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HOLOMORPHIC VECTOR BUNDLES ON HOLOMORPHICALLY CONVEX COMPLEX MANIFOLDS

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Abstract. Let X be a holomorphically convex complex manifold and $\mathrm{Exc}(X) \subseteq X$ the union of all positive dimensional compact analytic subsets of X. We assume that $\mathrm{Exc}(X) \neq X$ and X is not a Stein manifold. Here we prove the existence of a holomorphic vector bundle E on X such that $(E|U) \oplus \mathcal{O}_U^m$ is not holomorphically trivial for every open neighborhood U of $\mathrm{Exc}(X)$ and every integer $m \geq 0$. Furthermore, we study the existence of holomorphic vector bundles on such a neighborhood U, which are not extendable across a 2-concave point of $\partial(U)$.

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1. Introduction

A famous theorem of Grauert states that on a complex Stein space the holomorphic and the topological classification of vector bundles are the same. In particular every holomorphic vector bundle on a one-dimensional or a contractible Stein space is holomorphically trivial. A suitable extension of Grauert's theorem to 0-convex complex manifolds was proved by G. Henkin and J. Leiterer (see [5] and [3]). J. Winkelmann proved that on any n-dimensional compact complex manifold there is a non-trivial holomorphic vector bundle of rank at most n ([6] and [7], Theorem 7.13.1). Let X be a connected holomorphically convex complex manifold and $f : X \to Z$ be its Remert reduction $f : X \to Z$. We recall that f is proper, Z is a Stein space, $f_*(\mathcal{O}_X) = \mathcal{O}_Z$ and that the pair (Z, f) is uniquely determined by these properties. Furthermore, f is surjective and for any $P \in X$ the fiber $f^{-1}(f(P))$ is the union of all irreducible compact analytic subsets of X containing P. X is Stein if and only if f is an isomorphism. X is compact if and only if Z is a point. Let $\mathrm{Exc}(X) := \mathrm{Exc}(f) := \{P \in X : f$ is not a local isomorphism at $P\}$ be the exceptional locus of f. $\mathrm{Exc}(f)$ is the union of all the positive dimensional irreducible compact analytic subsets of X. We will also call it the exceptional subset of X. $\mathrm{Exc}(X) = X$ if and only if $\dim(Z) < \dim(X)$. In this paper we will only consider the case $\dim(X) = \dim(Z)$. For the case $1 \leq \dim(X) - \dim(Z) \leq 2$, see [2], Theorem 1.2. In Section 2 we will prove the following two theorems.

Theorem 1. *Let X be a holomorphically convex complex manifold such that $\mathrm{Exc}(X) \neq X$ and $\mathrm{Exc}(X)$ contains a hypersurface of X. Then there is a*

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holomorphic line bundle L on X such that $(L|U) \oplus \mathcal{O}_U^m$ is not holomorphically trivial for every open neighborhood U of $\mathrm{Exc}(X)$ *and every integer $m \geq 0$.*

Theorem 2. *Let X be a holomorphically convex complex manifold such that* $\mathrm{Exc}(X) \neq X$ *and* $\mathrm{Exc}(X)$ *contains no hypersurface of X. Then the tangent bundle TX is not holomorphically trivial. Furthermore, for every open neighborhood U of* $\mathrm{Exc}(X)$ *and every integer $m \geq 0$ on U the vector bundle $TX|U \oplus \mathcal{O}_U^{\oplus m}$ is not holomorphically trivial.*

From Theorems 2 and 1 we obviously obtain the following result.

Corollary 1. *Let X be a holomorphically convex complex manifold such that* $\mathrm{Exc}(X) \neq X$*. Then there exists a holomorphic vector bundle E on X such that* $\mathrm{rank}(E) \leq \dim(X)$ *and for every open neighborhood U of* $\mathrm{Exc}(X)$ *and every integer $m \geq 0$ the holomorphic vector bundle $E|U \oplus \mathcal{O}_U^{\oplus m}$ is not holomorphically trivial.*

For the results corresponding to Corollary 1 when X is 0-convex, i.e. when $f(\mathrm{Exc}(X))$ is a finite set, see [2], Theorem 1.2.

Now we will drop the assumption $\mathrm{Exc}(X) \neq X$ and consider the problem of the existence of non-trivial holomorphic vector bundles on certain open subsets of X. We will obtain a very easy extension of [1] to this set-up. Let $U \subseteq X$ be an open subset of X. We will say that U is f-saturated if $U = f^{-1}(f(U))$. Since f is proper and surjective, the set $f(U)$ is open in Z for every f-saturated open subset of X. Let X be a complex space and $U \subset X$ be an open subset of X, E a holomorphic vector bundle on U and $P \in \partial(U)$. We will say that E extends across P if there are an open neighborhood W of P in X and a holomorphic vector bundle F on W such that $F|U \cap W \cong E|U \cap W$. Since a holomorphic vector bundle is locally trivial, by restricting, if necessary, W we may assume that F is trivial. Hence E extends across P if and only if there is an open neighborhood A of P in X such that $E|U \cap A$ is trivial.

