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CONTROLLABILITY OF SOBOLEV TYPE SEMILINEAR FUNCTIONAL DIFFERENTIAL AND INTEGRODIFFERENTIAL INCLUSIONS WITH AN UNBOUNDED DELAY

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Abstract. In this paper, sufficient conditions are established for the controllability of Sobolev type semilinear functional differential and integrodifferential inclusions with an unbounded delay in Banach spaces. The main results are obtained by using the fixed point theorem for condensing maps due to Martelli.

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1. Introduction

This paper is concerned mainly with the controllability of Sobolev type functional differential inclusions with an unbounded delay

$$(Ey(t))' - Ay(t) \in F(t, y_t) + (Bu)(t), \quad t \in J, \tag{1.1}$$

$$y_0 = \phi \in \mathcal{B}, \tag{1.2}$$

and of functional integrodifferential inclusions with an unbounded delay

$$(Ey(t))' - Ay(t) \in \int_0^t \vartheta(t,s) F(s, y_s)\, ds + (Bu)(t), \quad t \in J, \tag{1.3}$$

$$y_0 = \phi \in \mathcal{B}, \tag{1.4}$$

where $J = [0, b]$, the state $y(\cdot)$ takes values in a Banach space X with the norm $|\cdot|$, A and E are linear operators with domains $D(A)$ and $D(E)$ contained in the Banach space X and ranges contained in a Banach space Y, the control function $u(\cdot)$ is given in $L^2(J, U)$, a Banach space of admissible control functions with U as a Banach space, B is a bounded linear operator from U into Y, $F : J \times \mathcal{B} \to \mathcal{P}(Y)$ is a multivalued map, $\mathcal{P}(Y)$ is the family of all subsets of Y, and

$$\vartheta : D \to \mathbb{R}, D = \{(t, s) \in J \times J : t \geq s\}.$$

The histories

$$y_t : (-\infty, 0] \to X, y_t(\theta) = y(t + \theta), \quad \theta \leq 0,$$

belong to some abstract phase space $\mathcal{B}$, i.e., to a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$ in $\mathcal{B}$.

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The problem of the controllability for functional differential and integrodifferential inclusions in Banach spaces has been studied extensively, see for instance, [1, 2, 3] and the references therein. However, all these results are concerned only with bounded delays. Since many systems arising from realistic models can be described as functional differential inclusions with an unbounded delay, it is natural to discuss this kind of problems. Moreover, to the best of our knowledge, there are few papers in the literature dealing with the controllability of Sobolev type semilinear functional differential and integrodifferential inclusions with an unbounded delay. In the present paper, we investigate the controllability of systems (1.1)–(1.2) and (1.3)–(1.4) by using the fixed point theorem for condensing maps due to Martelli [10].

In the next section, we recall briefly some basic definitions and preliminary facts which will be used throughout this paper.

2. Preliminaries

Let $C(J, X)$ be the Banach space of continuous functions from J into X with the norm

$$||y||_\infty := \sup\{|y(t)| : t \in J\}$$

and $B(X)$ denote the Banach space of bounded linear operators from X into itself.

A measurable function $y : J \to X$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For the properties of a Bochner integral see Yosida [12].)

Let $L^1(J, X)$ be the Banach space of measurable functions $y : J \to X$ which are Bochner integrable and normed by

$$||y||_{L^1} = \int_0^b |y(t)|dt \quad \text{for all} \quad y \in L^1(J, X).$$

Let $(X, |\cdot|)$ be a Banach space. Then a multivalued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on the bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for any bounded set B of X (i.e., $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$).

G is called upper semicontinuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set B of X containing $G(x_0)$, there exists an open neighborhood V of x_0 such that $G(V) \subseteq B$.

G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset B of X.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e.,

$$x_n \to x_*, \;\; y_n \to y_*, \;\; y_n \in Gx_n \quad \text{imply} \quad y_* \in Gx_*.$$

In what follows $BCC(X)$ denotes the set of all nonempty bounded, closed and convex subsets of X.

A multivalued map $G : J \to BCC\,(X)$ is said to be measurable if, for each $x \in X$, the function $h : J \to \mathbb{R}$ defined by

$$h(t) = d(x, G(t)) = \inf\{|x - z| : z \in G(t)\}$$

belongs to $L^1(J, \mathbb{R})$.

An upper semicontinuous map $G : X \to \mathcal{P}\,(X)$ is said to be condensing if for any subset $B \subseteq X$ with $\alpha\,(B) \neq 0$, we have $\alpha\,(G\,(B)) < \alpha\,(B)$, where α denotes the Kuratowski measure of noncompactness [4].

