac{1}{2}\right) \right| : 0 \le k \le n-1,\ 1 \le i \le n-k,\ r_k^i \ne m - \frac{1}{2} \right\}. \tag{2.11}$$

Then we have

(a) *For $\lambda \in (m - \frac{1}{2} - \varepsilon_0, m - \frac{1}{2} + \varepsilon_0)$, $\Gamma_n^{(2k)}(\lambda)$ is alternating in sign, i.e.,*

$$\Gamma_n^{(2k-2)}(\lambda)\Gamma_n^{(2k)}(\lambda) < 0 \tag{2.12}$$

for $k = 1, 2, \ldots, m$;

(b) *For any $\lambda \in (m - \frac{1}{2} - \varepsilon_0, m - \frac{1}{2})$ and $(m - \frac{1}{2}, m - \frac{1}{2} + \varepsilon_0)$, we also have*

$$\Gamma_n^{(2k-1)}(\lambda)\Gamma_n^{(2k+1)}(\lambda) < 0, \tag{2.13}$$

for $k = 1, 2, \ldots, m - 1$.

Proof. (a) Since $\Gamma_n(\lambda)$ is a polynomial, $\Gamma_n^{(k)}(\lambda)$ has only a finite number of isolated zeros, so $\varepsilon_0 > 0$ is clearly defined. To prove (a), we need only to show that $\Gamma_n^{(2k-2)}(\lambda)\Gamma_n^{(2k)}(\lambda) < 0$ for $k = 1, 2, \ldots, m$ at $\lambda = m - \frac{1}{2}$. It follows by continuity that the same conclusion holds for λ satisfying $|\lambda - (m - \frac{1}{2})| < \varepsilon_0$. Assume to the contrary that $\Gamma_n^{(2k-2)}(m - \frac{1}{2})\Gamma_n^{(2k)}(m - \frac{1}{2}) > 0$ for some k, $1 \le k \le m$. Note that $\Gamma_n^{(2k-1)}(m - \frac{1}{2} + \lambda)$ is an odd function by Lemma 2(a), hence

$\Gamma_n^{(2k-1)}(m-\frac{1}{2})=0$, so $m-\frac{1}{2}$ is an extremum of $\Gamma_n^{(2k-2)}(\lambda)$ and must satisfy $\Gamma_n^{(2k-2)}(m-\frac{1}{2})\Gamma_n^{(2k)}(m-\frac{1}{2})<0$, which is the desired contradiction.

(b) Let $\lambda\in(m-\frac{1}{2},m-\frac{1}{2}+\varepsilon_0)$. By the definition of ε_0, we note that $\Gamma_n^{(2k-1)}(\lambda)$ has only one sign, say $\Gamma_n^{(2k-1)}(\lambda)>0$ on the interval $(m-\frac{1}{2},m-\frac{1}{2}+\varepsilon_0)$. Again $\Gamma_n^{(2k-1)}(m-\frac{1}{2})=\Gamma_n^{(2k-3)}(m-\frac{1}{2})=0$ implies that both $\Gamma_n^{(2k-2)}(\lambda)$ and $\Gamma_n^{(2k)}(\lambda)$ attain extrema at $\lambda=m-\frac{1}{2}$. By (a) $\Gamma_n^{(2k-2)}(\lambda)\Gamma_n^{(2k)}(\lambda)<0$, $\Gamma_n^{(2k-1)}(\lambda)>0$ implies that $\Gamma_n^{(2k-2)}(\lambda)<0$ attains its minimum at $m-\frac{1}{2}$. On the other hand, $\Gamma_n^{(2k)}(\lambda)$ must attain its maximum at $m-\frac{1}{2}$ by virtue of Lemma 1 (c). In this case, $\Gamma_n^{(2k+1)}(\lambda)<0$, hence $\Gamma_n^{(2k-1)}(\lambda)\Gamma_n^{(2k+1)}(\lambda)<0$.

The zeros r_k^i of $\Gamma_n^{(k)}(\lambda)$ are finite in number and this shows that $\varepsilon_0>0$ exists. Since $\Gamma_n(m)=0$, we have $\varepsilon_0\le\frac{1}{2}$. It can be supposed that $\Gamma_k^{(i)}(\lambda)$ can have other zeros, apart from $\lambda=m-\frac{1}{2}$, in the interval $(m-\frac{1}{2},m+\frac{1}{2})$. The next lemma shows that this does not happen. □

Lemma 5. $\varepsilon_0=\frac{1}{2}$.

Proof. Since the characteristic polynomial $\Gamma_{2m}(\lambda)$ and its derivatives $\Gamma_{2m}^{(k)}(\lambda)$ have certain symmetry properties around $m-\frac{1}{2}$, to prove $\varepsilon_0=\frac{1}{2}$, it suffices to prove

$$\operatorname{sgn}\Gamma_{2m}^{(k)}\left(m-\frac{1}{2}+\lambda\right)=(-1)^{m+[\frac{k+1}{2}]},\quad 0<\lambda<\frac{1}{2}\,,\tag{2.14}$$

for all $k=0,1,\dots,2m$, where $[k]$ denotes the largest integer less than or equal to k. We shall use identity (2.6) and proceed to prove (2.14) by induction on m. Because $\Gamma_{2m+2}^{(k)}(\lambda)$ is symmetric around $\lambda=m+\frac{1}{2}$, we need to prove

$$\operatorname{sgn}\Gamma_{2m+2}^{(k)}\left(m+\frac{1}{2}+\lambda\right)=(-1)^{m+[\frac{k+1}{2}]},\quad 0<\lambda<\frac{1}{2}\,.\tag{2.15}$$

Firstly , we note that (2.14) is equivalent to

$$\operatorname{sgn}\Gamma_{2m}^{(k)}(m+\lambda)=(-1)^{m+[\frac{k+1}{2}]},\quad -\frac{1}{2}<\lambda<0.\tag{2.16}$$

Using (2.16) in (2.6), we find for $k=odd$ that

$$\operatorname{sgn}\Gamma_{2m+1}^{(k)}(m+\lambda)=(-1)^{m+1+[\frac{k+1}{2}]}=(-1)^{m+[\frac{k}{2}]},\quad -\frac{1}{2}<\lambda<0.\tag{2.17}$$

Since $\Gamma_{2m+1}^{(k)}(m+\lambda)$ is an even function in λ, (2.17) also is valid for the interval $0<\lambda<\frac{1}{2}$, i.e.,

$$\operatorname{sgn}\Gamma_{2m+1}^{(k)}(m+\lambda)=(-1)^{m+1+[\frac{k+1}{2}]}=(-1)^{m+[\frac{k}{2}]},\quad 0<\lambda<\frac{1}{2}\,.\tag{2.18}$$

We claim that (2.18) also holds for $k=even$. Note that differentiating an even function induces the negative sign for $0<\lambda<\frac{1}{2}$. We now differentiate the even function $\Gamma_{2m+1}^{(k)}(\lambda)$ with $k=odd$ and by (2.18) we find

$$\operatorname{sgn}\Gamma_{2m+1}^{(k+1)}(m+\lambda)=(-1)^{m+1+[\frac{k}{2}]}.$$

Writing $k+1=2l$ in the above equation, we obtain

$$\operatorname{sgn}\Gamma_{2m+1}^{(2l)}(m+\lambda)=(-1)^{m+1+[l-\frac{1}{2}]}=(-1)^{m+[\frac{2l}{2}]},$$

which shows that (2.18) holds for all $k=0,1,\ldots,2m$.

Now using (2.18) and identity (2.6), we have for even k

$$\operatorname{sgn}\Gamma_{2m+2}^{(k)}(m+\lambda)=(-1)^{m+1+[\frac{k}{2}]},\quad 0<\lambda<\frac{1}{2},$$

which in fact is equivalent to

$$\operatorname{sgn}\Gamma_{2m+2}^{(k)}(m+\frac{1}{2}+\lambda)=(-1)^{m+1+[\frac{k}{2}]},\quad -\frac{1}{2}<\lambda<0. \tag{2.19}$$

For even k, $\Gamma_{2m+2}^{(k)}(m+\frac{1}{2}+\lambda)$ is an even function, so (2.19) remains valid for $0<\lambda<\frac{1}{2}$, i.e.,

$$\operatorname{sgn}\Gamma_{2m+2}^{(k)}\left(m+\frac{1}{2}+\lambda\right)=(-1)^{m+1+[\frac{k}{2}]}=(-1)^{m+1+[\frac{k+1}{2}]},\ \ 0<\lambda<\frac{1}{2}. \tag{2.20}$$

This proves (2.15) for even k. Differentiating $\Gamma_{2m+2}^{(k)}(m+\frac{1}{2}+\lambda)$ once, we obtain from (2.20)

$$\operatorname{sgn}\Gamma_{2m+2}^{(k+1)}\left(m+\frac{1}{2}+\lambda\right)=(-1)^{m+[\frac{k+1}{2}]}. \tag{2.21}$$

Rewriting (2.21) for $k=odd$, we find

$$\operatorname{sgn}\Gamma_{2m+2}^{(k)}\left(m+\frac{1}{2}+\lambda\right)=(-1)^{m+[\frac{k}{2}]},\quad 0<\lambda<\frac{1}{2},$$

which is in fact (2.15) and the proof of Lemma 5 is completed. □

Here $\Gamma_n(\lambda)$ is known as the characteristic polynomial associated with the nth order equations:

$$y^{(n)}(t)-\frac{k}{t^n}\,y(t)=0 \tag{2.22}$$

and

$$y^{(n)}(t)+\frac{k}{t^n}\,y(t)=0, \tag{2.23}$$

where $k\geq 0$. As an immediate application of Lemma 1 concerning $\Gamma_n(\lambda)$, we have

Proposition 1. *For the case $n=2m$, where m is even, every solution of equation (2.22) is nonoscillatory if and only if $k\leq\prod_{j=1}^{m}(\frac{2j-1}{2})^2$. For the case $n=2m$, m is odd, every solution of equation (2.23) is nonoscillatory if and only if $k\leq\prod_{j=1}^{m}(\frac{2j-1}{2})^2$.*

Proof. We only prove the first case. By transformation (2.1), equation (2.22) becomes

$$\sum_{k=0}^{n} \frac{1}{k!}\Gamma_n^{(k)}(\lambda)\frac{d^k w}{dx^k} - kw = 0. \tag{2.24}$$

Substituting $w(x) = e^{ux}$ into (2.24), we obtain its algebraic characteristic equation

$$\Gamma_n(\lambda + u) = k. \tag{2.25}$$

The characteristic polynomial Γ_n in (2.25) has and only has n real zeros if and only if $k \leq \prod_{j=1}^{m}(\frac{2j-1}{2})^2$. The proof is complete. □

Using the well-known comparison theorem, see, e.g., Levin [31], Kim [27] and Kiguradze and Chanturia [25; p. 32, Theorem 2.6], we can also extend Proposition 1 to a more general linear ordinary differential equation $y^{(n)}(t) \pm p(t)y(t) = 0$. For other useful results concerning the Euler equations and the associated characteristic polynomial $\Gamma_n(\lambda)$, we refer the reader to the monograph by Elias [6]. Proposition 1 can also be used to deduce results concerning the nonoscillation and ultimate disconjugacy of the linear equation (2.22) (G. D. Jones [19], U. Elias [6], Z. Nehari [34] and Trench [39]).

