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ON THE FOURIER EXPANSIONS OF EISENSTEIN SERIES OF SOME TYPES

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Abstract. Bases of the spaces of Eisenstein series $E_k(\Gamma_0(4N),\chi)$ ($k \in \mathbb{N}$, $k \geq 3$, N is an odd natural and square-free) and $E_{k/2}(\widetilde{\Gamma}_0(4N),\chi)$ ($k \in \mathbb{N}$, $2 \nmid k$, $k \geq 5$, N is an odd natural and square-free) are constructed for any Dirichlet character mod $4N$ and Fourier expansions of these series are obtained.

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We will mostly use the notions and notation from [1]. Let $E_k(\Gamma_0(4N),\chi)$ ($k \in \mathbb{N}$) denote the space of Eisenstein series of weight k and character χ with respect to $\Gamma_0(4N)$; $E_{k/2}(\widetilde{\Gamma}_0(4N),\chi)$ ($k \in \mathbb{N}$) denotes the space of Eisenstein series of weight $\frac{k}{2}$ and character χ with respect to $\widetilde{\Gamma}_0(4N)$. Van Asch in [2] constructed a basis for the space $E_{k/2}\left(\widetilde{\Gamma}_0(4p),\left(\frac{4p}{\cdot}\right)\right)$ (p is an odd prime, $k \in \mathbb{N}$, $2 \nmid k$; $k \geq 3$ if $p \geq 13$; $k \geq 5$ if $p = 11$; $k \geq 7$ if $p = 7$; $k \geq 9$ if $p = 3$ or 5; $\left(\frac{4p}{\cdot}\right)$ is the Kronecker symbol) using theta-series of some positive quadratic forms. These series are given in the form of infinite products. Pei in [6] constructed bases for the spaces $E_{3/2}(\Gamma_0(4N),(\frac{l}{\cdot}))$, $E_{3/2}(\Gamma_0(8N),(\frac{l}{\cdot}))$, $E_{3/2}(\Gamma_0(8N),(\frac{2l}{\cdot}))$ and $E_{3/2}(\Gamma_0(2^e),1)$, N being an odd natural and square-free number, e an integer ≥ 4, $l \mid N$, using transforms of some Eisenstein series of special kind and obtained Fourier expansions of these series.

In the present paper, for N specified above, bases of the spaces $E_k(\Gamma_0(4N),\chi)$ ($k \in \mathbb{N}$, $k \geq 3$) and $E_{k/2}(\widetilde{\Gamma}_0(4N),\chi)$ ($k \in \mathbb{N}$, $k \geq 5$, $2 \nmid k$) are constructed for any character mod $4N$ using the Eisenstein series, and their Fourier expansions are obtained.

In what follows, p is an odd prime, q is prime; M, n, d, k, h, m, u, r, s, t, α, β, δ, ω are integers; $\mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$, $z = \exp(2\pi i\tau)$, $\tau \in \mathbb{H}$; (m,n) denotes g.c.d. of m and n; $n \bmod N$ means that n runs through the full residue system modulo N; if $\tau \in \mathbb{C}$, then $\overline{\tau}$ denotes the complex conjugate of τ. If χ is a character mod N, then χ_0 denotes the main character mod N.

1. Lemma 1 ([4], p. 13). *If $\chi(n)$ is a character* mod N *and $N = N_1 \cdot N_2 \cdots N_m$, $(N_r,N_s) = 1$ when $r \neq s$, then there is a unique system of characters $\chi_1,\chi_2,\dots,\chi_m$ with modules $N_1,N_2,\dots,N_m$, respectively, such that $\chi(n) = \chi_1(n)\chi_2(n)\cdots\chi_m(n)$.*

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With mod 4 there are only two characters: the main character and the primitive character $\chi(n)=\left(\frac{-1}{n}\right)=(-1)^{(n-1)/2}$ if $2\nmid n$ and $\chi(n)=0$ if $2\mid n$.

If χ is a character mod p, not the main character, then χ is primitive (see [4], p. 22). With mod p there exists only one real primitive character $\chi(n)=\left(\frac{n}{p}\right)$ ($\left(\frac{n}{p}\right)$ is the generalized Legendre symbol).

In what follows $\phi(n)$ denotes a character mod 4. If χ is a character mod N, then let

$$g(\chi)=\frac{1}{\sqrt{N}}\sum_{n \bmod N}\chi(n)\exp(2\pi in/N).$$

Lemma 2 ([4], p. 45)**.** *If χ is a primitive character* mod N, *then*

$$\sum_{n \bmod N}\chi(n)\exp(2\pi imn/N)=\overline{\chi}(m)\sqrt{N}\,g(\chi),\quad |g(\chi)|=1.$$

Lemma 3 ([4], p. 50)**.** *If χ is a real primitive character* mod N, *then*

$$g(\chi)=\begin{cases}1 & \textit{when } \chi(-1)=1,\\ i & \textit{when } \chi(-1)=-1.\end{cases}$$

Lemma 4 ([4], p. 52)**.** *Let $\chi(n)=\chi_1(n)\chi_2(n)$, where χ_r is a character* mod N_r, *$r=1,2$, and $(N_1,N_2)=1$, then*

$$\sum_{n \bmod N}\chi(n)\exp(2\pi imn/N)=\chi_1(N_2)\chi_2(N_1)\sum_{n_1 \bmod N_1}\chi_1(n_1)\exp(2\pi imn_1/N_1)\times\sum_{n_2 \bmod N_2}\chi_2(n_2)\exp(2\pi imn_2/N_2).$$

Using Lemma 2 it is easy to verify (see [1], Ch. IV, §2, [4], p. 39) that if ψ is a character mod p, then

$$\sum_{r \bmod p}\psi(r)\exp(2\pi inr/p^\beta)=\begin{array}{l}0 \text{ if } \psi\neq\psi_0,\ p^\beta\mid n;\\ p-1 \text{ if } \psi=\psi_0,\ p^\beta\mid n;\\ -1 \text{ if } \psi=\psi_0,\ p^\beta\nmid n,\ p^{\beta-1}\mid n;\\ \sqrt{p}\overline{\psi}(n/p^{\beta-1})g(\psi) \text{ if } \psi\neq\psi_0,\ p^\beta\nmid n,\ p^{\beta-1}\mid n.\end{array}\tag{1.1}$$

$$\sum_{r \bmod 4}\phi(r)\exp(2\pi inr/2^\alpha)=\begin{array}{l}0 \text{ if } \phi\neq\phi_0,\ 2^{\alpha-1}\mid n;\\ 0 \text{ if } \phi=\phi_0,\ 2^{\alpha-2}\mid n,\ 2^{\alpha-1}\nmid n;\\ 2 \text{ if } \phi=\phi_0,\ 2^\alpha\mid n;\\ -2 \text{ if } 2^{\alpha-1}\mid n,\ \phi=\phi_0,\ 2^\alpha\nmid n;\\ 2i\phi(n/2^{\alpha-2}) \text{ if } \phi\neq\phi_0,\ 2^{\alpha-2}\mid n,\ 2^{\alpha-1}\nmid n.\end{array}\tag{1.2}$$

$$\sum_{r \bmod \omega}\exp(2\pi inr/\omega)=\begin{cases}\omega & \text{if } \omega\mid n;\\ 0 & \text{if } \omega\nmid n.\end{cases}\tag{1.3}$$