G has a fixed point if there is $x \in X$ such that $x \in G\,(x)$. For more details on multivalued maps see the books of Deimling [5] and Hu and Papageorgious [8].

In this article, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato [6] and follow the terminology used in [7]. Assume that $\mathcal{B}$ satisfies the following axioms:

(A1) If $y : (-\infty, \sigma + a) \to X$, $a > 0$, is continuous on $[\sigma, \sigma + a)$ and $y_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + a)$, the following conditions hold:

(i) y_t is in $\mathcal{B}$;

(ii) $|y\,(t)| \leq H\,\|y_t\|_{\mathcal{B}}$, where $H \geq 0$ is a constant and independent of $y\,(\cdot)$.

(iii) $\|y_t\|_{\mathcal{B}} \leq K\,(t - \sigma) \sup\{|y\,(s)| : \sigma \leq s \leq t\} + M\,(t - \sigma)\,\|y_\sigma\|_{\mathcal{B}}$, where $K, M : [0, +\infty) \to [0, +\infty)$, K is continuous and M is locally bounded, and K, M are independent of $y\,(\cdot)$.

(A2) For the function $y\,(\cdot)$ in (A1), y_t is a $\mathcal{B}$-valued continuous function on $[\sigma, \sigma + a]$.

(A3) The space $\mathcal{B}$ is complete.

The operators $A : D\,(A) \subset X \to Y$ and $E : D\,(E) \subset X \to Y$ satisfy the following conditions:

(C1) A and E are closed linear operators,

(C2) $D\,(E) \subset D\,(A)$ and E is bijective,

(C3) $E^{-1} : Y \to D\,(E)$ is continuous.

The above facts and the closed graph theorem imply the boundedness of the linear operator $AE^{-1} : Y \to Y$. Furthermore, $-AE^{-1}$ generates a uniformly continuous semigroup $T\,(t)$, $t \geq 0$.

To set the framework for our main controllability results, we need to introduce the following definitions.

Definition 2.1. A function $y : (-\infty, b] \to X$ is called a mild solution of system (1.1)–(1.2) if $y_0 = \phi \in \mathcal{B}$ on $(-\infty, 0]$, the restriction of $y\,(\cdot)$ to the interval $[0, b]$ is continuous and the following integral inclusion

$$y\,(t) \in E^{-1}T\,(t)\,E\phi\,(0) + \int_0^t E^{-1}T\,(t - s)\,F\,(s, y_s)\,ds + \int_0^t E^{-1}T\,(t - s)\,(Bu)\,(s)\,ds$$

is satisfied.

Definition 2.2. A function $y : (-\infty, b] \to X$ is called a mild solution of system (1.3)–(1.4) if $y_0 = \phi \in \mathcal{B}$ on $(-\infty, 0]$, the restriction of $y\,(\cdot)$ to the interval $[0,b]$ is continuous and the following integral inclusion

$$y(t) \in E^{-1}T(t)E\phi(0) + \int_0^t E^{-1}T(t-s)\int_0^s \vartheta(s,\tau)F(\tau,y_\tau)\,d\tau ds + \int_0^t E^{-1}T(t-s)(Bu)(s)\,ds$$

is satisfied.

Definition 2.3. System (1.1)–(1.2) is said to be controllable on the interval J if, for every continuous initial state $\phi \in \mathcal{B}, y_1 \in X$, there exists a control $u \in L^2(J,U)$ such that the mild solution $y(t)$ of (1.1)–(1.2) satisfies $y(b) = y_1$.

Definition 2.4. System (1.3)–(1.4) is said to be controllable on the interval J if, for every continuous initial state $\phi \in \mathcal{B}, y_1 \in X$, there exists a control $u \in L^2(J,U)$ such that the mild solution $y(t)$ of (1.3)–(1.4) satisfies $y(b) = y_1$.

The main tool in our approach is the following fixed point theorem due to Martelli.

Lemma 2.1 ([10])**.** *Let X be a Banach space and let $G : X \to BCC(X)$ be a condensing map. If the set*

$$\Omega = \{x \in X : \lambda x \in Gx \ \ \textit{for some } \lambda > 1\}$$

is bounded, then G has a fixed point.

We remark that a completely continuous multivalued map is the easiest example of a condensing map. We also need the following lemma.

