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A COMMON FIXED POINT THEOREM CONNECTED TO A RESULT OF V. POPA AND H. K. PATHAK

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Abstract. In this paper, we prove a common fixed point theorem for two pairs of weakly compatible self-mappings of a complete metric space without requiring continuity. Our result generalizes a theorem obtained by V. Popa and H. K. Pathak in 1998 and a result obtained by B. Fisher and S. Sessa in 1986.

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1. Introduction

Throughout this note, M will be a metric space endowed with the metric d. Let T and J be two selfmappings of M. Sessa [17] defines T and J to be weakly commuting if $d(TJx, JTx \leq d(Jx, Tx)$ for all $x \in M$. Jungck [6] defines T and J to be compatible if $\lim_{n\to\infty} d(TJx_n, JTx_n) = 0$ whenever $\{x_n\}$ is a sequence in M such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Jx_n = x$ for some $x \in M$. Clearly, commuting mappings are weakly commuting and weakly commuting mappings are compatible, but none of these implications is reversible (Example 1 [18] and Example 2.2 [6]).

Jungck, Murthy and Cho (see [8]) define T and J to be compatible of type (A) if $\lim_{n\to\infty} d(TJx_n, J^2x_n) = 0$ and $\lim_{n\to\infty} d(JTx_n, T^2x_n) = 0$ whenever $\{x_n\}$ is a sequence in M such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Jx_n = x$ for some $x \in M$. Clearly, weakly commuting mappings are compatible of type (A). By [8, Example 2.2] it follows that the converse of this implication is not true. By [8, Example 2.1 and Example 2.2] it follows that the notions of compatible mappings and compatible mappings of type (A) are independent of each other.

Recently, H. K. Pathak and M. S. Khan [13] have defined T and J to be compatible of type (B) if

$$\lim_{n\to\infty} d(JTx_n, T^2x_n) \leq \frac{1}{2}\left[\lim_{n\to\infty} d(JTx_n, Jx) + \lim_{n\to\infty} d(Jx, J^2x_n)\right]$$

and

$$\lim_{n\to\infty} d(TJx_n, J^2x_n) \leq \frac{1}{2}\left[\lim_{n\to\infty} d(TJx_n, Tx) + \lim_{n\to\infty} d(Tx, T^2x_n)\right]$$

whenever $\{x_n\}$ is a sequence in M such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Jx_n = x$ for some $x \in M$. It is clear that compatible mappings of type (A) are compatible

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of type (B). By Example 2.4 [13], the implication is not reversible. One can find examples (see [4]) to see that the notions of compatible mappings and compatible mappings of type (A)(and consequently of type (B)) are different if the continuity is dropped. Indeed, there are examples (see [13] and [8]) showing that the notions of compatible mappings, compatible mappings of types (A) and (B) are not equivalent.

G. Jungck and B. E. Rhoades (see [9]) define T and J to be weakly compatible if T and J commute at their coincidence points. Clearly, if T and J are weakly commuting or compatible, then they are weakly compatible. To see that compatible mappings of type (A) or compatible mappings of type (B) are weakly compatible, we can use the following proposition on compatible mappings of type (B) established in [13].

Proposition 1.1 ([13])**.** *Let S and J be compatible mappings of type (B) from a metric space (M,d) into itself. If $Tx = Jx$ for some $x \in M$, then we have*

$$TJx = T^2x = J^2x = JTx.$$

We can say that the notion of weakly compatible mappings is the weakest version of the commutativity amongst the previous notions.

We point out that several authors used these concepts to prove common fixed point theorems and study periodic points (see, e.g., [1]–[19] and references therein).

Let $\mathcal{H}$ (see [14]) be the set of all real functions $f : ([0,\infty))^5 \mapsto [0,\infty)$ satisfying the following conditions:

(H_1): is upper semi-continuous and nondecreasing in variables x_4 and x_5;
(H_2): $f(u,0,0,u,u) < u$, for all $u > 0$;
(H_3): there exist $0 \leq h < 1$ such that for every $u, v \in [0,\infty)$ with
$(H_{3,a})$: $u \leq f(v,v,u,u+v,0)$, or
$(H_{3,b})$: $u \leq f(v,u,v,0,u+v)$
we have $u \leq h\,v$.