Next we shall state the important lemma due to Kiguradze [23] and [24] concerning nonoscillatory solutions of (E_1^n) and (E_2^n).

Lemma 6. *Suppose that $y(t)$ is a solution of (E_1^n) (or (E_2^n)), where $y(t) > 0$ for $t \geq t_0 \geq 0$. Then there exist numbers $t_1 \in [t_0, \infty)$ and $l \in \{0, 1, \ldots, n-1\}$ such that $l + n$ is even (odd) and*

$$\begin{cases} y^{(i)}(t) > 0 & \text{for } t \geq t_1, \quad i = 0, 1, \ldots, l-1, \\ (-1)^{i+1}y^{(i)}(t) > 0 & \text{for } t \geq t_1, \quad i = l, l+1, \ldots, n\,. \end{cases} \tag{2.26}$$

Almost all the results on the oscillation of equations (E_1^n) and (E_2^n) and similar equations in a more general form are in one way or other based upon Lemma 6. This result is commonly known as Kiguradze's lemma, see, e.g., [1] as well as [14] by Grimmer, [11] by Foster, [12] by Foster and Grimmer and [30] by Leizaroutz and Bracket.

The next two lemmas are concerned with the equation (E_2^n) only. Lemma 7 deals with the existence of proper solutions and Lemma 8 with the behavior of a proper nonoscillatory solution near infinity. Both lemmas are intended to control the growth of the solution by establishing a bound on $y^{(n-1)}(t)$. In the case of (E_1^n), this situation does not arise, since $y(t) > 0$ implies $y^{(n)}(t) < 0$ and therefore every nonoscillatory solution is extendable throughout $[t_0, \infty)$, i.e., it must be proper, the fact explicitly pointed out in [11; pp. 116–117].

Lemma 7. *Let $n = 2m$ and $p(t) > 0$. For any $c > 0$, there exists a solution of (E_2^n) satisfying the initial condition*

$$y(t_0) = y'(t_0) = \cdots = y^{(m-1)}(t) = 0, \quad y^{(m)}(t_0) = c \tag{2.27}$$

which is either oscillatory in a certain neighborhood of τ, $t_0 \le \tau \le \infty$, *satisfying*

$$\limsup_{t\to\tau} |y(t)| = \infty \tag{2.28}$$

or extendable throughout $[t_0, \infty)$ *and satisfies*

$$\liminf_{t\to\infty} |y^{(n-1)}(t)| = 0. \tag{2.29}$$

This lemma is modelled after a similar result given by Kura [28] who attributes it to Kitamura for the case $n = 4$.

Proof. We denote by $y(t, d)$ a solution of the equation (E_2^n) satisfying the initial conditions

$$\begin{gathered} y(t_0) = y'(t_0) = \cdots = y^{(m-1)}(t_0) = 0, y^{(m)}(t_0) = c, \\ y^{(m+1)}(t_0) = y^{(m+2)}(t_0) = \cdots = y^{(n-2)}(t_0) = 0, y^{(n-1)}(t_0) = d. \end{gathered} \tag{2.30}$$

where d is a real number. It is clear that in the interval of the existence of both $y(t, d_1)$ and $y(t, d_2)$

$$y^{(i)}(t, d_1) < y^{(i)}(t, d_2), \quad i = 0, 1, \ldots, n-1, \quad \text{if} \quad d_1 < d_2, \quad t > t_0. \tag{2.31}$$

Define the set A^+ and A^- by

$$A^+ = \{d : y^{(i)}(t, d) > 0, \quad i = 0, 1, \ldots, n-1, \quad \text{for some} \quad t > t_0\}$$

and

$$A^- = \{d : y^{(i)}(t, d) < 0, \quad i = 0, 1, \ldots, n-1, \quad \text{for some} \quad t > t_0\}.$$

From (2.31) and the continuity of solutions of (E_2^n) with respect to the initial values, it is clear that A^+ and A^- are open intervals, $A^+ \cap A^-$ is empty and $0 \in A^+$. We claim that A^+ is bounded from below. Indeed, there exists a positive constant δ such that $y(t, 0)$ is defined on $[t_0, t_0 + 2\delta]$. Choose $d_1 < 0$ such that

$$\frac{c}{m!}\delta^m + \frac{d_1}{(n-1)!}\delta^{n-1} + \frac{\delta^{n-1}}{(n-1)!} y^\alpha(t_0 + \delta, 0) \int_{t_0}^{t_0+\delta} p(t)dt < 0.$$

Then

$$\begin{aligned} y(t_0 + \delta, d_1) &= \frac{c}{m!}\delta^m + \frac{d_1}{(n-1)!}\delta^{n-1} \\ &\quad + \frac{1}{(n-1)!} \int_{t_0}^{t_0+\delta} (t_0 + \delta - t)^{n-1} p(t) y^\alpha(t, d_1) \operatorname{sgn} y(t, d_1) dt \\ &\le \frac{c}{m!}\delta^m + \frac{d_1}{(n-1)!}\delta^{n-1} + \frac{\delta^{n-1}}{(n-1)!} y^\alpha(t_0 + \delta, 0) \int_{t_0}^{t_0+\delta} p(t)dt \\ &< 0. \end{aligned}$$

Similarly, we have

$$y^{(i)}(t_0+\delta, d_1) < 0, \quad 1 \le i \le n-1.$$

This implies $d_1 \in A^-$, and A^+ is bounded from below by d_1. Denote $d_0 = \inf\{d : d \in A^+\}$. Since A^+ and A^- are open, $d_0 \notin A^+ \cup A^-$. If $y(t, d_0)$ cannot be extended to $+\infty$, then $y(t, d_0)$ must be oscillatory in a certain neighborhood of some τ. On the other hand, if $y(t, d_0)$ can be extended to $+\infty$, we can now show that (2.29) holds, i.e.,

$$\liminf_{t\to\infty} |y^{(n-1)}(t, d_0)| = 0.$$

On the other hand, if $\liminf_{t\to\infty} |y^{(n-1)}(t, d_0)| > 0$, then there exists $t > t_0$ such that $y^{(i)}(t, d_0)y(t, d_0) > 0$, $i = 0, 1, \ldots, n-1$, which contradicts the definition of d_0. □

Lemma 7 shows that under certain conditions, the solutions of (E_2^n) cannot be "fast growing", i.e.,

$$\lim_{t\to\infty} |y^{(n-1)}(t)| = \infty,$$

a result due to Kiguradze [24], see also [23, p. 35, Theorem 2.13]. We shall prove the lemma by an entirely different method.

Lemma 8. *Let n be any positive integer. Suppose that $p(t)$ satisfies*

$$\liminf_{t\to\infty} t^{1+(n-1)\alpha} p(t) = c > 0. \tag{2.32}$$

Then every nonoscillatory solution $y(t)$ of (E_2^n) must satisfy

$$\limsup_{t\to\infty} |y^{(n-1)}(t)| < \infty. \tag{2.33}$$

Proof. Without loss of generality, we assume that $y(t) > 0$ for $t \ge t_0$ in which case $y^{(n)}(t) > 0$ so that $y^{(n-1)}(t)$ is increasing. Suppose that $y^{(n-1)}(t) > 0$ for $t \ge t_1 \ge t_0$, then it is easy to see that there exists $t_2 \ge t_1 \ge t_0$ so that $y^{(i)}(t) > 0$ for $i = 0, 1, \ldots, n$.

Now multiply (E_2^n) by $y'(t)$ and integrate by parts from t_2 to t to obtain

$$y^{(n-1)}(t)y'(t) - \int_{t_2}^{t} y^{(n-1)}y''(s)ds \ge y^{(n-1)}(t_2)y'(t_2) + \int_{t_2}^{t} p(s)y^{\alpha}(s)y'(s)ds. \tag{2.34}$$

Using (2.32), we integrate by parts the last integral in (2.34) and obtain

$$\begin{aligned} \int_{t_2}^{t} p(s)y^{\alpha}(s)y'(s)ds &\ge c\int_{t_2}^{t} s^{-1-(n-1)\alpha}y^{\alpha}(s)y'(s)ds \\ &\ge \left.\frac{ct^{-1-(n-1)\alpha}y^{\alpha+1}}{1+\alpha}\right|_{t_2}^{t} + c\frac{(1+(n-1)\alpha)}{1+\alpha}\int_{t_2}^{t} y^{\alpha+1}(s)s^{-2-(n-1)\alpha}ds. \end{aligned} \tag{2.35}$$

Combining (2.34) and (2.35), we can find $c_1 > 0$ such that

$$y^{(n-1)}(t)y'(t) \geq c_1 t^{-1-(n-1)\alpha} y^{\alpha+1}(t). \tag{2.36}$$

Multiplying (2.36) again by $y'(t)$ and integrating by parts once more, we find $c_2 > 0$ such that

$$y^{(n-2)}(t)(y'(t))^2 \geq c_2 t^{-1-(n-1)\alpha} y^{\alpha+2}(t). \tag{2.37}$$

Proceeding like in the case of (2.37), we finally obtain after $n-1$ times

$$(y'(t))^n \geq c_{n-1} t^{-1-(n-1)\alpha} y^{\alpha+n-1}(t), \tag{2.38}$$

which is reduced to

$$y'(t) \geq M_0 t^{\frac{-1-(n-1)\alpha}{n}} (y(t))^{\frac{\alpha+n-1}{n}}. \tag{2.39}$$

Dividing both sides of (2.39) by $(y(t))^\beta$, $\beta = (\alpha + n - 1)/n$, and integrating from t to ∞ we obtain

$$y(t) \leq M_0 t^{n-1},$$

which establishes (2.33) and completes the proof. □

We can prove that condition (2.32) implies that every proper nonoscillatory solution $y(t)$ of (E_2^n) must satisfy

$$\lim_{t\to\infty} y^{(n-1)}(t) = 0. \tag{2.40}$$

Indeed, suppose the contrary, then there exists a constant $b > 0$ such that $y^{(n-1)}(t) \geq b$, which implies that $y(t) \geq b_1 t^{n-1}$. Using this in (E_2^n), we have

$$y^{(n)}(t) \geq p(t) b_1^\alpha t^{\alpha(n-1)} \geq c b_1^\alpha t^{-1}. \tag{2.41}$$

Since $\lim_{t\to\infty} y^{(n-1)}(t) = \hat{b} < \infty$, we can integrate (2.41) from t to ∞ and obtain

$$\hat{b} - y^{(n-1)}(t) \geq \int_t^\infty c b_1^\alpha s^{-1} ds = \infty,$$

which is the desired contradiction. So (2.40) must hold. Indeed, $y(t) > 0$ for $t \geq t_0$ implies that there exists $t_3 \geq t_0$ such that $y^{(n-1)}(t) < 0$, for $t \geq t_3$.