Lemma 2.2 ([11])**.** *Let $S(t)$ be a uniformly continuous semigroup and let A be its infinitesimal generator. If the resolvent operator $R(\lambda,A)$ of A is compact for every $\lambda \in \rho(A)$, then $S(t)$ is a compact semigroup.*

3. Main Results

To establish the controllability result for system (1.1)–(1.2), we need the following hypotheses:

(H1) The resolvent operator $R(\lambda,-AE^{-1})$ is compact for some $\lambda \in \rho(-AE^{-1})$, the resolvent set of $-AE^{-1}$, such that $T(t)$ is a compact semigroup. Let $M_1 = \max_{t\in J}\|T(t)\|$.

(H2) $F : J \times \mathcal{B} \to BCC(Y); (t,\phi) \to F(t,\phi)$ is measurable with respect to t for each $\phi \in \mathcal{B}$, u.s.c. with respect to ϕ for each $t \in J$, and for each fixed $\phi \in \mathcal{B}$ the set

$$S_{F,\phi} = \left\{f \in L^1(J,Y) : f(t) \in F(t,\phi) \ \ \text{for a.e. } t \in J\right\}$$

is nonempty.

(H3) The linear operator $W : L^2(J,U) \to X$ defined by

$$Wu = \int_0^b E^{-1}T(b-s)(Bu)(s)\,ds$$

has the induced inverse operator W^{-1} which takes values in $L^2(J,U)/\ker W$ (the restriction of W to the domain $L^2(J,U)/\ker W$ is invertible), and there exist positive constants M_2, M_3, M_4 such that $M_2 = \|E^{-1}\|, \|B\| \le M_3, \|W^{-1}\| \le M_4$.

(H4) $\|F(t,\phi)\| := \sup\{|v| : v(t) \in F(t,\phi)\} \le p(t)\psi(\|\phi\|_{\mathcal{B}})$ for a.e. $t \in J$ and all $\phi \in \mathcal{B}$, where $p \in L^1(J,\mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \to (0,\infty)$ is continuous and nondecreasing with

$$\max_{t\in J} K(t) M_1 M_2 \int_0^b p(s)\,ds \le \int_{\overline{M}}^{\infty} \frac{1}{\psi(s)}ds,$$

where

$$\overline{M} = \sup_{t\in J} M(t)\|\phi\|_{\mathcal{B}} + \max_{t\in J} K(t)\Lambda,$$

$$\Lambda = M_1|\phi(0)| + bM_1M_2M_3M_4\left[(|y_1| + M_1|\phi(0)|) + M_1M_2\int_0^b p(s)\psi(\|y_s\|_{\mathcal{B}})\,ds\right].$$

Remark 3.1. $S_{F,\phi}$ is nonempty if and only if the function $h : J \to \mathbb{R}$ defined by $h(t) = \inf\{|v| : v \in F(t,\phi)\}$ belongs to $L^1(J,\mathbb{R})$, see [8].

Remark 3.2. Condition (H1) is satisfied by virtue of Lemma 2.2.

First, from (H3) for an arbitrary function $y(\cdot)$, define the control

$$u(t) = W^{-1}\left\{y_1 - E^{-1}T(b)E\phi(0) - \int_0^b E^{-1}T(b-s)f(s)\,ds\right\}(t), \qquad (3.1)$$

where

$$f \in S_{F,y} = \left\{f \in L^1(J,Y) : f(t) \in F(t,y_t),\ \text{for a.e. } t \in J\right\}.$$

Let $\mathcal{B}_0$ be the space of all functions $y : (-\infty,b] \to X$ such that $y_0 \in \mathcal{B}$ and the restriction $y : [0,b] \to X$ is continuous. Let $\|\cdot\|_0$ be a seminorm in $\mathcal{B}_0$ defined by

$$\|y\|_0 = \|y_0\|_{\mathcal{B}} + \sup\{|y(t)| : t \in J\}, \quad y \in \mathcal{B}_0.$$

Using (3.1), we define the multivalued map $N : \mathcal{B}_0 \to 2^{\mathcal{B}_0}$ by $Ny = \{h \in \mathcal{B}_0\}$, where

$$h(t) = \begin{cases} \phi(t), & t \in (-\infty,0], \\ E^{-1}T(t)E\phi(0) + \int_0^t E^{-1}T(t-s)f(s)\,ds & \\ \quad + \int_0^t E^{-1}T(t-s)(Bu)(s)\,ds, & f \in S_{F,y},\ t \in J. \end{cases}$$

We will show that N has a fixed point, which in turn is a mild solution of system (1.1)–(1.2). Obviously, $y_1 \in (Ny)(b)$.