To generalize the works [5] and [1], V. Popa and H. K. Pathak [14] established

Theorem 1.1 ([14])**.** *Let I, J, S and T be mappings from a complete metric space (M,d) into itself satisfying the conditions:*

(a) *$S(M) \subset J(M)$ and $T(M) \subset I(M)$.*
(b) *One of the mappings I, J, S or T is continuous.*
(c) *The pairs (S,I) and (T,J) are compatible of type (A).*
(d) *The inequality*

$$d(Sx,Ty) \leq f\left(d(Ix,Jy)), d(Ix,Sx), d(Jy,Ty), d(Ix,Ty), d(Jy,Sx)\right) \quad (C(f))$$

holds for all $x, y \in M$, where $f \in \mathcal{H}$. Then I, J, S and T have a unique common fixed point. Furthermore, z is the unique common fixed point of S and I and of T and J.

The aim of this paper is to establish a common fixed point theorem for four self-mappings I, J, S or T (extending Theorem 1.1) in a more general case where the pairs (S, I) and (T, J) are merely weakly compatible. Our result (see Theorem 2.1 below) does not appeal to the continuity.

2. The Result

As before, M denotes a metric space endowed with the metric d. For any subset A of M, the closure of A will be denoted by $\overline{A}$. Our main result is

Theorem 2.1. *Let* $f \in \mathcal{H}$ *and let* I, J, S *and* T *be mappings from a complete metric space* (M, d) *into itself satisfying the contractive condition* $(C(f))$. *Suppose that*

(A 1) $S(M) \subset J(M)$ *and* $T(M) \subset I(M)$.

(A 2) *The pairs* (S, I) *and* (T, J) *are weakly compatible.*

(A 3) $\overline{S(M)} \cap \overline{T(M)} \subset I(M) \cup J(M)$.

Then I, J, S *and* T *have the unique common fixed point* z. *Furthermore, we have* $\{z\} = \mathrm{Fix}\{S, I\} = \mathrm{Fix}\{T, J\}$, *and if* I *(resp.* J*) is continuous at* z, *then* S *(resp.* T*) is continuous at* z.

Proof. Let x_0 be an arbitrary point in M. Set $y_0 = Sx_0$. Since $S(M) \subset J(M)$, we can find a point $x_1 \in M$ such that $y_0 = Jx_1$. Set $y_1 = Sx_1$. By induction it is easy to construct two sequences (x_n) and (y_n) in M satisfying for each nonnegative integer n,

$$y_{2n} = Sx_{2n} = Jx_{2n+1} \quad \text{and} \quad y_{2n+1} = Tx_{2n+1} = Ix_{2n+2} \tag{2.1}$$

For each nonnegative integer n, we set $d_n := d(y_n, y_{n+1})$. Using the inequality $(C(f))$, for all nonnegative integer n, we have

$$\begin{aligned} d_{2n+1} = d(y_{2n+2}, y_{2n+1}) = d(Sx_{2n+2}, Tx_{2n+1}) &\le f\big(d(Ix_{2n+2}, Jx_{2n+1}), \\ &\quad d(Ix_{2n+2}, Sx_{2n+2}), d(Jx_{2n+1}, Tx_{2n+1}), d(Ix_{2n+2}, Tx_{2n+1}), d(Jx_{2n+1}, Sx_{2n+2})\big) \\ &= f\big(d(y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n+!2}), d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+1}), d(y_{2n}, y_{2n+2})\big) \\ &= f(d_{2n}, d_{2n+1}, d_{2n}, 0, d(y_{2n}, y_{2n+2})) \le f(d_{2n}, d_{2n+1}, d_{2n}, 0, d_{2n} + d_{2n+1}). \end{aligned}$$

By $(H_{3,b})$, $d_{2n+1} \le h\, d_{2n}$. Similarly, by $(H_{3,a})$ we have $d_{2n} \le h\, d_{2n-1}$ for all positive integers n. We deduce that

$$d_n \le h^n\, d_0,$$

for all nonnegative integers n. By routine manipulations it follows that sequence (2.1) is a Cauchy one. Since M is complete, the sequence $\{y_n\}$ converges to some element $z \in M$. Thus, we have

$$z = \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Jx_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Ix_{2n+2}. \tag{2.2}$$

By assumption (A 3), we have $z \in I(M) \cup J(M)$.