Lemma 5 ([1], Ch. IV, §2). *If $\tau \in \mathbb{H}$, $a \in \mathbb{R}$, $a > 1$, then*

$$\sum_{h=-\infty}^{+\infty} (\tau + h)^{-a} = (2\pi)^a \exp(-\pi i a/2)\,(\Gamma(a))^{-1} \sum_{n=1}^{+\infty} n^{a-1} \exp(2\pi i \tau n),$$

where Γ is the Euler function and when $\tau \in \mathbb{H}$, $\tau^a = \exp(a \ln \tau)$; also $\ln \tau$ denotes the branch of logarithm for which $0 < \operatorname{Im}(\ln \tau) < \pi$.

Let σ_∞ denote the number of cusps with respect to $\Gamma_0(N)$. Then (see [5], p. 102)

$$\sigma_\infty = \sum_{t|N} \varphi\left((t, N/t)\right), \tag{1.4}$$

where φ is the Euler function.

Lemma 6. *If $N = p_1 \cdots p_j$, then $\Gamma_0(4N)$ has $3 \cdot 2^j$ cusps: $\zeta_1 = \infty$, $\sigma_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$; $\zeta_2 = 0$, $\sigma_2 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$; $\{\zeta = -\frac{1}{N_1}$, $\sigma = \left(\begin{smallmatrix} 1 & 0 \\ -N_1 & 1 \end{smallmatrix}\right) \,\big|\, N_1 \mid 4N$, $N_1 \neq 1$, $N_1 \neq 4N\}$; $\zeta_r = \sigma_r \infty$ $(r = 1, 2)$, $\zeta = \sigma \infty$.*

Proof directly follows from (1.4) and the definition of a cusp. □

Lemma 7 ([1], Ch. IV, §2). *If $\omega = \omega_0 \omega_1^2$, $2 \nmid \omega$, ω_0 and n are square-free, then*

$$\sum_{r \bmod \omega} \left(\frac{r}{\omega}\right) \exp(2\pi i n r/\omega) = \begin{cases} 0 & \textit{when } \omega_1 \nmid n; \\ \varepsilon_\omega \left(\frac{n}{\omega_0}\right) \sqrt{\omega_0}\, \mu(\omega_1) \omega_1 & \textit{when } \omega_1 \mid n, \end{cases}$$

where

$$\varepsilon_\omega = \begin{cases} 1 & \textit{if } \omega \equiv 1 \pmod 4; \\ i & \textit{if } \omega \equiv -1 \pmod 4), \end{cases} \tag{1.5}$$

$\left(\frac{r}{\omega}\right)$ is the generalized Jacobi symbol and μ is the Möbius function.

In what follows, let χ be the Dirichlet character mod N; $\zeta = \sigma\infty$, $\sigma \in SL_2(\mathbb{Z})$, $\Gamma_\zeta = \{\gamma \in \Gamma_0(N) \mid \gamma\zeta = \zeta\}$; if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, $\tau \in \mathbb{H}$, $k \in \mathbb{N}$, let $J_k(\gamma, \tau) = (c\tau + d)^k$, $f(\tau)|_k \gamma = J_k(\gamma, \tau)^{-1} f(\gamma\tau)$.

Lemma 8 (see [1], Ch. III, §2; [3], Ch. II, §1). a) *Let $\chi(-1) = (-1)^k$,*

$$E(\tau; k, N, \chi) = \sum_{\gamma \in \Gamma_\zeta \setminus \Gamma_0(N)} \overline{\chi}(d) J_k(\sigma^{-1}\gamma, \tau)^{-1}, \tag{1.6}$$

where $\Gamma_\zeta \setminus \Gamma_0(N)$ denotes the set of right cosets of $\Gamma_0(N)$ by Γ_ζ. Then $E(\tau; k, N, \chi) \in M_k(\Gamma_0(N), \chi)$ for any $k \geq 3$.

b) *If $E(\tau; k, N, \chi) \not\equiv 0$, then $E(\tau; k, N, \chi) \neq 0$ only at the cusp ζ and vanishes at the remaining cusps.*

In what follows, let $\tau^{k/2} = (\sqrt{\tau})^k$, $-\frac{\pi}{2} < \operatorname{Arg} \sqrt{\tau} \leq \frac{\pi}{2}$ for any $k \in \mathbb{Z}$; if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, $\tau \in \mathbb{H}$, suppose $\varphi(\gamma, \tau)$ to be a holomorphic function on $\mathbb{H}$ such that $\varphi^2(\gamma, \tau) = t(c\tau + d)$, $t \in \{-1; 1\}$. If $\gamma \in \Gamma_0(4)$, then $\varphi(\gamma, \tau) = j(\gamma, \tau) = \left(\frac{c}{d}\right) \varepsilon_d^{-1} \sqrt{c\tau + d}$, where ε_d is defined by (1.5), $\left(\frac{c}{d}\right)$ is the generalized

Jacobi symbol when d is odd positive and if d is odd negative, then $\left(\frac{c}{d}\right) = \operatorname{sgn} c\left(\frac{c}{|d|}\right)$, also $\left(\frac{0}{\pm 1}\right) = 1$.

Let $G = \{(\gamma, \varphi(\gamma, \tau)) \mid \gamma \in SL_2(\mathbb{Z})\}$. If $(\gamma_1, \varphi(\gamma_1, \tau)), (\gamma_2, \varphi(\gamma_2, \tau)) \in G$, suppose $(\gamma_1, \varphi(\gamma_1, \tau)) \cdot (\gamma_2, \varphi(\gamma_2, \tau)) = (\gamma_1\gamma_2, \varphi(\gamma_1, \gamma_2\tau)\varphi(\gamma_2, \tau))$. It is known (see [1], Ch. IV, §1) that G is a group with respect to this operation.