For $\phi \in \mathcal{B}$, we define $\overline{\phi}$ by

$$\overline{\phi}(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ E^{-1}T(t)E\phi(0), & t \in J, \end{cases}$$

then $\overline{\phi} \in \mathcal{B}_0$. Set

$$y(t) = z(t) + \overline{\phi}(t), \quad t \in (-\infty, b].$$

It is clear that y is a mild solution of system (1.1)–(1.2) if and only if z satisfies $z_0 = 0$ and

$$z(t) = \int_0^t E^{-1}T(t-s) f(s)\, ds + \int_0^t E^{-1}T(t-\eta) BW^{-1}\Big\{ y_1 - E^{-1}T(b)E\phi(0) - \int_0^b E^{-1}T(b-s) f(s)\, ds \Big\}(\eta)\, d\eta, \quad t \in J.$$

Let $\widetilde{\mathcal{B}}_0 = \{z \in \mathcal{B}_0 : z_0 = 0 \in \mathcal{B}\}$. For any $z \in \widetilde{\mathcal{B}}_0$, we have

$$\|z\|_0 = \|z_0\|_{\mathcal{B}} + \sup\{|z(t)| : t \in J\} = \sup\{|z(t)| : t \in J\};$$

thus $\left(\widetilde{\mathcal{B}}_0, \|\cdot\|_0\right)$ is a Banach space. Set

$$B_r = \left\{ z \in \widetilde{\mathcal{B}}_0 : \|z\|_0 \le r \right\} \quad \text{for some} \quad r \ge 0,$$

then $B_r \subseteq \widetilde{\mathcal{B}}_0$ is uniformly bounded and, for some $r \ge 0$, we have

$$\begin{aligned} \left\| z_t + \overline{\phi}_t \right\|_{\mathcal{B}} &\le \|z_t\|_{\mathcal{B}} + \left\| \overline{\phi}_t \right\|_{\mathcal{B}} \\ &\le K(t) \sup\{|z(s)| : 0 \le s \le t\} + M(t)\|z_0\|_{\mathcal{B}} \\ &\quad + K(t) \sup\left\{ \left|\overline{\phi}(s)\right| : 0 \le s \le t \right\} + M(t) \left\|\overline{\phi}_0\right\|_{\mathcal{B}} \\ &\le \max_{t\in J} K(t)(r + M_1\|\phi(0)\|) + \sup_{t \in J} M(t)\|\phi\|_{\mathcal{B}} = r'. \end{aligned} \tag{3.2}$$

Define the multivalued map $\overline{N} : \widetilde{\mathcal{B}}_0 \to 2^{\widetilde{\mathcal{B}}_0}$ by $\overline{N}z = \left\{ \rho \in \widetilde{\mathcal{B}}_0 \right\}$, where

$$\rho(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ \int_0^t E^{-1}T(t-s) f(s)\, ds + \int_0^t E^{-1}T(t-\eta) BW^{-1}\Big\{ y_1 - E^{-1}T(b)E\phi(0) & \\ \quad - \int_0^b E^{-1}T(b-s) f(s)\, ds \Big\}(\eta)\, d\eta, \quad f \in S_{F,z}, \quad t \in J. & \end{cases}$$

The following lemmas are of great importance in the proof of our main theorems.

Lemma 3.1 ([9])**.** *Let I be a compact real interval and X be a Banach space. Let F be a multivalued map satisfying* (H2) *and let Γ be a linear continuous operator from $L^1(I,X)$ to $C(I,X)$, then the operator*

$$\Gamma \circ S_F : C(I,X) \to BCC(C(I,X)), \quad y \mapsto (\Gamma \circ S_F)(y) = \Gamma(S_{F,y}),$$

is a closed graph operator in $C(I,X) \times C(I,X)$.

Lemma 3.2. *If conditions* (H1)–(H4) *are satisfied, then the above defined map $\overline{N} : \widetilde{\mathcal{B}}_0 \to 2^{\widetilde{\mathcal{B}}_0}$ is a completely continuous multivalued map, u.s.c. with closed convex value.*

Proof. The proof will be given in several steps.

Step 1. $\overline{N}z$ is convex for each $z \in \widetilde{\mathcal{B}}_0$.

Indeed, if $\rho_1, \rho_2 \in \overline{N}z$, then there exist $f_1, f_2 \in S_{F,z}$ such that, for each $t \in J$, we have

$$\rho_i(t) = \int_0^t E^{-1}T(t-s) f_i(s)\,ds + \int_0^t E^{-1}T(t-\eta) BW^{-1}\Bigg\{ y_1 - E^{-1}T(b) E\phi(0) - \int_0^b E^{-1}T(b-s) f_i(s)\,ds \Bigg\}(\eta)\,d\eta, \quad i = 1, 2.$$