3. Oscillatory Solutions of the Equation (E_1^n)

In this section, we investigate the existence of oscillatory solutions of the equation (E_1^n) using the auxiliary lemmas given in Section 2.

Theorem 3.1. *Let $n = 2m$ and m be odd. If $p(t)$ satisfies*

$$\frac{d}{dt}\left(p(t) t^{n+(m-\frac{1}{2})(\alpha-1)}\right) \geq 0 \tag{3.1}$$

for $t \geq t_0$, then every solution $y(t)$ of (E_1^n) satisfying $y^{(i)}(t_0) = 0$, for $0 \leq i \leq m-1$, and $|y^{(m)}(t_0)|$ sufficiently large is oscillatory.

Proof. Let $y(t)$ be a nonoscillatory solution of (E_1^n). Suppose that $y(t)$ is not proper, i.e., it cannot be extended throughout $[t_0, \infty)$. Then $y(t)$ must be singular of second kind, i.e., $\lim_{t \to t_*} |y(t)| = \infty$ for some $t_0 \le t_* < \infty$. If $y(t)$ is not oscillatory, then it must satisfy $\lim_{t \to t_*} y(t) = \infty$ or $\lim_{t \to t_*} y(t) = -\infty$. Since $-y(t)$ is also a solution, we always have $y(t)y^{(n)}(t) < 0$ for $t \ge t_1 \ge t_0$. In this case, it is known that $y(t)$ cannot be of finite escape times (Foster [11; pp. 116–117]). Therefore $y(t)$ is proper.

Let $y(t)$ be the solution of (E_1^n) satisfying the initial condition

$$y(t_0) = y'(t_0) = \cdots = y^{(m-1)}(t_0) = 0,\ y^{(m)}(t_0) = c, \tag{3.2}$$

where $c > 0$ will be chosen sufficiently large later.

We may assume without loss of generality that $y(t) > 0$ for $t > T_0 \ge t_0$ and $y(T_0) = 0$, $y'(T_0) \ge 0$. By Kiguradze's lemma, we conclude that there exists an odd integer l such that for $t \ge T_0$ we have

$$\begin{cases} \text{(a)} \quad y^{(i)}(t) > 0, \quad 0 \le i \le l-1, \\ \text{(b)} \quad (-1)^{i+l} y^{(i)}(t) > 0, \\ \text{(c)} \quad 0 \le y^{(l)}(\infty) = \lim_{t \to \infty} y^{(l)}(t) < \infty,\ \lim_{t \to \infty} y^{(i)}(t) = 0,\ \ l \le i \le n-1\,. \end{cases} \tag{3.3}$$

Since $y(t) > 0$, the equation (E_1^n) implies $y^{(n)}(t) < 0$, so $y^{(n-1)}(t) > 0$ and decreases to a limit L which is a positive finite number. Integrating (E_1^n), we find

$$L - y^{(n-1)}(t) = -\int_t^\infty p(s) y^\alpha(s) ds. \tag{3.4}$$

Denote $P(t) = p(t) t^{n+h(\alpha-1)}$, where $h = m - \frac{1}{2}$. From (3.4) we obtain

$$y^{(n-1)}(t) \ge \int_t^\infty p(s) y^\alpha(s) ds.$$

Integrating the latter inequality $n-1$ times and using (a), (b), (c), we find

$$\begin{aligned} y(t) &\ge \sum_{i=0}^{i=l-1} \frac{y^{(i)}(T_0)}{i!}(t - T_0)^i + \frac{1}{l!} y^{(l)}(\infty)(t - T_0)^l \\ &\quad + \int_{T_0}^t \cdots \int_{T_0}^s (ds)^l \int_s^\infty \cdots \int_s^\infty p(s) y^\alpha(s) (ds)^{n-l} \\ &\ge \frac{1}{l!}(t - T_0)^l y^\alpha(t) P(t) \int_t^\infty \cdots \int_s^\infty s^{-n-h(\alpha-1)} (ds)^{n-l}. \end{aligned} \tag{3.5}$$

Denote $K = 2 \times l! \times (l + h(\alpha - 1))_n$ where $(a)_n = a(a+1)\cdots(a+n-1)$. Using this in (3.5), we obtain for some sufficiently large $T_1 \ge T_0 \ge t_0$

$$y(t) \ge K^{-1} y^\alpha(t) P(t) t^{-h(\alpha-1)}$$

or

$$y(t) \le K^{\frac{1}{(\alpha-1)}} P(t)^{-\frac{1}{\alpha-1}} t^h \tag{3.6}$$

for $t \ge T_1$. Using (3.5) and (3.6), we conclude either $l < \frac{n}{2}$ or $l = \frac{n}{2}$ with $y^{(l)}(\infty) = 0$. Furthermore, after substituting (3.6) into (E_1^n), we find

$$y^{(n)}(t) \ge -K^{\frac{\alpha}{(\alpha-1)}} P(t)^{-\frac{1}{\alpha-1}} t^{h-n}$$

or

$$y^{(n)}(t) \ge -C_n t^{h-n} \tag{3.7}$$

for $t \ge T_0$, where $C_n = K^{\alpha/(\alpha-1)} P(t)^{-1/(\alpha-1)}$. Integrating (3.7) once again, we have

$$y^{(n-1)}(T) - y^{(n-1)}(t) \ge -C_n \int_t^T s^{h-n} ds. \tag{3.8}$$

From (3.3) (b) and (3.6), we deduce $\lim_{T\to\infty} y^{(n-1)}(T) = 0$, hence (3.8) becomes

$$y^{(n-1)}(t) \le C_{n-1} t^{h-n+1}, \tag{3.9}$$

where $C_{n-1} = C_n(h-n+1)^{-1}$. Repeating the above process inductively, we obtain

$$(-1)^i y^{(i)}(t) \le C_i t^{h-i}, \quad i = l+1, \ldots, n, \tag{3.10}$$

and

$$y^{(l)}(t) \le y^{(l)}(\infty) + C_l t^{h-l}, \tag{3.11}$$

where C_i, $i = l, l+1, \ldots, n$, are positive constants depending only on K, α and $P(t_0)$ (but independent of c). Now, we can integrate (3.11) several times on $[T_1, t)$ and by virtue of (3.3)(c) we find

$$y^{(i)}(t) t^{-h+i} = O(1), \tag{3.12}$$

as $t \to \infty$ for $i = 1, 2, \ldots, l-1$. Here $O(1)$ is again dependent only on K, α and $P(t_0)$ and is independent of the choice of c.

Using the transformation $x = \log t$, $w(x) = t^{-h} y(t)$, equations (E_1^n) and (E_2^n) are transformed into the following two nonlinear nth order differential equations

$$\sum_{k=0}^{n} \Gamma_n^{(k)}(h) \frac{1}{k!} w^{(k)}(x) + f(x)|w(x)|^{\alpha-1} w(x) = 0 \tag{3.13}$$

and

$$\sum_{k=0}^{n} \Gamma_n^{(k)}(h) \frac{1}{k!} w^{(k)}(x) - f(x)|w(x)|^{\alpha-1} w(x) = 0, \tag{3.14}$$

where $f(x) = P(t) = p(t) t^{n+h(\alpha-1)}$. Since the transformation of y and t into w and x is "oscillation preserving", the question concerning the existence and nonexistence of oscillatory or nonoscillatory solutions of (E_1^n) and (E_2^n) can be transferred to equations (3.13) and (3.14) and vice versa. In particular, the relation between $y^{(i)}(t)$ and $w^{(j)}(x)$ is defined by the identity

$$y^{(i)}(t)t^{-h+i} = \sum_{k=0}^{i} \Gamma_i^{(k)}(h) w^{(k)}(x). \tag{3.15}$$

We can consider (3.15) as a matrix equation with column vectors

$$(y(t)t^{-h}, y'(t)t^{-h+1}, \ldots, y^{(n-1)}(t)t^{-h+n-1})$$

and $(w(x), \dot{w}(x), \ldots, w^{(n-1)}(x))$ connected by the matrix $\{\Gamma_i^{(k)}(h)\}$, where $i = 0, 1, \ldots, n-1$ and $k = 0, 1, \ldots, i$. Note that $\{\Gamma_i^k(h)\}$ is a triangular matrix and it is easy to verify that $\det(\Gamma_i^{(k)}(h)) = (-1)^n \neq 0$. Thus, we can express $w^{(i)}(x)$ in terms of linear combinations of $y^{(j)}t^{-h+j}$, $j = 0, 1, \ldots, i$, i.e.,

$$w^{(i)}(x) = \sum_{j=0}^{i} b_i(h) y^{(i)}(t) t^{-h+i}, \tag{3.16}$$

where $b_i(h)$ is a polynomial in h of degree $n - i$. In particular $b_0(h) = (-1)^n h^n$ and $b_{n-1}(h) = \sum_{j=0}^{n-1}(j - h)$. Hence, by (3.12) and (3.16) we can deduce

$$w^{(i)}(x) = O(1), t \to \infty\,, \quad for \quad 0 \le i \le n. \tag{3.17}$$

where $O(1)$ is dependent only on K, α and $P(t_0)$ but independent of c. Also by (3.6), we have

$$\begin{aligned} f(x)|w(x)|^{\alpha+1} &= P(t)|y(t)|^{\alpha+1} t^{-h(\alpha+1)} \\ &\le K^{\frac{\alpha+1}{\alpha-1}} P(t)^{-\frac{2}{\alpha-1}} = O(1) \end{aligned} \tag{3.18}$$

as $t \to \infty$.