If $\xi = (\gamma, \varphi(\gamma, \tau)) \in G$, f is some function defined on $\mathbb{H}$, let $f(\tau)|_{k/2}\xi = \varphi(\gamma, \tau)^{-k} f(\gamma\tau)$. If $4 \mid N$, suppose $\widetilde{\Gamma}_0(N) = \{(\gamma, j(\gamma, \tau)) \mid \gamma \in \Gamma_0(N)\}$.

Lemma 9 (see [1], Ch. IV, §2; [3], Ch. II, §1)**.** a) *Let* $4 \mid N$, $\chi(-1) = 1$,

$$E\left(\tau; \frac{k}{2}, N, \chi\right) = \sum_{\gamma \in \Gamma_\zeta \backslash \Gamma_0(N)} \overline{\chi}(d) \varphi(\sigma^{-1}\gamma, \tau)^{-k}, \tag{1.7}$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. *Then* $E\left(\tau; \frac{k}{2}, N, \chi\right) \in M_{k/2}(\widetilde{\Gamma}_0(N), \chi)$ *for any* $k \geq 5$.

b) $E\left(\tau; \frac{k}{2}, N, \chi\right) \neq 0$ *only at the cusp* ζ *and vanishes at the remaining cusps.*

In the following let

$$\rho_r(u, \chi) = \sum_{\delta d = u} \chi(\delta) d^r, \quad \sigma_r(u) = \sum_{d \mid u} d^r; \tag{1.8}$$

$$\mathcal{L}(k, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-k} = \prod_q \left(1 - \chi(q) q^{-k}\right)^{-1}; \tag{1.9}$$

$$\mathcal{L}\left(\frac{k-1}{2}, (-1)^{\frac{k-1}{2}} u\right) = \sum_{\substack{n=1 \\ 2 \nmid n}}^{\infty} \left(\frac{(-1)^{\frac{k-1}{2}} u}{n}\right) n^{\frac{1-k}{2}}$$

$$= \prod_{q > 2} \left(1 - \left(\frac{(-1)^{\frac{k-1}{2}} u}{q}\right) q^{\frac{1-k}{2}}\right)^{-1} \quad \text{(Dirichlet } \mathcal{L}\text{-function)}; \tag{1.10}$$

$$\zeta(k) = \sum_{n=1}^{\infty} n^{-k} = \prod_q (1 - q^{-k})^{-1} \quad \text{(Riemann } \zeta\text{-function)}. \tag{1.11}$$

2. In this section let $N = p_1 p_2 \cdots p_j$, $n = 2^s p_1^{t_1} p_2^{t_2} \cdots p_j^{t_j} u$, $(2N, u) = 1$. It follows from Lemma 1 that if $\chi(n)$ is a character $\bmod 4N$, then $\chi(n) = \phi(n)\psi(n) = \phi(n)\psi_1(n) \cdots \psi_j(n)$, where $\psi(n)$ is a character $\bmod N$ and ψ_l is a character $\bmod p_l$ $(l = 1, 2, \ldots, j)$.

Lemma 10. *Let*

$$Q_{1,k}(n, \chi) = \sum_{\substack{\omega=1 \\ (2N, \omega)=1}}^{\infty} \chi(\omega) \omega^{-k} \sum_{r \bmod \omega} \exp(2\pi i n r/\omega). \tag{2.1}$$

Then $Q_{1,k}(n, \chi) = u^{1-k} \rho_{k-1}(u, \chi)$.

Proof. Since $(2N, \omega) = 1$, from (1.3) and (1.8) we have

$$Q_{1,k}(n, \chi) = \sum_{\omega \mid u} \chi(\omega) \omega^{1-k} = u^{1-k} \sum_{\omega \mid u} \chi(\omega) \left(\frac{u}{\omega}\right)^{k-1} = u^{1-k} \rho_{k-1}(u, \chi).$$

The lemma is proved. □

Lemma 11. *Let* $\chi(n)=\phi(n)\psi(n)$, $M=p_1^{t_1}\cdots p_j^{t_j}$,

$$Q_{2,k}(n,\chi)=\sum_{\alpha=0}^{\infty}2^{-\alpha k}\psi^{\alpha}(2)\sum_{r \bmod 2^{\alpha+2}}\phi(r)\exp(2\pi inr/2^{\alpha+2}). \tag{2.2}$$

Then

a) *when* $\phi=\phi_0$, $Q_{2,k}(n,\chi)=0$ *if* $2\nmid n$ *and*

$$Q_{2,k}(2n,\chi)=2\cdot 2^{s(1-k)}\psi^s(2)\left(2^{k-1}\overline{\psi}(2)\cdot\frac{2^{s(k-1)}\overline{\psi}^s(2)-1}{2^{k-1}\overline{\psi}(2)-1}-1\right);$$

b) *when* $\phi\neq\phi_0$, $Q_{2,k}(n,\chi)=2\cdot 2^{s(1-k)}\psi^s(2)\phi(Mu)i$.

Proof. Any $r\in\mathbb{Z}/2^{\alpha+2}\mathbb{Z}$ is written uniquely as $r=r_1+4r_2$, $0\le r_1<4$, $0\le r_2<2^{\alpha}$. Then $\phi(r)=\phi(r_1)$ and

$$\sum_{r \bmod 2^{\alpha+2}}\phi(r)\exp(2\pi inr/2^{\alpha+2})$$
$$=\sum_{r_1 \bmod 4}\phi(r_1)\exp(2\pi inr_1/2^{\alpha+2})\sum_{r_2 \bmod 2^{\alpha}}\exp(2\pi inr_2/2^{\alpha}). \tag{2.3}$$

a) Let $\phi=\phi_0$. By virtue of (1.2), (1.3), from (2.2), (2.3) we get

1) if $2\nmid n$, i.e., $s=0$, then

$$Q_{2,k}(n,\chi)=\sum_{\alpha=0}^{\infty}2^{-\alpha k}\psi^{\alpha}(2)\sum_{r_1 \bmod 4}\phi_0(r_1)\exp(2\pi iMur_1/2^{\alpha+2})$$
$$\times\sum_{r_2 \bmod 2^{\alpha}}\exp(2\pi iMur_2/2^{\alpha})=\sum_{r_1 \bmod 4}\phi_0(r_1)\exp(2\pi ir_1/4)=0;$$