Let $\alpha \in (0,1)$. Since the operators B and E^{-1} are linear, we obtain

$$\begin{aligned}(\alpha\rho_1 + (1-\alpha)\rho_2)(t) &= \int_0^t E^{-1}T(t-s)(\alpha f_1(s) + (1-\alpha) f_2(s))\,ds \\ &+ \int_0^t E^{-1}T(t-\eta) BW^{-1}\Bigg\{ y_1 - E^{-1}T(b) E\phi(0) \\ &- \int_0^b E^{-1}T(b-s)(\alpha f_1(s) + (1-\alpha) f_2(s))\,ds \Bigg\}(\eta)\,d\eta.\end{aligned}$$

Since $S_{F,z}$ is convex (because F has convex values), we get

$$\alpha\rho_1 + (1-\alpha)\rho_2 \in \overline{N}z.$$

Step 2. $\overline{N}$ maps bounded sets into bounded sets in $\widetilde{\mathcal{B}}_0$.

In fact, we only to show that there exists a positive constant c such that, for each $\rho \in \overline{N}z$, $z \in B_r = \Big\{ z \in \widetilde{\mathcal{B}}_0 : \|z\|_0 \le r \Big\}$, one has $\|\rho\|_0 \le c$. If $\rho \in \overline{N}z$, then there exists $f \in S_{F,z}$ such that, for each $t \in J$, we have

$$\rho(t) = \int_0^t E^{-1}T(t-s) f(s)\,ds + \int_0^t E^{-1}T(t-\eta) BW^{-1}\Bigg\{ y_1 - E^{-1}T(b) E\phi(0)$$

$$- \int_0^b E^{-1}T(b-s) f(s)\,ds \Bigg\} (\eta)\,d\eta. \tag{3.3}$$

From (H1)–(H4) and (3.2), we have, for each $t \in J$,

$$\begin{aligned} |\rho(t)| &\le M_1M_2 \int_0^b p(s)\,\psi\left(\left\|z_t + \overline{\phi}_t\right\|_{\mathcal{B}}\right) ds + bM_1M_2M_3M_4\left(|y_1| + M_1\left|\phi(0)\right|\right) \\ &\quad + b\left(M_1M_2\right)^2 M_3M_4 \int_0^b p(s)\,\psi\left(\left\|z_t + \overline{\phi}_t\right\|_{\mathcal{B}}\right) ds \\ &\le M_1M_2 \sup_{z\in[0,r']} \psi(z) \int_0^b p(s)\,ds + bM_1M_2M_3M_4\left(|y_1| + M_1\left|\phi(0)\right|\right) \\ &\quad + b\left(M_1M_2\right)^2 M_3M_4 \sup_{z\in[0,r']} \psi(z) \int_0^b p(s)\,ds \triangleq c. \end{aligned}$$

Thus, for each $\rho \in \overline{N}(B_r)$, we have

$$\|\rho\|_0 \le c.$$

Step 3. $\overline{N}$ maps bounded sets into equicontinuous sets of $\widetilde{\mathcal{B}}_0$.

Let $t_1, t_2 \in J, t_1 < t_2$ and $B_r = \left\{ z \in \widetilde{\mathcal{B}}_0 : \|z\|_0 \le r \right\}$ be a bounded set of $\widetilde{\mathcal{B}}_0$. For each $z \in B_r$ and $\rho \in \overline{N}z$, there exists $f \in S_{F,z}$ such that (3.3) holds. So,

$$\begin{aligned} &|\rho(t_1) - \rho(t_2)| \\ &\le \left| \int_0^{t_1} E^{-1}\left(T(t_1-s) - T(t_2-s)\right) f(s)\,ds \right| + \left| \int_{t_1}^{t_2} E^{-1}T(t_2-s) f(s)\,ds \right| \\ &\quad + \left| \int_0^{t_1} E^{-1}\left(T(t_1-\eta) - T(t_2-\eta)\right) BW^{-1}\Bigg\{ y_1 - E^{-1}T(b)\,E\phi(0) \right. \\ &\quad \left. - \int_0^b E^{-1}T(b-s) f(s)\,ds \Bigg\}(\eta)\,d\eta \right| + \left| \int_{t_1}^{t_2} E^{-1}T(t_2-s)\,BW^{-1}\Bigg\{ y_1 \right. \\ &\quad \left. - E^{-1}T(b)\,E\phi(0) - \int_0^b E^{-1}T(b-s) f(s)\,ds \Bigg\}(\eta)\,d\eta \right| \\ &\le M_2 \int_0^{t_1} \left|\left(T(t_1-s) - T(t_2-s)\right) f(s)\right| ds + M_1M_2 \int_{t_1}^{t_2} p(s)\,\psi(r)\,ds \end{aligned}$$

$$+ M_2M_3M_4\widetilde{M}\int_0^{t_2} \|T(t_1-\eta) - T(t_2-\eta)\| \, d\eta + M_1M_2M_3M_4\widetilde{M}(t_2-t_1),$$

where

$$\widetilde{M} = \left(|y_1| + M_1 |\phi(0)| + M_1M_2 \int_0^b p(s)\psi(\|z\|_0) \, ds \right).$$