Next, we introduce an energy function associated with (3.13) as follows:

$$F_n(x) = W_n(x) + S_n(x) + I_n(x) + \Phi_n(x),$$

where

$$\begin{cases} W_n(x) = \sum\limits_{i=1}^{m-1} (-1)^{i+1} w^{(i)}(x) \sum\limits_{k=i+1}^{n-i} \frac{1}{(i+k)!} \Gamma_n^{(i+k)}(h) w^{(k)}(x), \\ S_n(x) = \frac{1}{2} \sum\limits_{i=1}^{m} (-1)^{i+1} \frac{1}{(2i)!} \Gamma_n^{(2i)}(h) (w^{(i)})^2, \\ I_n(x) = \sum\limits_{i=1}^{m} (-1)^{i+1} \frac{1}{(2i-1)!} \Gamma_n^{(2i-1)}(h) \int\limits_{x_0}^{x} (w^{(i)})^2 dx, \text{ where } x_0 = \log t_0, \\ \Phi_n(x) = \frac{1}{(1+\alpha)} f(x) |w|^{\alpha+1} - \frac{1}{2} \Gamma_n(h) w^2. \end{cases} \tag{3.19}$$

Using assumption (3.1) in equation (3.13), we obtain

$$\frac{d}{dx} F_n(x) = \frac{1}{1+\alpha} \left[\frac{d}{dx} f(x) \right] |w(x)|^{\alpha+1} \ge 0. \tag{3.20}$$

Since $\Gamma_n^{(2i-1)}(h) = 0$, we have $I_n(x) \equiv 0$ and, by Lemma 1(c), $(-1)^{i+1}\Gamma_n^{(2i)}(h) < 0$. We also have $S_n(x) \le 0$. By a choice of initial conditions, we note that

$W_n(x_0) = \Phi_n(x_0) = 0$. Note that m is odd so that $(-1)^{m+1}\frac{1}{(2m)!}\Gamma_n^{(2m)}(h) = 1$. Hence we obtain from (3.19)and (3.20) that

$$F_n(x_0) = \frac{1}{2}|w^{(m)}(x_0)|^2 = \frac{1}{2}t_0c^2 \le F(\infty) = O(1), \tag{3.21}$$

where $F_n(\infty)$ is bounded from above by a constant independent of c. Thus, by choosing c sufficiently large in (3.21), we obtain the desired contradiction proving that $y(t)$ is oscillatory. □

Remark 3.1. When $n = 2$, Theorem 3.1 becomes Theorem B_1 of Jasný [18] and Kurzweil [29].

Theorem 3.2. *Let $n = 2m$ and m be even. If $p(t)$ satisfies*

$$\frac{d}{dt}(p(t)t^{n+(m-\frac{1}{2})(\alpha-1)}) \le 0, \quad t \ge t_0, \tag{3.22}$$

and in addition

$$\lim_{t\to\infty} p(t)t^{n+(m-\frac{1}{2})(\alpha-1)} = k > 0, \tag{3.23}$$

then every solution $y(t)$ of (E_1^n) satisfying $y^{(i)}(t_0) = 0$ for $0 \le i \le m-1$ and $|y^{(m)}(t_0)|$ sufficiently large is oscillatory.

Proof. We proceed in the same manner as for Theorem 3.1 with a solution $y(t)$ of (E_1^n) satisfying the initial conditions (3.2). In this case, since m is even $(-1)^{m+1}\frac{1}{(2m)!}\Gamma_n^{(2m)}(h) = -1$, by (3.20), we find

$$F(\infty) \le F(x_0) = -\frac{1}{2}|w^{(m)}(x_0)|^2 = -\frac{1}{2}t_0c^2. \tag{3.24}$$

We claim that $F(\infty)$ is bounded from below by a constant independent of c in which case the desired contradiction follows by choosing c sufficiently large. In other words, we need to establish (3.17) and (3.18) with assumptions (3.22) and (3.23) replacing (3.1).

Suppose that $y(t)$ is a nonoscillatory solution satisfying (3.2) and (3.3). Using (3.23) we find, instead of (3.5), that

$$y(t) \ge \frac{1}{l!}(t-T)^l y^a(t)k\int_t^\infty \cdots \int_s^\infty s^{-n-h(\alpha-1)}(ds)^{n-l} \tag{3.25}$$

for $t \ge T_0 \ge t_0$. By choosing $T_1 > T_0 \ge t_0$, we obtain for sufficiently large $t \ge T_1$

$$y(t) \ge K^{-1}y^\alpha(t)kt^{-h(\alpha-1)}, \quad h = m - \frac{1}{2}, \tag{3.26}$$

where $K = 2 \times l! \times (l + h(\alpha-1))_n$ and $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$. Since $\alpha > 1$, (3.26) implies

$$y(t) \le K^{\frac{1}{\alpha-1}}k^{\frac{1}{1-\alpha}}t^h, \tag{3.27}$$

which in turn implies that either $l < \frac{n}{2} = m$ or $y^{(l)}(\infty) = 0$ when $l = m$ (since if $y^{(l)}(\infty) = c_1 > 0 \Longrightarrow y(t) \ge c_2t^l$, where $c_1, c_2 > 0$, then by (3.27) $h = m - \frac{1}{2} \ge l$

and $l < m$; thus if $l = m$, then $y^{(l)}(\infty) = 0$). Now substituting (3.27) into (E_1^n) we find

$$y^{(n)}(t) \geq -K^{\frac{1}{\alpha-1}} k^{\frac{\alpha}{1-\alpha}} t^{h\alpha} p(t) \geq -K^{\frac{1}{\alpha-1}} k^{\frac{1}{1-\alpha}} t^{-n+h}. \tag{3.28}$$

Using (3.2) (b), we have $\lim_{t\to\infty} y^{(n-1)}(t) = 0$. So integrating (3.28) we obtain

$$y^{(n-1)}(t) \leq C_{n-1} t^{h-n+1},$$

where $C_{n-1} = K^{\alpha/(1-\alpha)} k^{1/(1-\alpha)} (h-n+1)^{-1}$. Repeating the above process, by induction we find

$$(-1)^{i-1} y^{(i)}(t) \leq C_i t^{h-i}, \quad i = l+1, \ldots, n, \tag{3.29}$$

and

$$y^{(l)}(t) \leq y^{(l)}(\infty) + C_l t^{h-l}, \tag{3.30}$$

where, like in (3.10) and (3.11), C_i, $i = l+1, \ldots, n$, are positive constants depending only on K, α, and k and independent of c. Since $l \leq m$ and $h - l$ is not an integer, we can integrate (3.30) to find once again

$$y^{(j)}(t) t^{-h+j} = O(1) \quad \text{as} \quad t \to \infty, \tag{3.31}$$

for $i = 1, 2, \ldots, l-1$ and $O(1)$ is again independent of c. We can then appeal to the identity

$$w^{(i)}(x) = \sum_{j=0}^{i} b_j(h) y^{(j)}(t) t^{-h+j}$$

to conclude that $w^{(i)}(x) = O(1)$, $x \to \infty$ for $i = 0, 1, \ldots, n$. Thus $F(\infty)$ is bounded from below by a constant independent of c and the proof of the theorem is complete. □

In addition to Theorems 3.1 and 3.2, we can also establish results on the existence of oscillatory solutions of the equations (E_1^n) for all n by a different choice of initial conditions irrespective of whether m is even or odd.

Theorem 3.3. *If $p(t)$ satisfies (3.1), then the equation (E_1^n) has oscillatory solutions.*

Proof. Let $y(t)$ be a solution of (E_1^n) satisfying the initial condition $\{y^{(i)}(t_0) : i = 0, 1, \ldots, m\}$ so that

$$\begin{cases} w^{(i)}(x_0) = 0, & i = 1, 2, \ldots, m \\ w(x_0) = t_0^{-h} y(t_0) = c > 0. \end{cases} \tag{3.32}$$

By an argument similar to that used in proving Theorem 3.1, we assert that condition (3.1) is sufficient to show that $w^{(j)}(x) = O(1)$ as $x \to \infty$ for all $i, i = 0, 1, \ldots, n$, where $O(1)$ is independent of c. In that case, we can also conclude from (3.20) that $F(\infty)$ is bounded from above by a constant independent of c. Now we observe that

$$F(\infty) \geq F(x_0) = \Gamma_n(h)\frac{w^2(x_0)}{2} + f(x_0)\frac{w^{\alpha+1}(x_0)}{\alpha+1}$$
$$= \Gamma_n(h)\frac{c^2}{2} + f(x_0)\frac{c^{\alpha+1}}{\alpha+1}. \tag{3.33}$$

Since $\alpha > 1$ and $f(x_0) > 0$, (3.33) gives the desired contradiction by choosing c sufficiently large, which completes the proof. □

Next we shall show that under a stronger assumption than that of condition (3.1) every proper solution $y(t)$ of the equation (E_1^n) such that $y^{(i)}(t_0) = 0$ for $0 \leq i \leq m-1, |y^{(m)}(t_0)| \neq 0$ is oscillatory. When $n = 2$, this amounts to proving the fact that every solution with a zero is oscillatory.

Theorem 3.4. *Let $n = 2m$, where m is odd. Suppose that there exists a constant $\varepsilon > 0$ such that*

$$\frac{d}{dt}(p(t)t^{n+(h-\varepsilon)(\alpha-1)}) \geq 0, \qquad h = m - \frac{1}{2}, \tag{3.34}$$

then every solution of (E_1^n) with the initial conditions $y^{(i)}(t_0) = 0, 0 \leq i \leq m-1$ and $|y^{(m)}(t_0)| \neq 0$ is oscillatory.

Proof. We may assume that $\varepsilon < \frac{1}{2}$ in (3.34), since if it holds for some $\varepsilon_1 > 0$, then it also does so for all $0 < \varepsilon < \varepsilon_1$. Assume to the contrary that there exists a proper, eventually positive solution $y(t)$ with prescribed initial conditions, say, $y(t) > 0$ for $t \geq T_0 \geq t_0$ and $y(T_0) = 0, y'(T_0) \geq 0$ as in the proof of Theorem 3.1. Again by Kiguradze's Lemma, we note that $y(t)$ satisfies (3.3), where l is an odd integer greater than one. Using (3.34) instead of (3.1), we obtain, similarly to (3.6), the following estimate

$$y(t) \leq K^{\frac{1}{\alpha-1}} P_\varepsilon(t)^{\frac{-1}{\alpha-1}} t^{h-\varepsilon}, \quad h = m - \frac{1}{2}, \tag{3.35}$$

where $K = 2 \times l! \times (l + h(\alpha-1))$, $(a)_n = a(a+1)\cdots(\alpha+n-1)$ and $P_\varepsilon(t) = p(t)t^{n+(m-\frac{1}{2}-\varepsilon)(\alpha-1)}$. Substituting (3.35) into (E_1^n), we repeat the argument as in Theorem 3.1 starting with (3.6) and ending with (3.12), namely,

$$y^{(i)}(t) = O(t^{h-\varepsilon-i}), \quad 0 \leq i \leq n-1, \tag{3.36}$$

where $O(t^{h-\varepsilon-i})$ is dependent only on $K, \alpha, P(t_0)$ and independent of the constant c.