2) $$Q_{2,k}(2n,\chi)=\sum_{\alpha=0}^{\infty}2^{-\alpha k}\psi^{\alpha}(2)\sum_{r_1 \bmod 4}\phi_0(r_1)\exp(2\pi i2^{s+1}Mur_1/2^{\alpha+2})$$
$$\times\sum_{r_2 \bmod 2^{\alpha}}\exp(2\pi i2^{s+1}Mur_2/2^{\alpha})$$
$$=\sum_{\alpha=0}^{s-1}2\cdot 2^{\alpha(1-k)}\psi^{\alpha}(2)-2\cdot 2^{s(1-k)}\psi^s(2)$$
$$=2\cdot\frac{1-2^{s(1-k)}\psi^s(2)}{1-2^{1-k}\psi(2)}-2\cdot 2^{s(1-k)}\psi^s(2)$$
$$=2\cdot 2^{s(1-k)}\psi^s(2)\left(2^{k-1}\overline{\psi}(2)\cdot\frac{2^{(k-1)s}\overline{\psi}^s(2)-1}{2^{k-1}\overline{\psi}(2)-1}-1\right).$$

b) Let $\phi\neq\phi_0$. Again using (1.2), (1.3), from (2.2), (2.3) we have

$$Q_{2,k}(n,\chi)=2^{-sk}\psi^s(2)\cdot 2^s\sum_{r_1 \bmod 4}\phi(r_1)\exp(2\pi iMur_1/4)$$

$$= 2^{(1-k)s}\psi^s(2)\cdot 2i\phi(Mu). \qquad \square$$

Lemma 12. *Let* $M_l = \prod\limits_{\substack{l_1=1\\ l_1\neq l}}^{j} p_{l_1}^{t_{l_1}}$ $(1\le l\le j)$, $\chi(n)=\phi(n)\psi_l(n)\psi_{2,l}(n)$, *where* ψ_l *is a character* mod p_l *and* $\psi_{2,l}$ *is a character* mod $\frac{N}{p_l}$,

$$Q_{3,k,l}(n,\overline{\chi}) = \sum_{\beta=0}^{\infty} p_l^{-\beta k}\phi^{\beta}(p_l)\overline{\psi}_{2,l}^{\beta+1}(p_l)\sum_{r \bmod p_l^{\beta+1}}\overline{\psi}_l(r)\exp(2\pi i n r/p_l^{\beta+1}). \tag{2.4}$$

In that case,

a) *if* $\overline{\psi_l}$ *is the main character, then*

$$Q_{3,k,l}(n,\overline{\chi}) = p_l^{(1-k)t_l}\phi^{t_l}(p_l)\overline{\psi}_{2,l}^{t_l+1}(p_l)$$
$$\times\left((p_l-1)\phi(p_l)\psi_{2,l}(p_l)p_l^{k-1}\cdot\frac{p_l^{(k-1)t_l}\phi^{t_l}(p_l)\psi_{2,l}^{t_l}(p_l)-1}{p_l^{k-1}\phi(p_l)\psi_{2,l}(p_l)-1}-1\right);$$

b) *if* $\overline{\psi}_l$ *is not the main character, then*

$$Q_{3,k,l}(n,\overline{\chi}) = p_l^{(1-k)t_l}\phi^{t_l}(p_l)\overline{\psi}_{2,l}^{t_l+1}(p_l)\sqrt{p_l}\psi_l^s(2)\psi_l(Mu)g(\overline{\psi}_l).$$

Proof. Any $r\in\mathbb{Z}/p_l^{\beta+1}\mathbb{Z}$ is written uniquely as $r=r_1+p_lr_2$, $0\le r_1<p_l$, $0\le r_2<p_l^{\beta}$. Then $\overline{\psi}_l(r)=\overline{\psi}_l(r_1)$ and

$$\sum_{r \bmod p_l^{\beta+1}}\overline{\psi}_l(r)\exp(2\pi i n r/p_l^{\beta+1})$$
$$= \sum_{r_1 \bmod p_l}\overline{\psi}_l(r_1)\exp(2\pi i n r_1/p_l^{\beta+1})\sum_{r_2 \bmod p_l^{\beta}}\exp(2\pi i n r_2/p_l^{\beta}). \tag{2.5}$$

a) If $\overline{\psi}_l=\psi_0$, by virtue of (1.1), (1.3), from (2.4), (2.5) we get

$$Q_{3,k,l}(n,\overline{\chi}) = (p_l-1)\sum_{\beta=0}^{t_l-1}p_l^{\beta(1-k)}\phi^{\beta}(p_l)\overline{\psi}_{2,l}^{\beta+1}(p_l) - p_l^{t_l(1-k)}\phi^{t_l}(p_l)\overline{\psi}_{2,l}^{t_l+1}(p_l)$$
$$= \overline{\psi}_{2,l}(p_l)\left((p_l-1)\cdot\frac{1-p_l^{t_l(1-k)}\phi^{t_l}(p_l)\overline{\psi}_{2,l}^{t_l}(p_l)}{1-p_l^{1-k}\phi(p_l)\overline{\psi}_{2,l}(p_l)} - p_l^{t_l(1-k)}\phi^{t_l}(p_l)\overline{\psi}_{2,l}^{t_l}(p_l)\right)$$
$$= p_l^{(1-k)t_l}\phi^{t_l}(p_l)\overline{\psi}_{2,l}^{t_l+1}(p_l)$$
$$\times\left((p_l-1)\phi(p_l)\psi_{2,l}(p_l)p_l^{k-1}\cdot\frac{p_l^{(k-1)t_l}\phi^{t_l}(p_l)\psi_{2,l}^{t_l}(p_l)-1}{p_l^{k-1}\phi(p_l)\psi_{2,l}(p_l)-1}-1\right).$$

b) If $\overline{\psi}_l\neq\psi_0$, using (1.1), (1.3), from (2.4), (2.5) we have

$$Q_{3,k,l}(n,\overline{\chi}) = p_l^{-t_lk}\phi^{t_l}(p_l)\overline{\psi}_{2,l}^{t_l+1}(p_l)p_l^{t_l}\sum_{r_1 \bmod p_l}\overline{\psi}_l(r_1)\exp(2\pi i 2^s M_l u r_1/p_l)$$
$$= p_l^{(1-k)t_l}\phi^{t_l}(p_l)\overline{\psi}_{2,l}^{t_l+1}(p_l)\sqrt{p_l}\psi_l(2^sM_lu)g(\overline{\psi}_l). \qquad \square$$