Proposition 1. *Let X be an irreducible holomorphically convex complex space such that* $\mathrm{Exc}(X) \neq X$ *and its Remmert reduction $f : X \to Z$ has the property that Z is biholomorphic open subset of $\mathbb{C}^n$, $n \geq 3$. Let U be an open subset of X containing* $\mathrm{Exc}(X)$ *and $P \in \partial(U)$ such that the domain $f(U)$ is 2-concave at $f(P)$. Then there is a holomorphic vector bundle E on U such that* $\mathrm{rank}(E) = n - 1$*, E does not extend across P, but it extends across every other point of $\partial(U)$.*

Proof. By [1], Theorem 1.1, there is a holomorphic vector bundle F on $f(U)$ such that F does not extend across $f(P)$, but it extends across all the other boundary points of $f(U)$ (seen as an open subset of $\mathbb{C}$). In particular F extends along all the boundary points of $f(U)$ inside Z, except $f(P)$. The proof of [1], Theorem 1.1, shows that there is such a vector bundle F with the additional property $\mathrm{rank}(F) = n - 1$. Set $E := f^*(E)$. E is a rank $n - 1$ holomorphic vector bundle on U. Since $\mathrm{Exc}(X) \subset X$ and $P \notin U$, f is biholomorphic in a neighborhood of P and hence $f^*(E)$ does not extend across P. Similarly (but this part is true even for more simpler reasons), f extends across all other points of $\partial(U)$. □

Remark 1. Take the set-up of Proposition 1 and its proof, but drop the assumption " $\mathrm{Exc}(X) \subset U$ ". Assume only that U is f-saturated. The properness of f, the fact that $f_*(\mathcal{O}_X) = \mathcal{O}_Z$ and the proof of Proposition 1 show that there is no f-saturated open subset W of P in X such that $f^*(F)|W \cap U$ is trivial.

By the proofs of Proposition 1, Remark 1 and [1], Theorem 1.2, we immediately get the following result.

Proposition 2. *Let X be a holomorphically convex complex space, $f : X \to Z$ its Remmert reduction, U an open f-saturated subset of X and $P \in \partial(U)$ such that $f(U)$ is 2-concave at $f(P)$ and Z has dimension at least three at P. Then there exists a holomorphic vector bundle E on U such that E extends across every point of $\partial(U)\backslash\{P\}$, but there is no f-saturated open neighborhood W of $f^{-1}(f(P))$ in X such that $E|U \cap W$ is trivial. If $\mathrm{Exc}(X) \subset U$, then E does not extend across P.*

2. The proofs

Proof of Theorem 1. By assumption, there is an irreducible component D of $\mathrm{Exc}(X)$ which is a closed hypersurface of X. Since X is smooth, the sheaf $\mathcal{I}_{D,X}$ is a holomorphic line bundle, L, on X. Fix a neighborhood U of $\mathrm{Exc}(X)$ and an integer $m \geq 0$ and assume $(L|U) \oplus \mathcal{O}_U^{\oplus m}$ trivial. Since $L|U$ is the determinant of $(L|U) \oplus \mathcal{O}_U^{\oplus m}$, $L|U$ must be trivial. Hence its dual $L^*|U$ is trivial. Hence there is a holomorphic function g on U whose zero-locus is scheme-theoretically exactly D. Since $\mathrm{Exc}(U) \subset U$ and $f_*(\mathcal{O}_X) = \mathcal{O}_Z$, $f(U)$ is an open subset of Z and there is a holomorphic function h on $f(U)$ such that $g = h \circ f$. Since X is normal, Z is normal and in particular every local ring $\mathcal{O}_{Z,Q}$, $Q \in Z$, is an integral domain. We have $f(D) = \{h = 0\}$. Notice that $\{h = 0\}$ is an effective Cartier divisor of Z and hence it has pure codimension one in Z at each of its points. Since $D \subseteq \mathrm{Exc}(X)$, we have $\dim(f(D)) < \dim(D) = \dim(X) - 1 = \dim(Z) - 1$, a contradiction. □

Proof of Theorem 2. Fix an integer $m \geq 0$ and assume the existence of an open neighborhood U of $\mathrm{Exc}(X)$ such that $TU \oplus \mathcal{O}_U^{\oplus m}$ is trivial. Since $U\backslash \mathrm{Exc}(X) \cong f(U)\backslash f(\mathrm{Exc}(X))$, the restriction to $f(U)\backslash f(\mathrm{Exc}(X))$ of $\Theta_{f(U)} \oplus \mathcal{O}_{f(U)}^{\oplus m}$ is trivial, where $\Theta_{f(U)}$ denotes the tangent sheaf of $f(U)$. By its very definition $\Theta_{f(U)}$ is the dual of the cotangent sheaf of $f(U)$ and in particular it is isomorphic to a dual of a coherent analytic sheaf with rank $\dim(Z)$ at each smooth point of $f(U)$. In particular $\Theta_{f(U)}$ is the so-called reflexive sheaf. Since X is smooth, it is normal. Hence the universal properties of the normalization and of the Remmert reduction imply that Z is normal. On a normal complex space any reflexive sheaf is uniquely determined by its restriction to an open subset whose complementary is a closed analytic subset with codimension at least two. Since the trivial vector bundle $\mathcal{O}_{f(U)}^{\oplus \dim(X)+m}$ is an extension of $\Theta_{f(U)} \oplus \mathcal{O}_{f(U)}^{\oplus m}|f(U)\backslash f(\mathrm{Exc}(X))$, $\Theta_{f(U)} \oplus {\mathcal{O}_{f(U)}}^{\oplus m}$ is trivial. Hence $\Theta_{f(U)}$ is locally free.

Since $\dim(f(\mathrm{Exc}(X))) < \dim(\mathrm{Exc}(X)) \le \dim(X) - 2$, $f(U)$ is smooth ([4], Corollary at p. 318). The holomorphic map $f : U \to f(U)$ is a holomorphic map between smooth manifolds which is an isomorphism outside a closed analytic subset of U with codimension at least two. This implies that f is an isomorphism (use the determinant of the differential $df : TX \to f^*(TZ)$). By the universal property of the Remmert reduction, this implies X is Stein, a contradiction. □

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