Turning to the transformed equation (3.13), under the transformation $w(x) = t^{-m+\frac{1}{2}}y(t)$, $x = \log t$, we note that $f(x) = p(t)t^{n+(m-\frac{1}{2})(\alpha-1)} = P_\varepsilon(t)t^{\varepsilon(\alpha-1)}$. Also, by (3.35) we have

$$f(x)|w(x)|^{\alpha+1} = P_\varepsilon(t)t^{\varepsilon(\alpha-1)}(t^{-h}y(t))^{\alpha+1} \leq K^{\frac{\alpha+1}{\alpha-1}} P_\varepsilon(t)^{\frac{2}{1-\alpha}} t^{-2\varepsilon} = o(1), \tag{3.37}$$

and by (3.36)

$$w^{(i)}(x) = O(t^{-\varepsilon}) = o(1) \quad \text{as} \quad x \to \infty, \quad 0 \leq i \leq n. \tag{3.38}$$

Now, on account of (3.37) and (3.38), the energy function $F_n(x)$ defined by (3.19) satisfies the condition $\lim_{x\to\infty} F_n(x) = 0$. On the other hand, condition (3.34) implies trivially that (3.1) holds and thus we have

$$0 = \lim_{x\to\infty} F_n(x) \geq F_n(x_0) = (-1)^{m+1}\Gamma_n^{2m}(h)w^{(m)}(x_0) = \frac{1}{2}t_0|y^{(m)}(t_0)|^2 > 0,$$

which is the desired contradiction. This completes the proof of the Theorem. □

Remark 3.2. Consider the special Emden–Fowler equation $y^{(n)} + t^\beta|y|^\alpha \operatorname{sgn} y = 0$ (E_0^n). Condition (3.1) requires that $\beta \geq -n - (\frac{n-1}{2})(\alpha - 1) = -\frac{n-1}{2}\alpha - \frac{n+1}{2}$ for the existence of oscillatory solutions of the equation (E_1^n). When $n = 2$, $\beta \geq -\frac{\alpha+3}{2}$, which also follows from the classical result of Jasny and Kurzweil in Theorem B_1. For $n = 4$, $\beta \geq -\frac{3\alpha+5}{2}$, which is the same as given by Kura [28] in Theorem D_1 for the equation (E_2^n). Applying Theorem 4 to (E_0^2), we obtain that if $\beta > -\frac{\alpha+3}{2}$, then every solution with a zero is oscillatory. This conclusion is weaker than the results of Heidel and Hinton [16], Coffman and Wong [5], where it was shown that $\beta \geq -\frac{\alpha+3}{2}$ will suffice.

4. Existence of Oscillatory Solutions of (E_2^n)

We now turn our attention to the existence of oscillatory solutions of the equation (E_2^n). It is well known that (E_2^n) always has nonoscillatory solutions because of the sign condition on $p(t)$ that it is positive. Unlike equation (E_1^n), a nonoscillatory solution of (E_2^n) may be singular, i.e., it may have finite escape time in the sense that

$$\lim_{t\to t_*} y(t) = +\infty \quad \text{or} \quad -\infty, \tag{4.1}$$

where $t_0 \leq t_* < \infty$. On the other hand, since (E_2^n) is superlinear with $\alpha > 1$, any singular solution must be of second type, the term introduced by Kiguradze, see [25; p. 205, Theorem 11.5], i.e.,

$$\lim_{t\to t_*} \sup |y(t)| = +\infty, \tag{4.2}$$

where $t_0 \leq t_* < \infty$. To prove the existence of oscillatory solutions of (E_2^n), we need to choose solutions satisfying a certain set of initial conditions so that they be first of all not singular in the sense of (4.1) and could be extended throughout the entire interval $[t_0, \infty)$. One such a sufficient condition is given by Lemma 6.

Theorem 4.1. *Let $n = 2m$ and m be even. If $p(t)$ satisfies* (3.1) *for $t \geq t_0$,then there exist a solution $y(t)$ of (E_2^n) satisfying $y^{(i)}(t_0) = 0$ for $0 \leq i \leq m-1$, $y^{(m)}(t_0) = c$, which is oscillatory when c is sufficiently large.*

Proof. By Lemma 7 there exists a solution $y(t)$ of (E_2^n) with the prescribed initial condition (3.2) which is oscillatory or otherwise satisfies $\liminf_{t\to\infty} |y^{(n-1)}(t)| = 0$. In the later case we know that $y(t)$ can be extended throughout the semi-infinite interval $[t_0, \infty)$. We shall show that this situation cannot occur under condition (3.1). Without loss of generality, assume the contrary and let $y(t) > 0$

for $t \geq t_1 \geq t_0$. From Kiguradze's theorem, Lemma 6, and by the fact that $y(t)$ has a zero at $t = t_0$, there exists an even integer $l > 0$ such that (2.26) holds for some $t_2 \geq t_1$. More precisely,

$$\begin{cases} y^{(i)}(t) > 0, & 1 \leq i \leq l-1, \\ (-1)^i y^{(i)}(t) > 0, & l \leq i \leq n-1, \\ 0 \leq y^{(l)}(\infty) = \lim_{t\to\infty} y^{(l)}(t) < \infty, & \lim_{t\to\infty} y^{(i)}(t) = 0, \quad l+1 \leq i \leq n-1. \end{cases}$$

Note that condition (3.1) implies (2.32), so every proper nonoscillatory solution satisfies $y^{(n-1)}(t) < 0$ for $t \geq t_2$ and $\lim_{t\to\infty} y^{(n-1)}(t) = 0$ by Lemma 8. We now define the constant K and the function $P(t)$ by

$$K = 2 \times l! \times (l + h(\alpha-1))_n, \quad P(t) = p(t)t^{n+h(\alpha-1)}, \tag{4.3}$$

where $h = m - \frac{1}{2}$, $(a)_n = a(a+1)\cdots(a+n-1)$. Integrating (E_2^n) $n-l$ times from t to ∞ and l times from t_2 to t, we obtain

$$y(t) = \sum_{i=0}^{l-1} \frac{y^{(i)}(t_2)}{i!}(t-t_2)^i + \frac{1}{l!}y^{(l)}(\infty)(t-t_2)^l + \int_{t_2}^{t}\cdots\int_{t_2}^{s}(ds)^l\int_{s}^{\infty}\cdots\int_{s}^{\infty} p(\sigma)y^\alpha(\sigma)(d\sigma)^{n-l}. \tag{4.4}$$

Using (4.3), we can estimate (4.4) as follows:

$$\begin{aligned} y(t) &> \int_{t_2}^{t}\cdots\int_{t_2}^{s}(ds)^l\int_{t}^{\infty}\cdots\int_{s}^{\infty} p(\sigma)y^\alpha(\sigma)(d\sigma)^{n-l} \\ &\geq \frac{1}{l!}(t-t_2)^l\int_{t}^{\infty}\cdots\int_{s}^{\infty} p(\sigma)y^\alpha(\sigma)(d\sigma)^{n-l} \\ &\geq \frac{1}{l!}(t-t_2)^l y^\alpha(t)P(t)\int_{t}^{\infty}\cdots\int_{s}^{\infty} \sigma^{-n-h(\alpha-1)}(d\sigma)^{n-l} \\ &\geq K^{-1}y^\alpha(t)P(t)t^{-h(\alpha-1)}. \end{aligned} \tag{4.5}$$

Simplifying (4.5), we find

$$y(t) \leq K^{\frac{1}{\alpha-1}}P(t)^{\frac{-1}{\alpha-1}}t^h, \tag{4.6}$$

which when combined with (4.4) implies

$$\text{either} \quad l < m \quad \text{or} \quad l = m \quad \text{with} \quad y^{(l)}(\infty) = 0. \tag{4.7}$$

Substituting (4.6) into (E_2^n), we have

$$y^{(n)}(t) \leq K^{\alpha/(\alpha-1)}P(t)^{-1/(\alpha-1)}t^{h-n}, \quad t \geq t_2 \geq t_0,$$

or, simply,

$$y^{(n)}(t) \leq C_n t^{h-n}, \tag{4.8}$$

where $C_n = K^{\alpha/(\alpha-1)}P(t_0)^{-1/(\alpha-1)}$. Integrating (4.8) $n-l-1$ times over $[t,\infty)$, $t \geq t_2$, we obtain inductively

$$(-1)^i y^{(i)}(t) \leq C_i t^{h-i}, \quad i = l+1,\dots,n-1, \tag{4.9}$$

and

$$y^{(l)}(t) \leq y^{(l)}(\infty) + C_l t^{h-l}, \quad t \geq t_2, \tag{4.10}$$

where $C_i, i = l+1,\dots,n-1$, are positive constants depending on K, α and $P(t_0)$, but independent of c. We now integrate (4.10) l times from t_2 to t and obtain

$$y^{(i)}(t)t^{-h+i} = O(1), \quad 1 \leq i \leq l-1, \tag{4.11}$$

as $t \to \infty$, where $O(1)$ is dependent on C_i but independent of c.

We now turn to the transformed equation (3.14) and the energy function defined by (3.19). If $w(x)$ is a solution of (E_2^n), then condition (3.1) implies that

$$\frac{d}{dx}F_n(x) = \frac{-1}{\alpha+1}\left(\frac{d}{dx}(f(x)\right)|w(x)|^{\alpha+1} \leq 0, \tag{4.12}$$

where $f(x) = P(t) = p(t)t^{n+h(\alpha-1)}$. The initial conditions (3.2) imply by (3.16) that $w^{(i)}(x_0) = 0$, $0 \leq i \leq m-1$. Hence $W_n(x_0)$ in (3.19) is equal to zero, and so is $\Phi_n(x_0)$. Since $\Gamma_n^{(2i-1)}(h) = 0$ for $1 \leq i \leq m$, $I_n(x) \equiv 0$ for all x. Note that m is even and so $S_n(x_0) = \frac{1}{2}(-1)^{m+1}|w^{(m)}(x_0)|^2$. From (4.12) we deduce

$$O(1) = F_n(\infty) \leq F_n(x_0) = S_n(x_0) = \frac{-1}{2}|w^{(m)}(x_0)|^2 = -\frac{1}{2}t_0c^2. \tag{4.13}$$

Now that $F_n(\infty)$ is bounded from below by a constant independent of c because $w^{(i)}(x) = O(1)$, $0\leq i \leq n$, by (4.11). Hence, by choosing c sufficiently large in (4.13), we obtain the desired contradiction. □

Remark 4.1. For $n = 4$, Theorem 4.1 reduces to the result of Kura [28; p. 658, Theorem 3].

Theorem 4.2. *Let $n = 2m$ and m be odd. If $p(t)$ satisfies* (3.25) *and* (3.26) *in Theorem* 3.2, *then there exists a solution $y(t)$ of (E_2^n), satisfying the initial condition* (3.2), *which is oscillatory when $|y^{(m)}(t_0)|$ is sufficiently large.*

Proof. We proceed in the same manner as in Theorem 4.1 and by a similar argument based upon Theorem 3.2 but applied to equation (E_2^n), we conclude (4.11) which implies that the solution $w(x)$ of the transformed equation (3.14) satisfies $w^{(i)}(x) = O(1)$, $0 \leq i \leq n$, where $O(1)$ depends only on the constants K, α, and $P(t_0)$ as given in (4.3) but is independent of c.