Proposition 1. *Let* $\chi(n) = \phi(n)\psi_{1,l}(n)\psi_{2,l}(n)$, *where* $\psi_{1,l}$ *is a character* $\bmod N_l$ $(N_l \mid N)$ *and* $\psi_{2,l}$ *is a character* $\bmod (N/N_l)$; $\chi(-1) = (-1)^k$. *Then*

a) *if* $\phi = \phi_0$, *then the system of the functions*

$$E_1(\tau; k, 4N, \chi) = \sum_{\substack{4N|m \\ n>0,\, (m,n)=1}} \overline{\chi}(n)(m\tau + n)^{-k}, \tag{2.6}$$

$$E_{2,l}(\tau; k, 4N, \chi) = \sum_{\substack{m>0, n \\ (2N_l m, Nn/N_l)=1}} \phi(n)\overline{\psi}_{1,l}(n)\psi_{2,l}(m)(4N_l m\tau + n)^{-k} \tag{2.7}$$

$$(N_l \mid N,\ \ N_l \neq N,\ \ l = 1, \dots, 2^j - 1,\ \ N_{l_1} \neq N_{l_2},\ \ \textit{when}\ \ l_1 \neq l_2),$$

$$E_{3,l}(\tau; k, 4N, \chi) = \sum_{\substack{m>0, n \\ (N_l m, 2nN/N_l)=1}} \phi(m)\psi_{2,l}(m)\overline{\psi}_{1,l}(n)(N_l m\tau + n)^{-k} \tag{2.8}$$

$$(N_l \mid N,\ \ l = 1, \dots, 2^j,\ \ N_{l_1} \neq N_{l_2},\ \ \textit{when}\ \ l_1 \neq l_2),$$

$$E_{4,l}(\tau; k, 4N, \chi) = \sum_{\substack{2\nmid m,\, m>0 \\ n, (2N_l m, nN/N_l)=1}} \phi(n)\psi_{2,l}(m)\overline{\psi}_{1,l}(n)(2N_l m\tau + n)^{-k} \tag{2.9}$$

$$(N_l \mid N,\ \ l = 1, \dots, 2^j,\ \ N_{l_1} \neq N_{l_2},\ \ \textit{when}\ \ l_1 \neq l_2)$$

is the basis of the space $E_k(\Gamma_0(4N), \chi)$ *for any* $k \geq 3$;

b) *if* $\phi \neq \phi_0$, *then the system of functions* (2.6)–(2.8) *is the basis of the space* $E_k(\Gamma_0(4N), \chi)$ *for any* $k \geq 3$.

Proof. It is well known (see [1], [5]) that to each regular cusp of the group $\Gamma_0(4N)$ there corresponds an Eisenstein series and these functions form the basis of the space $E_k(\Gamma_0(4N), \chi)$. Now the result follows from Lemmas 6 and 8 after easy calculations. □

Next, we derive Fourier expansions of functions (2.6)–(2.9).

Let $a_k(\chi) = \frac{\pi^k}{i^k \Gamma(k)\mathcal{L}(k,\chi)}$, $\chi_1(n) = \phi(n)\psi_{2,l}(n)\overline{\psi}_{1,l}(n)$, $\chi_2(n) = \phi_0(n)\psi_{2,l}(n)\overline{\psi}_{1,l}(n)$.

1) Since $\overline{\chi}(-1) = (-1)^k$, we have

$$\sum_{\substack{m<0 \\ 4N|m,\, n>0 \\ (m,n)=1}} \overline{\chi}(n)(m\tau + n)^{-k} = \sum_{\substack{m>0 \\ 4N|m,\, n>0 \\ (m,n)=1}} \overline{\chi}(n)(-m\tau + n)^{-k} = \sum_{\substack{m>0 \\ 4N|m,\, n<0 \\ (m,n)=1}} \overline{\chi}(n)(m\tau + n)^{-k}.$$

Thus, because of the absolute convergence, we have

$$\begin{aligned} E_1(\tau; k, 4N, \chi) &= 1 + \sum_{\substack{m>0 \\ n,\, (m,n)=1}} \overline{\chi}(n)(4Nm\tau + n)^{-k} \\ &= 1 + \left(\sum_{d=1}^{\infty} \overline{\chi}(d)d^{-k}\right)^{-1} \sum_{\substack{m>0, \\ n}} \overline{\chi}(n)(4Nm\tau + n)^{-k}. \end{aligned}$$

Let $n = r + 4Nmh$, $r \bmod 4Nm$, $h \in \mathbb{Z}$. Then $\overline{\chi}(n) = \overline{\chi}(r) = \phi(r)\overline{\psi}(r)$. By Lemma 5 and (1.9) we get

$$\begin{aligned}
E_1(\tau; k, 4N, \chi) &= 1 + (4N)^{-k}\mathcal{L}(k,\overline{\chi})^{-1} \\
&\quad\times \sum_{m=1}^{\infty}\Bigg(m^{-k}\sum_{r \bmod 4Nm}\overline{\chi}(r)\sum_{h=-\infty}^{\infty}\Big(\tau + \frac{r}{4Nm} + h\Big)^{-k}\Bigg) \\
&= 1 + (4N)^{-k}(2\pi)^k\exp(-\pi i k/2)\left(\Gamma(k)\mathcal{L}(k,\overline{\chi})\right)^{-1} \\
&\quad\times \sum_{n=1}^{\infty}\Bigg(n^{k-1}\sum_{m=1}^{\infty}m^{-k}\sum_{r \bmod 4Nm}\overline{\chi}(r)\exp(2\pi i n r/(4Nm))\Bigg)z^n \\
&= 1 + (2N)^{-k}a_k(\overline{\chi})\sum_{n=1}^{\infty}\Bigg(n^{k-1}\sum_{\alpha=0}^{\infty}\sum_{\beta_1=0}^{\infty}\cdots\sum_{\beta_j=0}^{\infty}\sum_{\substack{\omega=1\\(2N,\omega)=1}}^{\infty}(2^{\alpha}p_1^{\beta_1}\cdots p_j^{\beta_j}\omega)^{-k} \\
&\quad\times \sum_{r \bmod 2^{\alpha+2}p_1^{\beta_1+1}\cdots p_j^{\beta_j+1}\omega}\phi(r)\overline{\psi}(r)\exp(2\pi i n r/(2^{\alpha+2}p_1^{\beta_1+1}\cdots p_j^{\beta_j+1}\omega))\Bigg)z^n. \qquad (2.10)
\end{aligned}$$