As $t_2 \to t_1$, the right-hand side of the above inequality is independent of $z \in B_r$ and tends to zero. The equicontinuities for the cases $t_1 < t_2 \le 0$ and $t_1 \le 0 < t_2$ are obvious. Thus the set $\left\{\overline{N}z : z \in B_r\right\}$ is equicontinuous.

As a consequence of Step 2, Step 3 and (H1) together with the Ascoli-Arzela theorem, we can conclude that $\overline{N} : \widetilde{\mathcal{B}}_0 \to 2^{\widetilde{\mathcal{B}}_0}$ is completely continuous and therefore is a condensing map.

Step 4. $\overline{N}$ has a closed graph.

Let

$$z_n \to z_*, \rho_n \in \overline{N}z_n, \quad \rho_n \to \rho_*.$$

We will prove that $\rho_* \in \overline{N}z_*$. Indeed, $\rho_n \in \overline{N}z_n$ means that there exists $f_n \in S_{F,z_n}$ such that

$$\rho_n(t) = \int_0^t E^{-1}T(t-s) f_n(s) \, ds + \int_0^t E^{-1}T(t-s)(Bu_{z_n})(s) \, ds, \quad t \in J,$$

where

$$u_{z_n}(t) = W^{-1}\left\{ y_1 - E^{-1}T(b)E\phi(0) - \int_0^b E^{-1}T(b-s) f_n(s) \, ds \right\}(t).$$

We must show that there exists $f_* \in S_{F,z_*}$ such that

$$\rho_*(t) = \int_0^t E^{-1}T(t-s) f_*(s) \, ds + \int_0^t E^{-1}T(t-s)(Bu_{z_*})(s) \, ds, \quad t \in J,$$

where

$$u_{z_*}(t) = W^{-1}\left\{ y_1 - E^{-1}T(b)E\phi(0) - \int_0^b E^{-1}T(b-s) f_*(s) \, ds \right\}(t).$$

Denote

$$\widehat{u}_z(t) = W^{-1}\left\{ y_1 - E^{-1}T(b)E\phi(0) \right\}(t).$$

Since f and W^{-1} are continuous, we have

$$\widehat{u}_{z_n}(t) \to \widehat{u}_{z_*}(t) \quad \text{for} \quad t \in J.$$

Clearly,

$$\left\| \left(\rho_n(t) - \int_0^t E^{-1}T(t-s)\left(B\widehat{u}_{z_n}\right)(s)\,ds \right) - \left(\rho_*(t) - \int_0^t E^{-1}T(t-s)\left(B\widehat{u}_{z_*}\right)(s)\,ds \right) \right\|_0 \to 0 \quad \text{as} \quad n \to \infty.$$

Consider the linear continuous operator $\Gamma : L^1(J,X) \to C(J,X)$ defined by

$$\Gamma(f)(t) = \int_0^t E^{-1}T(t-s)\left[f(s) - BW^{-1}\left(\int_0^b E^{-1}T(b-\tau) f(\tau)\,d\tau \right)(s) \right] ds.$$

Clearly, Γ is continuous. Indeed, one has

$$\|\Gamma f\|_\infty \leq \left(M_1M_2 + b\left(M_1M_2\right)^2 M_3M_4 \right) \|f\|_{L^1}.$$

In view of Lemma 3.1, we deduce that $\Gamma \circ S_F$ is a closed graph operator. Moreover, we have

$$\rho_n(t) - \int_0^t E^{-1}T(t-s)\left(B\widehat{u}_{z_n}\right)(s)\,ds \in \Gamma\left(S_{F,z_n}\right).$$

Since $z_n \to z_*$, it follows from Lemma 3.1 that

$$\begin{aligned} \rho_*(t) &- \int_0^t E^{-1}T(t-s)\left(B\widehat{u}_{z_*}\right)(s)\,ds \\ &= \int_0^t E^{-1}T(t-s)\left[f_*(s) - BW^{-1}\left(\int_0^b E^{-1}T(b-\tau) f_*(\tau)\,d\tau \right)(s) \right] ds, \end{aligned}$$

for some $f_* \in S_{F,z_*}$.