Now condition (3.22) implies that condition (3.20) holds for the energy function and $F_n(x)$ is nondecreasing. Similarly to Theorem 4.1, we have $F_n(x_0) = S_n(x_0) = \frac{1}{2}(-1)^{m+1}|w^{(m)}(x_0)|^2$. Since m is odd, we find by (3.16)

$$F_n(x_0) = \frac{1}{2}|w^{(m)}(x_0)|^2 = \frac{1}{2}t_0c^2 \leq F_n(\infty) = O(1). \tag{4.14}$$

Choosing c sufficiently large in (4.14), we obtain the desired contradiction and complete the proof. □

We can use the same method as the one by which we proved Theorem 3.5 to obtain for the equation (E_2^n) the following result.

Theorem 4.3. *Let $n = 2m$, where m is odd. Suppose that $p(t)$ satisfies for some $\varepsilon > 0$, condition* (3.34)*, then every proper solution of (E_2^n) such that $y^{(i)}(t_0) = 0, 0 \le i \le m-1$ and $y^{(m)}(t_0) \ne 0$ is oscillatory.*

Remark 4.2. When $n = 4$, Theorem 4.3 reduces to Theorem 4 in [28] by Kura.

Remark 4.3. In the book by Kiguradze and Chanturia [25; p. 236, Theorem 15.5] we find a result which gives the existence of oscillatory solutions for the equations (E_1^n) and (E_2^n) for $n = 2m$ and $n > 1$:

(i) if m is odd, then $\int^\infty t^{n+\alpha-2}p(t)dt = \infty$ implies that (E_2^n) has an $(m-1)$ parameter family of oscillatory solutions;

(ii) if m is even, then $\int^\infty t^{n-1}p(t)dt = \infty$ implies that (E_1^n) has an m parameter family of oscillatory solutions.

Kiguradze and Chanturia's result is also valid when n is odd. In case n is even, statement (ii) is already covered by Theorem F of Ličko and Švec [32], which in fact guarantees that all solutions of (E_1^n) are oscillatory when m is odd. As for an odd m, statement (i) can be compared with Theorem 4.2. Let $p(t) = t^\beta$, then (i) requires that $\beta \ge -n-\alpha+1$. On the other hand, condition (3.1) and Theorem 4.2 require that $\beta \ge -n-(m-\frac{1}{2})(\alpha-1)$ for (E_2^n), which is implied by $\beta \ge -n-\alpha+1$ when $m > 1$.

5. Nonoscillatory Solutions and Nonoscillation

In this section, we study nonoscillatory solutions of both equations (E_1^n) and (E_2^n) and prove that solutions bounded by certain powers of t must be nonoscillatory. Unlike the second order equation (E_1^2), there is no known result on the nonoscillation of (E_1^n) for $n \ge 4$. On the other hand, the equation (E_2^2) is nonoscillatory without additional assumptions on $p(t)$ except for $p(t) > 0$. There is also but one result on the nonoscillation of (E_2^4) which is given by Kura [28] in Theorem D_1 referred to in Section 1. Kura proved that the equation (E_2^4) has no proper oscillatory solutions. So, strictly speaking, Theorem D_1 states that the equation (E_1^4) is "properly" nonoscillatory. For definiteness, throughout this section we shall assume that solutions of (E_1^n) and (E_2^n) are always proper.

Returning to the equation (E_1^n), it is known that for $n = 2$ there are numerous results on the nonoscillation of (E_1^2) (e.g., Gollwitzer [13], Wong [42], Erbe [7], Erbe and Lu [10]). Theorems C_1 and C_2 referred to in Section 1 are however the only two known results which are sharp in the sense that when the condition on $p(t)$ is relaxed by setting $\varepsilon = 0$, the equation (E_1^2) has proper oscillatory solutions. It is useful to recall that the proofs of Theorems C_1 and C_2 depend heavily on a certain a priori bound on the oscillatory solutions of (E_1^2) in terms of certain powers of t using only the monotonicity conditions on $p(t)$. This is a characteristic typical of second order equations. That this is not available for higher order equations, was demonstrated by the fourth order linear equation

$y^{(iv)} + p(t)y = 0$ which can have oscillatory solutions growing faster than any powers of t. Indeed, the fourth order equation $y^{(iv)} + \tilde{k}t^{-4}y = 0$ which has solutions in the form $y(t) = t^{\lambda}$, where λ are the zeros of the algebraic equation $\lambda(\lambda-1)(\lambda-2)(\lambda-3) + \tilde{k} = 0$. For $0 < \tilde{k} < 1$ there are four real zeros, but for $\tilde{k} > 1$ the four zeros are

$$\lambda_1 = \overline{\lambda}_2 = \frac{3}{2} - \left(\frac{5}{4} \pm i\sqrt{\tilde{k}-1}\right)^{1/2}$$

and

$$\lambda_3 = \overline{\lambda}_4 = \frac{3}{2} + \left(\frac{5}{4} \pm i\sqrt{\tilde{k}-1}\right)^{1/2},$$

where $\overline{\lambda}_i$ denotes the complex conjugate of λ_i. Clearly, λ_3, λ_4 grow without bound when $\tilde{k}$ becomes large, and therefore it is not possible to impose a bound on λ without restricting the value of $\tilde{k}$. By contrast, solutions of the second order Euler equation $y'' + kt^{-2}y = 0$ are bounded by the power $t^{1/2}$ irrespective of the value of k. It is therefore natural to introduce the concept of "wildly oscillatory " solutions. An oscillatory solution of (E_1^n) and (E_2^n) is said to be wildly oscillatory with respect to $\lambda > 0$ if

$$\limsup_{t\to\infty} t^{-\lambda}|y(t)| = \infty. \tag{5.1}$$

Thus, by definition, a wildly oscillatory solution must be proper. Furthermore, if a solution $y(t)$ is wildly oscillatory with respect to λ then it is also wildly oscillatory with respect to μ for any $\mu < \lambda$.

We note that every solution $y(t)$ of (E_1^n) or (E_2^n) can also be considered as a solution of the linear equations

$$z^{(n)} + p(t)|y(t)|^{\alpha-1}z = 0 \tag{5.2}$$

and

$$z^{(n)} - p(t)|y(t)|^{\alpha-1}z = 0. \tag{5.3}$$

It is well known that the linear nth order equation

$$z^{(n)} + q(t)z = 0 \tag{5.4}$$

is nonoscillatory if either $\lim_{t\to\infty} t^n q(t) = 0$ [25, p. 56, Corollary 2.9] or $\int^{\infty} t^{n-1}|q(t)|dt < \infty$ ([6, p. 86, Corollary 6.22]). These conditions are related to the so-called de la Vallee Poussin's disconjugacy criteria, see, e.g., the papers by Swanson [38] and Willett [40]. A solution $z(t)$ of the linear equation (5.4) is said to be disconjugate on $[t_0, \infty)$ if it has at most $n-1$ zeros. Equation (5.4) is said to be eventually disconjugate if there exists $T > t_0$ such that all its solutions are disconjugate on $[T, \infty)$. Clearly, if equation (5.4) is eventually disconjugate on $[t_0, \infty)$, then it is also nonoscillatory. For the binomial equation (5.4), the converse is also true, i.e., nonoscillation implies eventual disconjugacy. This was proved by Nehari [34] for even n and the odd case was considered by Elias in [6]. Therefore, the criteria for eventual disconjugacy of (5.4) imply nonoscillation and vice versa. When $y(t)$ of (E_1^n) or (E_2^n) is not wildly oscillatory, i.e., $y(t)$

is bounded by a power of t, then we can use the linear equation (5.4) to show that it is nonoscillatory. More specifically, if $y(t)$ satisfies $|y(t)| \le M_0 t^\lambda$ and

$$\lim_{t\to\infty} p(t)t^{n+\lambda(\alpha-1)} = 0, \tag{5.5}$$

then $y(t)$ is nonoscillatory. Indeed, our first result asserts a somewhat stronger conclusion in the absence of wildly oscillatory solutions.

Theorem 5.1. *Let $\lambda > 1$ and $p(t)$ satisfy* (5.5)*, then every proper solution of (E_1^n) or (E_2^n) is either nonoscillatory or satisfies for any $\mu, 0 < \mu < \lambda$,*

$$\limsup_{t\to\infty} t^{-\mu+i}|y^{(i)}(t)| = \infty \tag{5.6}$$

for $i = 0, 1, \dots, n$.

Remark 5.1. Clearly, every bounded solution of (E_1^n) and (E_2^n) does not satisfy (5.6) and therefore must be nonoscillatory. In a restricted sense, this is a nonoscillation result concerning higher order equations (E_1^n) and (E_2^n), valid for all bounded solutions. Some related results are given by Lovelady in [33].

Proof of Theorem 5.1. Let $y(t)$ be a proper oscillatory solution of (E_1^n) or (E_2^n) which does not satisfy (5.6). We shall prove that $y(t)$ is nonoscillatory. Suppose that for some i, $0 \le i < n$, we have

$$\limsup_{t\to\infty} t^{-\mu+i}|y^{(i)}(t)| = K_i < \infty, \tag{5.7}$$

where $0 < \mu < \lambda$, or $y^{(i)}(t) = O(t^{\mu-i})$ as $t \to \infty$. When μ is not an integer, it is easy to see that $y^{(j)} = O(t^{\mu-j})$ as $t \to \infty$ for $j = 0, 1, \dots, i$ since $y(t)$ is oscillatory. When μ is an integer, say, $\mu = j - 1$, then $y^{(j)}(t) = O(t^{\lambda-j})$ for $j = 0, 1, \dots, i$, since $\mu < \lambda$. In particular, $y(t) = O(t^\lambda)$ as $t \to \infty$. Using this in either equation (E_1^n) or equation (E_2^n), we deduce that

$$y^{(j)}(t) = O(t^{\lambda-j}), \quad j = 0, 1, \dots, n, \tag{5.8}$$

as $t \to \infty$. Note that $y(t)$ also satisfies the related nth order linear equations (5.1) and 5.2). Using (5.5) and (5.8) with $j = 0$ and applying the well known nonoscillation criteria $\lim_{t\to\infty} t^n q(t) = 0$ for equation (5.4) to equations (5.2) and (5.3), we conclude that $y(t)$ is nonoscillatory. This completes the proof of the theorem. □

We now turn to a more delicate problem when condition (5.5) is replaced by

$$\lim_{t\to\infty} p(t)t^{n+\lambda(\alpha-1)} = k > 0, \tag{5.9}$$

where $\lambda > 0$. This is satisfied in the case of the Euler equation with $q(t) = kt^{-n}$ and $\alpha = 1$. In this case we have the nonoscillation of equation (5.4) if $k \le \prod_{j=1}^{m} (j - \frac{1}{2})^2$. For large values of k, the Euler equations $z^{(n)} + kt^{-n}z = 0$ and $z^{(n)} - kt^{-n}z = 0$ always have oscillatory solutions, see, e.g., [19] by Jones. So when we impose condition (5.9) on the linear equations (5.2) and (5.3), i.e., on (E_1^n) and (E_2^n), we cannot determine the nonoscillation of their solutions even if they are bounded by t^λ, i.e., $y(t) = O(t^\lambda)$, $t \to \infty$. However, if we suitably restrict λ in the neighborhood $m - \frac{1}{2}$, the point of reflection about which the polynomial $\Gamma_n(\lambda)$ is symmetric, then we can prove that such solutions are nonoscillatory for equations (E_1^n) and (E_2^n) with $\alpha > 1$. This is a nonlinear phenomenon of superlinear equations, since in the linear case we can have oscillatory solutions of the Euler equation satisfying (5.9) which grow faster than any power t^λ, $\lambda > m - \frac{1}{2}$. We are now ready to prove our main results concerning nonoscillatory solutions.