Any $r \in \mathbb{Z}/2^{\alpha+2}p_1^{\beta_1+1}\cdots p_j^{\beta_j+1}\omega\mathbb{Z}$ is written uniquely as

$$r = \omega\cdot 2^{\alpha+2}\sum_{h=1}^{j}r_h\prod_{\substack{h_1=1\\h_1\neq h}}^{j}p_{h_1}^{\beta_{h_1}+1} + \omega r_{j+1}\prod_{h=1}^{j}p_h^{\beta_h+1} + 2^{\alpha+2}r_{j+2}\prod_{h=1}^{j}p_h^{\beta_h+1},$$

$$0 \le r_{j+2} < \omega,\quad 0 \le r_h < p_h^{\beta_h+1},\quad 1 \le h \le j,\quad 0 \le r_{j+1} < 2^{\alpha+2}.$$

Then

$$\phi(r) = \phi(r_{j+1})\phi(\omega)\prod_{l=1}^{j}\phi(p_l^{\beta_l+1}),$$

$$\overline{\psi}_l(r) = \overline{\psi}_l(\omega)\overline{\psi}_l(4)\overline{\psi}_l^{\alpha}(2)\overline{\psi}_l(r_l)\prod_{\substack{l_1=1\\l_1\neq l}}^{j}\overline{\psi}_l(p_{l_1}^{\beta_{l_1}+1}),\quad \phi(\omega)\prod_{l=1}^{j}\overline{\psi}_l(\omega) = \overline{\chi}(\omega),$$

$$\prod_{l=1}^{j}\overline{\psi}_l(2^{\alpha}) = \overline{\psi}^{\alpha}(2),\qquad \prod_{l=1}^{j}\overline{\psi}_l(4) = \overline{\psi}(4),$$

$$\exp(2\pi i n r/(2^{\alpha+2}p_1^{\beta_1+1}\cdots p_j^{\beta_j+1}\omega)) = \exp(2\pi i n r_{j+2}/\omega)$$

$$\times\prod_{l=1}^{j}\exp(2\pi i n r_l/p_l^{\beta_l+1})\exp(2\pi i n r_{j+1}/2^{\alpha+2})$$

and from (2.10) we obtain

$$E_1(\tau; k, 4N, \chi) = 1 + (2N)^{-k}a_k(\overline{\chi})\overline{\psi}(4)\phi(N)$$

$$\times \sum_{n=1}^{\infty} \Bigg(n^{k-1} \sum_{\substack{\omega=1 \\ (2N,\omega)=1}}^{\infty} \overline{\chi}(\omega)\omega^{-k} \sum_{r_{j+2} \bmod \omega} \exp(2\pi i n r_{j+2}/\omega)$$

$$\times \sum_{\alpha=0}^{\infty} 2^{-\alpha k}\overline{\psi}^{\alpha}(2) \sum_{r_{j+1} \bmod 2^{\alpha+2}} \phi(r_{j+1}) \exp(2\pi i n r_{j+1}/2^{\alpha+2})$$

$$\times \prod_{l=1}^{j} \Bigg(\sum_{\beta_l=0}^{\infty} p_l^{-\beta_l k} \phi^{\beta_l}(p_l) \prod_{\substack{l_1=1 \\ l_1 \neq l}}^{j} \overline{\psi}_{l_1}(p_l^{\beta_l+1})$$

$$\times \sum_{r_l \bmod p_l^{\beta_l+1}} \overline{\psi}_l(r_l) \exp(2\pi i n r_l/p_l^{\beta_l+1}) \Bigg) \Bigg) z^n$$

$$= 1 + a_k(\overline{\chi})\overline{\psi}(4)\phi(N)2^{-k}N^{-k}$$

$$\times \sum_{n=1}^{\infty} n^{k-1} Q_{1,k}(n,\overline{\chi}) Q_{2,k}(n,\overline{\chi}) \prod_{l=1}^{j} Q_{3,k,l}(n,\overline{\chi}) z^n. \qquad (2.11)$$

2) $$E_{2,l}(\tau;k,4N,\chi) = \sum_{\substack{m>0,n \\ (m,n)=1}} \phi(n)\overline{\psi}_{1,l}(n)\psi_{2,l}(m)(4N_l m\tau + n)^{-k}$$

$$= \frac{1}{\mathcal{L}(k,\chi_1)} \sum_{m=1}^{\infty} \psi_{2,l}(m) \sum_{n=-\infty}^{\infty} \phi(n)\overline{\psi}_{1,l}(n)(4N_l m\tau + n)^{-k}.$$

Let $n = r + 4N_l mh$, $r \bmod 4N_l m$, $h \in \mathbb{Z}$. Then $\phi(n) = \phi(r)$, $\overline{\psi}_{1,l}(n) = \overline{\psi}_{1,l}(r)$, and by Lemma 5 we get

$$E_{2,l}(\tau;k,4N,\chi) = (4N_l)^{-k}\mathcal{L}(k,\chi_1)^{-1}$$

$$\times \sum_{m=1}^{\infty} \Bigg(\psi_{2,l}(m)m^{-k} \sum_{r \bmod 4N_l m} \phi(r)\overline{\psi}_{1,l}(r) \sum_{h=-\infty}^{\infty} \Big(\tau + \frac{r}{4N_l m} + h \Big)^{-k} \Bigg)$$

$$= (2N_l)^{-k} a_k(\chi_1) \sum_{n=1}^{\infty} \Bigg(n^{k-1} \sum_{\substack{m=1 \\ (N/N_l,m)=1}}^{\infty} \psi_{2,l}(m)m^{-k}$$

$$\times \sum_{r \bmod 4N_l m} \phi(r)\overline{\psi}_{1,l}(r) \exp(2\pi i n r/(4N_l m)) \Bigg) z^n.$$