(1) Suppose that z belongs to the set $J(M)$. Then we can find a point $u \in M$ such that $z = Ju$. We set $\tau := d(Tu, z)$. To get a contradiction, let us suppose that $\tau > 0$. Then by using $(C(f))$ we get

$$d(Tu, z) \le d(Tu, Sx_{2n}) + d(Sx_{2n}, z)$$

$$\leq f\big(d(Ix_{2n},z),d(Ix_{2n},Sx_{2n}),d(z,Tu),d(Ix_{2n},Tu),d(z,Sx_{2n})\big)+d(Sx_{2n},z).$$

Taking the limits as $n\to\infty$ yields $0<\tau\leq f(0,0,\tau,\tau,0)$ which is a contradiction to $(H_{3,a})$. Thus we have $z=Ju=Tu$. Since $T(M)\subset I(M)$, there exists a point $v\in M$ such that $z=Iv$. Then, again by using $(C(f))$, we get

$$d(Sv,z)=d(Sv,Tu)\leq f\left(0,d(z,Sv),0,0,d(Sv,z)\right).$$

From $(H_{3,b})$ it follows that $Sv=z$. So we have proved that $Ju{=}Tu{=}z{=}Sv{=}Iv$. Since the pairs (T,J) and (S,I) are weakly compatible, we have

$$TJu=JTu \quad \text{and} \quad SIv=ISv. \tag{2.3}$$

From (2.3) we obtain

$$Tz=Jz \quad \text{and} \quad Sz=Iz. \tag{2.4}$$

Now, we will show that z is a fixed point of S. We set $\tau:=d(Sz,z)$ and suppose that $\tau>0$. Then by using $(C(f))$ and (2.4), we have

$$\begin{aligned}0<\tau=d(Sz,Tu)&\leq f\left(d(Iz,Ju),d(Iz,Sz),d(Ju,Tu),d(Iz,Tu),d(Sz,Ju)\right)\\&=f\left(\tau,0,0,\tau,\tau\right),\end{aligned}$$

contradicting (H_2). Thus $Sz=z$. Therefore $Sz=Iz=z$. Set $\rho:=d(z,Tz)$ and suppose that $\rho>0$. Then by using the condition $(C(f))$ and (2.4), we have

$$\begin{aligned}0<\rho&=d(z,Tz)=d(Sz,Tz)\\&\leq f\left(d(Iz,Jz),d(Iz,Sz),d(Jz,Tz),d(Iz,Tz),d(Jz,Sz)\right)\\&=f\left(d(z,Tz),d(z,z),d(Tz,Tz),d(z,Tz),d(Tz,z)\right)=f\left(\rho,0,0,\rho,\rho\right),\end{aligned}$$

a contradiction to (H_2). Thus $Jz=Tz=z$. Therefore z is the common fixed point of the mappings I,J,S and T.

(2) Suppose that z belongs to the set $I(M)$. Then by similar arguments, one can prove as in (1) that z is the common fixed point of the mappings I,J,S and T.

(3) The uniqueness of z is a consequence of the conditions $(C(f))$ and (H_2).

(4) Now we show that z is the unique common fixed point of the mappings S and I. Suppose that there exists $w\in\text{Fix}\{S,I\}$ with $w\neq z$. Then by using $(C(f))$, we have

$$\begin{aligned}0<d(z,w)&=d(Sw,Tz)\\&\leq f\left(d(Iw,Jz),d(Iw,Sw),d(Jz,Tz),d(Iw,Tz),d(Jz,Sw)\right)\\&=f\left(d(w,z),0,0,d(w,z),d(w,z)\right),\end{aligned}$$

contradicting (H_2). Therefore $\{z\}=\text{Fix}\{S,I\}$. Suppose that there exists $v\in\text{Fix}\{T,J\}$ with $v\neq z$. Then by using the condition $(C(f))$, we have

$$\begin{aligned}0<d(z,v)&=d(Sz,Tv)\\&\leq f\left(d(Iz,Jv),d(Iz,Sz),d(Jv,Tv),d(Iz,Tv),d(Jv,Sz)\right)\\&=f\left(d(z,v),0,0,d(z,v),d(v,z)\right),\end{aligned}$$

contradicting (H_2). Therefore $\{z\}=\text{Fix}\{T,J\}$.