Therefore, $\overline{N}$ is a completely continuous multivalued map, u.s.c. with convex closed values. The proof is complete. □

Next, in order to apply Lemma 2.1, we will consider the following auxiliary system with a parameter $\lambda > 1$:

$$(Ey(t))' - Ay(t) \in \left(\frac{1}{\lambda}\right) F(t, y_t) + \left(\frac{1}{\lambda}\right)(Bu)(t), \quad t \in J, \tag{3.4}$$

$$y_0 = \phi \in \mathcal{B}. \tag{3.5}$$

Thus, by Definition 2.1, a mild solution of system (3.4)–(3.5) can be written as

$$y(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \left(\frac{1}{\lambda}\right) E^{-1}T(t)E\phi(0) + \left(\frac{1}{\lambda}\right) \int_0^t E^{-1}T(t-s) f(s)\, ds \\ \quad + \left(\frac{1}{\lambda}\right) \int_0^t E^{-1}T(t-\eta) BW^{-1}\Big\{ y_1 - E^{-1}T(b)E\phi(0) \\ \quad - \int_0^b E^{-1}T(b-s) f(s)\, ds \Big\}(\eta)\, d\eta, & t \in J, \end{cases} \tag{3.6}$$

where

$$f \in S_{F,y} = \left\{ f \in L^1(J, Y) : f(t) \in F(t, y_t) \text{ for a.e. } t \in J \right\}.$$

We have the following lemma for system (3.4)–(3.5).

Lemma 3.3. *Assume that* (H1)–(H4) *hold. Let* $y(t)$ *be a mild solution of system* (3.4)–(3.5). *Then there exists an a priori bound* $C > 0$ *such that* $\|y_t\|_{\mathcal{B}} \le C$, $t \in J$, *where* C *depends only on the constant* b *and the functions* $p(\cdot)$ *and* $\psi(\cdot)$.

Proof. Since $y(t)$ is a mild solution of system (3.4)–(3.5), from (3.6) we have

$$\begin{aligned} |y(t)| &\le M_1 |\phi(0)| + M_1 M_2 \int_0^t p(s)\psi(\|y_s\|_{\mathcal{B}})\, ds + bM_1M_2M_3M_4(|y_1| \\ &\quad + M_1|\phi(0)|) + b(M_1M_2)^2 M_3M_4 \int_0^b p(s)\psi(\|y_s\|_{\mathcal{B}})\, ds \\ &\triangleq \Lambda + M_1M_2 \int_0^t p(s)\psi(\|y_s\|_{\mathcal{B}})\, ds, \quad t \in J, \end{aligned}$$

from which and axiom (A1) it follows that

$$\begin{aligned} \|y_t\|_{\mathcal{B}} &\le K(t) \sup\{|y(s)| : 0 \le s \le t\} + M(t)\|\phi\|_{\mathcal{B}} \\ &\le \max_{t\in J} K(t) \sup\{|y(s)| : 0 \le s \le t\} + \sup_{t\in J} M(t)\|\phi\|_{\mathcal{B}} \\ &\le \max_{t\in J} K(t) M_1M_2 \int_0^t p(s)\psi(\|y_s\|_{\mathcal{B}})\, ds + \sup_{t\in J} M(t)\|\phi\|_{\mathcal{B}} + \max_{t\in J} K(t)\Lambda \\ &\triangleq \max_{t\in J} K(t) M_1M_2 \int_0^t p(s)\psi(\|y_s\|_{\mathcal{B}})\, ds + \overline{M}, \quad t \in J. \end{aligned}$$

Let

$$\mu(t) = \sup\{\|y_s\|_{\mathcal{B}} : 0 \le s \le t\}.$$

Then the function $\mu(t)$ is continuous and nondecreasing in J by axiom (A2). So we have

$$\mu(t) \leq \max_{t\in J} K(t) M_1 M_2 \int_0^t p(s)\psi(\mu(s))\,ds + \overline{M}, \quad t \in J.$$

Denoting by $v(t)$ the right-hand side of the above inequality, we obtain

$$v(0) = \overline{M}, \quad \mu(t) \leq v(t), \quad t \in J,$$

and

$$v'(t) = \max_{t\in J} K(t) M_1 M_2 p(t)\psi(\mu(t)) \leq \max_{t\in J} K(t) M_1 M_2 p(t)\psi(v(t)),$$

which implies that

$$\int_{v(0)}^{v(t)} \frac{1}{\psi(s)}ds \leq \max_{t\in J} K(t) M_1 M_2 \int_0^b p(s)\,ds \leq \int_{v(0)}^{\infty} \frac{1}{\psi(s)}ds.$$

This inequality implies that $v(t) < \infty$. So there exists a constant C such that $v(t) \leq C$, $t \in J$. Thus,

$$\|y_t\|_{\mathcal{B}} \leq \mu(t) \leq v(t) \leq C, \quad t \in J,$$

where K depends only on b and the functions $p(\cdot)$ and $\psi(\cdot)$. □

Theorem 3.1. *Assume that* (H1)–(H4) *are satisfied. Then system* (1.1)–(1.2) *is controllable on* J.