Let $N_l = p_{\delta_1} \cdots p_{\delta_d}$, $1 \le d \le j-1$, or $N_l = 1$, then

$$E_{2,l}(\tau;k,4N,\chi) = (2N_l)^{-k} a_k(\chi_1)$$

$$\times \sum_{n=1}^{\infty} \Bigg(n^{k-1} \sum_{\alpha=0}^{\infty} \sum_{\beta_1=0}^{\infty} \cdots \sum_{\beta_d=0}^{\infty} \sum_{\substack{\omega=1 \\ (2N,\omega)=1}}^{\infty} \psi_{2,l}(2^{\alpha} p_{\delta_1}^{\beta_1} \cdots p_{\delta_d}^{\beta_d}\omega)(2^{\alpha} p_{\delta_1}^{\beta_1} \cdots p_{\delta_d}^{\beta_d}\omega)^{-k}$$

$$\times \sum_{r \bmod 2^{\alpha+2} p_{\delta_1}^{\beta_1+1} \cdots p_{\delta_d}^{\beta_d+1} \omega} \phi(r) \overline{\psi}_{1,l}(r) \exp(2\pi i n r / (2^{\alpha+2} p_{\delta_1}^{\beta_1+1} \cdots p_{\delta_d}^{\beta_d+1} \omega)) \Big) z^n.$$

Let

$$r = \omega \cdot 2^{\alpha+2} \sum_{h=1}^{d} r_h \prod_{\substack{h_1=1 \\ h_1 \neq h}}^{d} p_{\delta_{h_1}}^{\beta_{h_1}+1} + \omega r_{d+1} \prod_{h=1}^{d} p_{\delta_h}^{\beta_h+1} + 2^{\alpha+2} r_{d+2} \prod_{h=1}^{d} p_{\delta_h}^{\beta_h+1},$$

$$0 \le r_{d+2} < \omega, \quad 0 \le r_h < p_{\delta_h}^{\beta_h+1}, \quad 1 \le h \le d, \quad 0 \le r_{d+1} < 2^{\alpha+2}.$$

Then

$$\phi(r) = \phi(r_{d+1}) \phi(\omega) \phi(p_{\delta_1}^{\beta_1+1} \cdots p_{\delta_d}^{\beta_d+1}),$$

$$\overline{\psi}_{1,l}(r) = \prod_{h=1}^{d} \overline{\psi}_{\delta_h}(r) = \prod_{h=1}^{d} \Big(\overline{\psi}_{\delta_h}(\omega) \overline{\psi}_{\delta_h}(r_h) \overline{\psi}_{\delta_h}(2^{\alpha+2}) \prod_{\substack{h_1=1 \\ h_1 \neq h}}^{d} \overline{\psi}_{\delta_h}(p_{\delta_{h_1}}^{\beta_{h_1}+1}) \Big)$$

and

$$\begin{aligned} E_{2,l}(\tau; k, 4N, \chi) &= (2N_l)^{-k} \overline{\psi}_{1,l}(4) \phi(N_l) \overline{\psi}_{2,l}(N_l) a_k(\chi_1) \\ &\times \sum_{n=1}^{\infty} \Big(n^{k-1} \sum_{\substack{\omega=1 \\ (2N,\omega)=1}}^{\infty} \chi_1(\omega) \omega^{-k} \sum_{r_{d+2} \bmod \omega} \exp(2\pi i n r_{d+2}/\omega) \\ &\quad \times \sum_{\alpha=0}^{\infty} 2^{-\alpha k} \overline{\psi}_{1,l}^{\alpha}(2) \psi_{2,l}^{\alpha}(2) \sum_{r_{d+1} \bmod 2^{\alpha+2}} \phi(r_{d+1}) \exp(2\pi i n r_{d+1}/2^{\alpha+2}) \\ &\quad \times \prod_{h=1}^{d} \Big(\sum_{\beta_h=1}^{\infty} p_{\delta_h}^{-\beta_h k} \phi^{\beta_h}(p_{\delta_h}) \psi_{2,l}(p_{\delta_h}^{\beta_h+1}) \prod_{\substack{h_1=1 \\ h_1 \neq h}}^{d} \overline{\psi}_{\delta_{h_1}}(p_{\delta_h}^{\beta_h+1}) \\ &\quad \times \sum_{r_h \bmod p_{\delta_h}^{\beta_h+1}} \overline{\psi}_{\delta_h}(r_h) \exp(2\pi i n r_h / p_{\delta_h}^{\beta_h+1}) \Big) \Big) z^n \\ &= (2N_l)^{-k} \overline{\psi}_{1,l}(4) \phi(N_l) \overline{\psi}_{2,l}(N_l) a_k(\chi_1) \\ &\times \sum_{n=1}^{\infty} \Big(n^{k-1} Q_{1,k}(n, \chi_1) Q_{2,k}(n, \chi_1) \prod_{h=1}^{d} Q_{3,k,\delta_h}(n, \chi_1) \Big) z^n. \end{aligned} \tag{2.12}$$

3) If $N_l = 1$, $d = 0$ or $N_l = p_{\delta_1} \cdots p_{\delta_d}$, $1 \le d \le j$, then similarly to the above we obtain

$$\begin{aligned} E_{3,l}(\tau; k, 4N, \chi) = 2^k N_l^{-k} a_k(\chi_1) \overline{\psi}_{2,l}(N_l) \sum_{n=1}^{\infty} \Big(n^{k-1} Q_{1,k}(n, \chi_1) \\ \times \prod_{h=1}^{d} Q_{3,k,\delta_h}(n, \chi_1) \Big) z^n \end{aligned} \tag{2.13}$$

and

$$E_{4,l}(\tau;k,4N,\chi)=N_l^{-k}a_k(\chi_2)\overline{\psi}_{1,l}(2)\overline{\psi}_{2,l}(N_l)\sum_{n=1}^{\infty}\Big(n^{k-1}(-1)^nQ_{1,k}(n,\chi_2)$$
$$\times\prod_{h=1}^{d}Q_{3,k,\delta_h}(n,\chi_2)\Big)z^n. \tag{2.14}$$

For $N=p$, we obtain the following results.