Theorem 5.2. *Let $n = 2m$ with m even. Suppose that $p(t)$ satisfies* (5.9) *and also*

$$\frac{d}{dt}(p(t)t^{n+\lambda(\alpha-1)}) \le 0, \tag{5.10}$$

where $m - \frac{1}{2} < \lambda < m$, then every proper solution of (E_2^n) is nonoscillatory or satisfies

$$\limsup_{t\to\infty} t^{-\lambda+i}|y^{(i)}(t)| = \infty, \quad i = 0, 1, \ldots, n. \tag{5.11}$$

Remark 5.2. For $n = 4$, Theorem 5.2 reduces in part to Kura's [28] Theorem D_2, where $0 < \varepsilon < \alpha - 1$. We note for Kura's Theorem D_2 that when $\varepsilon \ge \alpha - 1$, his proof requires the boundedness of $y''(t)$. This can be proved for the equation (E_2^4) when $p'(t) \le 0$, which is implied by condition (5.10). When $n = 2$, the condition that $p'(t) \le 0$ implies that the derivatives of solutions of (E_1^2) are bounded. For general nth order equations, this amounts to the requirement that $y^{(m)}(t)$ be bounded when $n = 2m$. We know no such results for higher order equations when $n \ge 6$.

Proof of Theorem 5.2. Let $y(t)$ be a proper oscillatory solution of (E_2^n) which does not satisfy (5.11). We shall prove that $y(t)$ must be nonoscillatory. Suppose that for some $i, 0 \le i \le n$, we have

$$\limsup_{t\to\infty} t^{-\lambda+i}|y^{(i)}(t)| = K_i < \infty. \tag{5.12}$$

Since $m - \frac{1}{2} < \lambda < m$, we have $-\lambda + i \ne 0$ for $i = 0, 1, \ldots, n$. Now because $y(t)$ is oscillatory, it is easy to see that $y^{(j)}(t) = O(t^{\lambda-j})$, as $t \to \infty$ for $j = 0, 1, \ldots, i$. In particular, $y(t) = O(t^\lambda)$ as $t \to \infty$. Using this in equation (E_2^n), we can deduce as in Theorem 3.1 that

$$y^{(j)}(t) = O(t^{\lambda-j}), \quad t \to \infty,$$

for $j = 0, 1, \ldots, n$. We consider the following equations under the "oscillation preserving" transformation $w(x) = t^{-\lambda}y(t), x = \log t$ like (3.13) and (3.14), namely,

$$\sum_{k=0}^{n} \Gamma_n^{(k)}(\lambda)\frac{1}{k!}w^{(k)}(x) + f(x)|w(x)|^{\alpha-1}w(x) = 0, \tag{5.13}$$

and

$$\sum_{k=0}^{n} \Gamma_n^{(k)}(\lambda)\frac{1}{k!}w^{(k)}(x) - f(x)|w(x)|^{\alpha-1}w(x) = 0, \tag{5.14}$$

where $f(x) = p(t)t^{n+\lambda(\alpha-1)}$. Introduce as in (3.19) the energy function

$$F_n(x) = W_n(x) + S_n(x) + I_n(x) + \Phi_n(x), \tag{5.15}$$

where

$$\begin{cases} W_n(x) = \sum\limits_{i=1}^{m-1}(-1)^{i+1}w^{(i)}(x)\sum\limits_{k=i+1}^{n-i}\frac{1}{(i+k)!}\Gamma_n^{(i+k)}(\lambda)w^{(k)}(x), \\ S_n(x) = \frac{1}{2}\sum\limits_{i=1}^{m}(-1)^{i+1}\frac{1}{(2i)!}\Gamma_n^{(2i)}(\lambda)(w^{(i)})^2, \\ I_n(x) = \sum\limits_{i=1}^{m}(-1)^{i+1}\frac{1}{(2i-1)!}\Gamma_n^{(2i-1)}(\lambda)\int\limits_{x_0}^{x}(w^{(i)})^2dx, \quad \text{where } x_0 = \log t_0, \\ \Phi_n(x) = \frac{1}{(1+\alpha)}f(x)|w|^{\alpha+1} - \frac{1}{2}\Gamma_n(\lambda)w^2. \end{cases} \tag{5.16}$$

From (3.16) and (5.12), we conclude that $w^{(i)}(x) = O(1)$, $x \to \infty$ for $i = 0, 1, \dots, n$. Hence the functions $W_n(x)$ and $S_n(x)$ are also bounded. By virtue of (5.10) and (5.16), we can differentiate $F_n(x)$ and obtain

$$\frac{d}{dx}F_n(x) = -\frac{1}{\alpha+1}\dot{f}(x)|w|^{\alpha+1} \geq 0, \tag{5.17}$$

where $\dot{f}(x) = \frac{d}{dx}f(x)$. By Lemmas 3 and 4, we know that $(-1)^{i+1}\Gamma_n^{(2i-1)}(\lambda) < 0$ for $i = 1, 2, \dots, m$, since $m - \frac{1}{2} < \lambda < m$ and $n = 2m$ with m even. Therefore (5.17) shows that $I_n(x)$ is also bounded, hence

$$\int\limits_{x_0}^{\infty}(w^{(i)}(x))^2dx < \infty, \quad i = 1, 2, \dots, m, \tag{5.18}$$

which together with $w^{(i)}(x) = O(1)$ for $i = 2, 3, \dots, m+1$ implies that

$$\lim_{x\to\infty} w^{(i)}(x) = 0, \qquad i = 1, 2, \dots, m. \tag{5.19}$$

We further claim that in fact

$$\lim_{x\to\infty} w(x) = 0. \tag{5.20}$$

Introduce an auxiliary energy function $V_n(x)$ by

$$V_n(x) = W_n(x) + S_n(x) + \Phi_n(x), \tag{5.21}$$

which on account of (5.17) satisfies

$$\frac{d}{dx}V_n(x) = -\frac{1}{\alpha+1}\dot{f}(x)|w|^{\alpha+1} - \sum_{i=1}^{m}(-1)^{i+1}\Gamma_n^{(2i-1)}(\lambda)(w^{(i)}(x))^2 \geq 0. \tag{5.22}$$

Hence $V_n(x)$ is also nondecreasing as is $F_n(x)$. Let $\{x_n\}$ be an increasing sequence of zeros of $w(x)$ where $x_n \to \infty$ as $n \to \infty$. Then by (5.22), $\lim\limits_{n\to\infty} V_n(x_n) = \lim\limits_{x\to\infty} V_n(x) = 0$, which by (5.19) in turn implies

$$\lim_{x\to\infty} \Phi_n(x) = 0. \tag{5.23}$$

Suppose that $\limsup_{x\to\infty} w(x) \geq \delta > 0$, where δ is a constant. Since $w(x)$ is oscillatory, for any N, $0 < N < \delta$, there exists a sequence $\{z_k\}_{k=1}^{\infty}$ such that $\lim_{n\to\infty} z_k = \infty$ and $w(z_k) = N$. Choose N so small that

$$\frac{1}{2}\Gamma_n(\lambda) - \frac{1}{\alpha+1}LN^{\alpha-1} > 0$$

where $L = f(x_0)$ and satisfies by (5.10)

$$f(x) \leq L, \quad x \geq x_0.$$

We then have for all k

$$\begin{aligned}\Phi_n(z_k) &= \frac{1}{2}\Gamma_n(\lambda)w(z_n)^2 - \frac{1}{\alpha+1}f(z_n)|w(z_n)|^{1+\alpha} \\ &\geq N^2\left(\frac{1}{2}\Gamma_n(\lambda) - \frac{1}{\alpha+1}LN^{\alpha-1}\right) > 0\end{aligned}$$

which contradicts (5.23). Hence $\lim_{x\to\infty} w(x) = 0$. Returning back to the original variables, we see that $y(t) = o(t^{\lambda})$ and

$$t^n p(t)|y(t)|^{\alpha-1} = p(t)t^{n+\lambda(\alpha-1)}o(1) = o(1) \tag{5.24}$$

as $t \to \infty$. Now once again $y(t)$ can be viewed as an oscillatory solution of the nth order linear equation (5.3). Thus conditions (5.9) and (5.24) imply by the well-known nonoscillation criteria (Kiguradze and Chanturia [25; p. 56, Corollary 2.9]) that equation (5.3) is nonoscillatory. This contradiction proves the theorem. □

Theorem 5.3. *Let $n = 2m$ with m odd. Suppose that $p(t)$ satisfies* (5.9) *and also*

$$\frac{d}{dt}(p(t)t^{n+\lambda(\alpha-1)}) \geq 0, \tag{5.25}$$

where $m - \frac{1}{2} < \lambda < m$, then every proper solution of (E_2^n) is nonoscillatory or otherwise satisfies (5.11).