Proof. Let

$$\Omega = \left\{ z \in \widetilde{\mathcal{B}}_0 : \lambda z \in \overline{N} z \text{ for some } \lambda > 1 \right\}.$$

Then, for any $z \in \Omega$, we have

$$z(t) = \left(\frac{1}{\lambda}\right) \int_0^t E^{-1}T(t-s) f(s)\,ds + \left(\frac{1}{\lambda}\right) \int_0^t E^{-1}T(t-\eta) BW^{-1} \Bigg\{ y_1$$

$$- E^{-1}T(b) E\phi(0) - \int_0^b E^{-1}T(b-s) f(s)\,ds \Bigg\} (\eta)\,d\eta, \quad f \in S_{F,z}, \quad t \in J,$$

which implies the function

$$y = z + \overline{\phi}$$

is a mild solution of system (3.4)–(3.5), for which we have proved in Lemma 3.3 that

$$\|y_t\|_{\mathcal{B}} \leq C, \quad t \in J.$$

Hence, by axiom (A1),

$$\begin{aligned}\|z\|_0 &= \|z_0\|_{\mathcal{B}} + \sup\{|z(t)| : t \in J\} = \sup\{|z(t)| : t \in J\} \\ &\leq \sup\{|y(t)| : t \in J\} + \sup\{|\overline{\phi}(t)| : t \in J\}\end{aligned}$$

$$\leq \sup\{H\|y_t\|_{\mathcal{B}} : t \in J\} + \sup\{|T(t)\phi(0)| : t \in J\} \leq HC + M_1|\phi(0)|,$$

which implies that Ω is bounded on J. Then it follows from Lemma 3.2 and Lemma 2.1 that the operator $\overline{N}$ has a fixed point $z^* \in \widetilde{\mathcal{B}}_0$. Let

$$y(t) = z^*(t) + \overline{\phi}(t), \quad t \in (-\infty, b],$$

then y is a fixed point of the operator N which is a mild solution of system (1.1)–(1.2). Thus system (1.1)–(1.2) is controllable on J. This ends of the proof. □

Finally, we consider the controllability for system (1.3)–(1.4). We list the following hypotheses:

(H5) (i) For each $t \in J, \vartheta(t,s)$ is measurable on $[0,t]$ and

$$\vartheta(t) = \operatorname{ess\,sup}\{\vartheta(t,s), \ 0 \leq s \leq t\},$$

is bounded on J.

(ii) The map $t \mapsto \vartheta_t$ is continuous from J to $L^\infty(J,\mathbb{R})$; here $\vartheta_t(s) = \vartheta(t,s)$.

(H6) $\|F(t,\phi)\| := \sup\{|v| : v(t) \in F(t,\phi)\} \leq p(t)\psi(\|\phi\|_{\mathcal{B}})$ for a.e. $t \in J$ and all $\phi \in \mathcal{B}$, where $p \in L^1(J,\mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \to (0,\infty)$ is continuous and nondecreasing with

$$b \sup_{t\in J} \vartheta(t) \max_{t\in J} K(t) M_1 M_2 \int_0^b p(s)\,ds \leq \int_{\overline{M}'}^{\infty} \frac{1}{\psi(s)} ds,$$

where

$$\overline{M}' = \sup_{t\in J} M(t)\|\phi\|_{\mathcal{B}} + \max_{t\in J} K(t)\Lambda',$$

$$\Lambda' = M_1|\phi(0)| + bM_1M_2M_3M_4\Bigg[(|y_1| + M_1|\phi(0)|) + bM_1M_2 \sup_{t\in J}\vartheta(t)\int_0^b p(s)\psi(\|y_s\|_{\mathcal{B}})\,ds\Bigg].$$

Hence, similar to the previous arguments in this section, by using the same methods as in the proofs of Lemma 3.2, Lemma 3.3 and Theorem 3.1, we can establish the lemmas similar to Lemmas 3.2 and 3.3 and show that our next Theorem 3.2 holds.

Theorem 3.2. *Suppose that* (H1)–(H3) *and* (H5)–(H6) *hold. Then system* (1.3)–(1.4) *is controllable on* J.

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