(5) Suppose that I is continuous at the point z. Let (z_n) be any sequence of points in M converging to z in M. Set $u_n := d(Sz_n, z)$ and $\alpha := \limsup_n u_n$. To conclude that S is continuous at t, it is enough to prove that $\alpha = 0$. By using the condition $(C(f))$, for all nonnegative integers n, we have

$$d(Sz_n, z) = d(Sz_n, Tz) \le f\left(d(Iz_n, z), d(Iz_n, Sz_n), 0, d(Iz_n, z), d(z, Sz_n)\right).$$

By using the upper semicontinuity of f and by taking the limits, we get

$$\alpha \le f(0, \alpha, 0, 0, \alpha).$$

Using $(H_{3,b})$, this implies that $\alpha = 0$. So S is continuous at z.

Suppose that J is continuous at the point z. Let (z_n) be any sequence of points in M converging to z in M. Set $v_n := d(Tz_n, z)$ and $\beta := \limsup_n v_n$. We want to prove that $\beta = 0$. By using the condition $(C(f))$, for all nonnegative integers n, we have

$$d(Tz_n, z) = d(Sz, Tz_n) \le f\left(d(z, Jz_n), 0, d(Jz_n, Tz_n), d(z, Tz_n), d(Jz_n, z)\right).$$

By using the upper semicontinuity of f and by taking the limits, we get

$$\beta \le f(0, 0, \beta, \beta, 0).$$

By $(H_{3,a})$, this implies that $\beta = 0$. So T is continuous at z. □

Corollary 2.1. *Let $f \in \mathcal{H}$ and let I, J, S and T be mappings from a complete metric space (M, d) into itself satisfying for some positive integers p, q, r and s,*

$$d(S^p x, T^q y) \le f\big(d(I^r x, J^s y)), \quad d(I^r x, S^p x), d(J^s y, T^q y), d(I^r x, T^q y), d(J^s y, S^p x)\big) \qquad (C(f))$$

for all $x \in M$. Suppose that

(A 1) *$S^p(M) \subset J^s(M)$ and $T^q(M) \subset I^r(M)$.*

(A 2) *The pairs (S, I) and (T, J) are commuting.*

(A 3) *$\overline{S^p(M)} \cap \overline{T^q(M)} \subset I^r(M) \cup J^s(M)$.*

Then I, J, S and T have the unique common fixed point z. Furthermore, we have $\{z\} = \mathrm{Fix}\{S, I\} = \mathrm{Fix}\{T, J\}$, and if I^r (resp. J^s) is continuous at z, then S^p (resp. T^q) is continuous at z.

Proof. Set $\tilde{S} := S^p$, $\tilde{T} := T^q$, $\tilde{I} := I^r$ and $\tilde{J} := J^s$. Then an application of Theorem 2.1 yields that the mappings $\tilde{I}, \tilde{J}, \tilde{S}$ and $\tilde{T}$ have a unique common fixed point z in M. We have $Sz = \tilde{S}(Sz)$ and by the commutativity of S and I we obtain $Sz = S(I^r z) = I^r(Sz) = \tilde{I}(Sz)$. Thus $Sz \in \mathrm{Fix}\{\tilde{S}, \tilde{I}\}$. By virtue of Theorem 2.1, we must have $Sz = z$. By similar arguments we have $Iz = z$ and $Jz = Tz = z$. By using the condition $(C(f))$ and the properties of f one can prove (as in Theorem 2.1) that if $\tilde{I}$ (resp. $\tilde{J}$) is continuous at z, then $\tilde{S}$ (resp. $\tilde{T}$) is also continuous at z. □

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