Proof. The proof follows largely from that of Theorem 5.2. We need only to note that when m is odd, $\Gamma_n(\lambda) < 0$ for $m - \frac{1}{2} < \lambda < m$, and in this case the energy function $F_n(x)$ satisfies instead of (5.17)

$$\frac{d}{dx}F_n(x) = -\frac{1}{\alpha+1}\dot{f}(x)|w|^{\alpha+1} \leq 0,$$

where $F_n(x)$ is nonincreasing and thus is hence from bounded above by $F_n(x_0)$. Using Lemmas 3 and 4, we find $(-1)^{i+1}\Gamma_n^{(2i-1)}(\lambda) > 0$ for $i = 1, 2, \ldots, m$ for $m - \frac{1}{2} < \lambda < m$, which implies (5.18) and hence (5.19), since the conclusion that $w^{(i)}(x) = O(1)$ as $x \to \infty$ can be derived as in the proof of Theorem 5.2 using (5.9) but without using condition (5.10). To prove that $\lim_{x\to\infty} w(x) = 0$, we proceed again with the auxiliary energy function $V_n(x)$ defined by (5.21) and note that, by condition (5.25), $V_n(x)$ is nonincreasing. Let $\{x_n\}$ be an increasing sequence of zeros of $w(x)$. Then $\lim_{j\to\infty} V_n(x_j) = 0$ implies $\lim_{x\to\infty} V_n(x) = 0$ and hence $\lim_{x\to\infty} \Phi_n(x) = 0$. Repeating the argument starting with (5.23), we can similarly deduce $\lim_{x\to\infty} w(x) = 0$. The remaining part of the proof repeats the proof of Theorem 5.2. □

Returning to the equation (E_1^n), it is clear that the method used in proving the nonoscillation of solutions in Theorem 5.2 can be adopted to prove

Theorem 5.4. *Let $n = 2m$ with m odd. Suppose that $p(t)$ satisfies* (5.9) *and* (5.10)*, where $m - \frac{1}{2} < \lambda < m$, then every proper solution of (E_1^n) is nonoscillatory or otherwise satisfies* (5.11)*.*

Theorem 5.5. *Let $n = 2m$ with m even. Suppose that $p(t)$ satisfies* (5.9) *and* (5.25)*, where $m - \frac{1}{2} < \lambda < m$, then every proper solution of (E_1^n) is nonoscillatory or otherwise satisfies* (5.11)*.*

The proofs of Theorems 5.4 and 5.5 are similar to those of Theorems 5.2 and 5.3 and are left for the interested reader.

When $p(t)$ does not satisfy condition (5.9) and is nevertheless small for large values of t, we can establish a nonoscillation theorem for solutions with small initial values, namely

Theorem 5.6. *Assume that*

$$\int_{t_0}^{\infty} t^{(n-1)\alpha} p(t)\,dt < \infty \tag{5.26}$$

and $y(t)$ is a proper solution of the equation (E_1^n) or (E_2^n) such that

$$\sum_{k=0}^{n-1} \frac{1}{T^{n-k-1}} \frac{y^{(k)}(T)}{k!} < M(T) \tag{5.27}$$

for some $T \geq t_0$, where

$$M(T) = \left(\frac{\alpha - 1}{(n-1)!} \int_T^{\infty} p(s) s^{(n-1)\alpha} ds \right)^{1/(1-\alpha)}. \tag{5.28}$$

Then $y(t)$ is nonoscillatory.

Proof. Integrating the equation (E_1^n) or (E_2^n) from T to t yields

$$y(t) = \sum_{k=0}^{n-1} \frac{y^{(k)}(T)}{k!}(t-T)^k \pm \frac{1}{(n-1)!} \int_T^t (t-s)^{n-1} p(s) y^{\alpha}(s) \operatorname{sgn} y(s)\,ds. \tag{5.29}$$

If we let $v(t) = \frac{|y|}{t^{n-1}}$, then dividing (5.29) by t^{n-1} we get

$$v(t) \leq \sum_{k=0}^{n-1} \frac{y^{(k)}(T)}{k!T^{n-k-1}} + \frac{1}{(n-1)!} \int_T^t s^{(n-1)\alpha} p(s) v^{\alpha}(s)\,ds.$$

So

$$0 \le v(t) \le C + \frac{1}{(n-1)!} \int_T^t s^{(n-1)\alpha} p(s) v^\alpha(s) ds,$$

where $C = \sum_{k=0}^{n-1} \frac{y^{(k)}(T)}{k! T^{n-k-1}}$. By the Gronwall's inequality, we have

$$v(t) \le \left(C^{1-\alpha} + \frac{1-\alpha}{(n-1)!} \int_T^t s^{(n-1)\alpha} p(s) ds \right)^{1/(1-\alpha)}. \tag{5.30}$$

In view of (5.27), we conclude from (5.30) that $v(t)$ is bounded on $[T, \infty)$. This fact means that

$$|y(t)| \le k t^{n-1}, \quad \text{for some } k > 0 \text{ and } t \ge T. \tag{5.31}$$

Now assume to the contrary that $y(t)$ is oscillatory and satisfies the linear equation

$$z^{(n)} \pm p(t)|y(t)|^{\alpha-1} z = 0. \tag{5.32}$$

Using (5.31), we obtain

$$\int_T^\infty s^{n-1} p(s) |y(s)|^{\alpha-1} ds \le k^{\alpha-1} \int_T^\infty s^{(n-1)\alpha} p(s) ds < \infty$$

which implies that equation (5.32) is nonoscillatory, see Elias[6, p. 86, Corollary 6.22]. This is a contradiction and the proof is complete. □

Remark 5.3. We note that Theorem 5.6 is also valid when n is odd. Indeed when $n = 3$, it reduces to a result of Erbe and Rao [9; p. 477, Theorem 3.1].

6. Concluding Remarks

In this last section, we give a few comments on the current status of our knowledge of Emden–Fowler equations of even order with regard to Classification Problems (III) and (IV) mentioned in Section 1, i.e., the existence of oscillatory solutions under the assumption of co-existence of nonoscillatory solutions and that of nonoscillation. It is also hoped that these remarks will present open problems for further research.

1. For Problem (III) relating to the existence of oscillatory solutions we impose certain monotonicity conditions on the function $\varphi(t) = t^{n+(m-\frac{1}{2})(\alpha-1)} p(t)$. This brings us to composition of the equations (E_1^n) and (E_2^n) with the nth order Euler equation, where the function $\varphi(t)$ becomes $t^n p(t)$, and an application of the well-known oscillation and nonoscillation criteria, see, e.g., [6; p. 130, Theorem 8.37], or [25; p. 56, Corollary 2.9]. For $n = 2$, we have recently shown that the condition $\varphi'(t) \ge 0$ can be relaxed to $\varphi'_-(t) \in L^1(t_0, \infty)$, see Ou and Wong [36]. We cannot obtain a similar result for the equations (E_1^n) and (E_2^n) when $n \ge 4$.

2. For Problem (IV) on the nonoscillation of (E_1^n) and (E_2^n), we find only the results of Kura [28] for the equation (E_2^4) and it will be a real challenge

to find another nonoscillation theorem for higher order equations. Even when we restrict our attention to solutions bounded by powers of t, our results are fragmentary is a sense in that we require that $m - \frac{1}{2} < \lambda < m$. When a solution $y(t)$ of either (E_1^n) or (E_2^n) is bounded by t^λ for $m - \frac{1}{2} < \lambda < m$, , we have shown that if $p(t)$ satisfies condition (5.9) together with an appropriate monotonicity condition, then $y(t)$ is nonoscillatory without any restriction on the constant k in (5.9). This is a nonlinear feature not characteristic of the linear Euler equation $z^{(iv)} + kt^{-4}z = 0$ which has oscillatory solutions when $k > 1$. When $n = 4$, we have shown in our earlier paper [37] that the range of λ can be extended to an interval with endpoints being the two furthest roots of the third order polynomical $\Gamma_4'(\lambda) = 0$. Unfortunately, for higher order equations, the derivatives $\Gamma_n^{(k)}(\lambda)$ lose the desired alternating sign property described in Lemma 3(a), 3(b), which is used throughout this paper. Indeed, the nearest zero of $\Gamma_n'(\lambda)$ from $\lambda = m - \frac{1}{2}$, where $\Gamma_n'(m - \frac{1}{2}) = 0$, is larger than $m + \frac{1}{2}$, and approaches $m + \frac{1}{2}$ when n becomes large, see Lemma 3(e). It is therefore of interest to prove similar results for solutions of both (E_1^n) and (E_2^n) which are bounded by t^λ for λ outside the range, $m - \frac{1}{2} < \lambda < m$ when $n \geq 6$.

3. A similar question concerning the monotonicity condition on $\Psi(t) = p(t)t^{n+\lambda(\alpha-1)}, m - \frac{1}{2} < \lambda < m$, for nonoscillation remains open for the equation (E_1^n) when $n \geq 4$. In the case of (E_1^2), it was recently shown that the condition $\Psi'(t) \leq 0$ can be improved to that of $\Psi_+' \in L^1(t_0, \infty)$, see Wong [44].

4. In Section 5, for the linear equation (5.32) the eventual disconjugacy is equivalent to nonoscillation. It is not known whether the same is true for the nonlinear equations (E_1^n) and (E_2^n) when $n \geq 2$.

5. For the linear equation $y^{(iv)} + p(t)y = 0$, $p(t) > 0$, it is known that either all solutions oscillate or none do (see [35] by Leighton and Nehari). This is not true when $n \geq 6$ ([19] by Jones, cf. also [17] by Hunt). Find a condition on $p(t)$ such that this is true for the nonlinear equation (E_1^4).

6. Problems regarding classifications (III) and (IV) for equations more complicated than the equations (E_1^n) and (E_2^n) remain at large. Indeed, we know no such results for second order equations such as

(i) equations with delay

$$y''(t) + p(t)|y^\alpha(t-1)| \operatorname{sgn} y(t-1) = 0;$$

(ii) equations with linear damping

$$y''(t) + r(t)y'(t) + p(t)|y(t)|^\alpha \operatorname{sgn} y(t) = 0;$$

(iii) equations under forcing

$$y''(t) + p(t)|y(t)|^\alpha \operatorname{sgn} y(t) = e(t).$$

The classification of these equations with respect to (I) and (II) can be found in a recent book of Agarwal, Grace, and O'Regan [1].

7. Rather simple equations which may commonly occur still fail to be classified into (III) and (IV). Below we give two cases as examples:

(i) Consider $y^{(iv)} + e^{-x}y^3 = 0$. This equation is simple enough and by Licko and Svec's Theorem F we know it must have nonoscillatory solutions. Since the coefficient function is exponentially small, we would expect that in fact all solutions are nonoscillatory. We are unaware of any results which can be applied to deduce this conclusion. The same applies to all higher order equations.

(ii) Consider $y^{(iv)} = e^{-x}(\sin x + \frac{1}{2})y^3$. Without the term "$\sin x$", this equation is nonoscillatory by Theorem D_2 of Kura [28]. In this case, $q(t)$ does not have the definite sign and all the existing results are not applicable. The numerical results for a limited range of initial conditions indicate that this equation is again nonoscillatory.

8. When $p(t)$ is not of definite sign, the oscillation of all solutions of (E_1^n) like Theorem F is known only when $n = 2$, see [21] by Kiguradze for an extension of Atkinson's Theorem A. It will be of interest to establish similar results for higher order equations.

9. Finally, we should point out that the results given in this paper are based upon the techniques related to the superlinear equations (E_1^n) and (E_2^n) with $\alpha > 1$. The sublinear case, i.e., when $0 < \alpha < 1$, has not been considered although some of the nonoscillation results could perhaps be reformulated in order to be valid for sublinear equations too. Likewise, the problem for odd order equations is wide open except for the third order equation. On the existence of oscillatory solutions on odd order equations, we refer to Heidel [15].

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