Corollary 1. *Let* $\chi(n)=\phi_0(n)\psi(n)$, $\chi(-1)=(-1)^k$, $\psi\neq\psi_0$, $n=2^sp^tu$, $(2p,u)=1$, $b_k(\chi)=\psi(2)2^{k-1}\cdot\frac{\psi^s(2)\cdot 2^{(k-1)s}-1}{\psi(2)2^{k-1}-1}-1$. *Then*

$$E_1(\tau;k,4p,\chi)=1+2^{1-k}a_k(\overline{\chi})\overline{\psi}(2)g(\overline{\psi})p^{\frac12-k}\sum_{n=1}^{\infty}b_k(\chi)\psi(u)\rho_{k-1}(u,\overline{\chi})z^{2n},$$

$$E_2(\tau;k,4p,\chi)=2^{1-k}a_k(\chi)\sum_{n=1}^{\infty}b_k(\overline{\chi})\psi^s(2)p^{(k-1)t}\rho_{k-1}(u,\chi)z^{2n},$$

$$E_{3,1}(\tau;k,4p,\chi)=2^ka_k(\chi)\sum_{n=1}^{\infty}2^{(k-1)s}p^{(k-1)t}\rho_{k-1}(u,\chi)z^n,$$

$$E_{3,2}(\tau;k,4p,\chi)=a_k(\overline{\chi})g(\overline{\psi})2^k\cdot p^{\frac12-k}\sum_{n=1}^{\infty}\psi^s(2)\psi(u)2^{(k-1)s}\rho_{k-1}(u,\overline{\chi})z^n,$$

$$E_{4,1}(\tau;k,4p,\chi)=a_k(\chi)\sum_{n=1}^{\infty}(-1)^n\cdot 2^{(k-1)s}p^{(k-1)t}\rho_{k-1}(u,\chi)z^n,$$

$$E_{4,2}(\tau;k,4p,\chi)=a_k(\overline{\chi})\overline{\psi}(2)g(\overline{\psi})p^{\frac12-k}\sum_{n=1}^{\infty}(-1)^n\psi^s(2)\psi(u)2^{(k-1)s}\rho_{k-1}(u,\overline{\chi})z^n.$$

Proof directly follows from (2.11)–(2.14) and Lemmas 10–12, replacing, in Lemma 11, n and s by $2n$ and $s+1$ to obtain Fourier expansions for (2.6) and (2.7); furthermore, $\phi(p)=1$. □

Corollary 2. *Let* $\chi(n)=\phi_0(n)\psi_0(n)$, $n=2^sp^tu$, $(2p,u)=1$, $b_k=2^{2k-1}\cdot\frac{2^{(2k-1)s}-1}{2^{2k-1}-1}-1$, $c_k=(p-1)p^{2k-1}\cdot\frac{p^{(2k-1)t}-1}{p^{2k-1}-1}-1$, $d_k=\frac{(-1)^k4k(2p)^{2k}}{(2^{2k}-1)(p^{2k}-1)B_k}$, *where* B_k *are Bernoulli numbers. Then*

$$E_1(\tau;2k,4p,\chi)=1+(2p)^{-2k}d_k\sum_{n=1}^{\infty}b_kc_k\sigma_{2k-1}(u)z^{2n},$$

$$E_2(\tau;2k,4p,\chi)=2^{-2k}d_k\sum_{n=1}^{\infty}b_kp^{(2k-1)t}\sigma_{2k-1}(u)z^{2n},$$

$$E_{3,1}(\tau;2k,4p,\chi)=d_k\sum_{n=1}^{\infty}2^{(2k-1)s}p^{(2k-1)t}\sigma_{2k-1}(u)z^n,$$

$$E_{3,2}(\tau;2k,4p,\chi)=p^{-2k}d_k\sum_{n=1}^{\infty}c_k\cdot 2^{(2k-1)s}\sigma_{2k-1}(u)z^n,$$

$$E_{4,1}(\tau;2k,4p,\chi)=2^{-2k}d_k\sum_{n=1}^{\infty}(-1)^n 2^{(2k-1)s}p^{(2k-1)t}\sigma_{2k-1}(u)z^n,$$

$$E_{4,2}(\tau;2k,4p,\chi)=(2p)^{-2k}d_k\sum_{n=1}^{\infty}(-1)^n c_k 2^{(2k-1)s}\sigma_{2k-1}(u)z^n.$$

Proof. It follows from (1.9) and (1.11) that $\mathcal{L}(2k,\chi_0)=(1-2^{-2k})(1-p^{-2k})\zeta(2k)$. It is known (see [7], Ch. I) that $\zeta(2k)=\frac{2^{2k-1}\pi^{2k}}{(2k)!}B_k$. Then the result immediately follows from (2.11)–(2.14) and Lemmas 10–12. □

Corollary 3. *Let* $\chi(n)=\phi(n)\psi(n)$, $\phi\neq\phi_0$, $\psi\neq\psi_0$, $n=2^s p^t u$, $(2p,u)=1$; *then*

$$E_1(\tau;k,4p,\chi)=1+i\cdot 2^{1-k}p^{\frac12-k}a_k(\overline{\chi})\overline{\psi}(4)\phi(p)g(\overline{\psi})\sum_{n=1}^{\infty}\chi(u)\rho_{k-1}(u,\overline{\chi})z^n,$$

$$E_2(\tau;k,4p,\chi)=2^{1-k}ia_k(\chi)\sum_{n=1}^{\infty}p^{(k-1)t}\psi^s(2)\phi^t(p)\phi(u)\rho_{k-1}(u,\chi)z^n,$$

$$E_{3,1}(\tau;k,4p,\chi)=2^k\cdot a_k(\chi)\sum_{n=1}^{\infty}2^{(k-1)s}p^{(k-1)t}\rho_{k-1}(u,\chi)z^n,$$

$$E_{3,2}(\tau;k,4p,\chi)=2^k\cdot p^{\frac12-k}a_k(\overline{\chi})g(\overline{\psi})\sum_{n=1}^{\infty}2^{(k-1)s}\phi^t(p)\psi^s(2)\psi(u)\rho_{k-1}(u,\overline{\chi})z^n.$$

Proof directly follows from (2.11)–(2.13) and Lemmas 10–12. □

Corollary 4. *Let* $\chi(n)=\phi(n)\psi_0(n)$, $\phi\neq\phi_0$, $n=2^s p^t u$, $(2p,u)=1$, $c_k=\phi(p)(p-1)p^{2k}\cdot\frac{\phi^t(p)p^{2kt}-1}{\phi(p)p^{2k}-1}-1$, $d_k=\frac{4(-1)^k}{(1-\phi(p)p^{-(2k+1)})E_k}$, *where* E_k *are Euler numbers. Then*

$$E_1(\tau;2k+1,4p,\chi)=1+\phi(p)p^{-(2k+1)}d_k\sum_{n=1}^{\infty}c_k\phi(u)\rho_{2k}(u,\chi)z^n,$$

$$E_2(\tau;2k+1,4p,\chi)=d_k\sum_{n=1}^{\infty}\phi^t(p)\phi(u)p^{2kt}\rho_{2k}(u,\chi)z^n,$$

$$E_{3,1}(\tau;2k+1,4p,\chi)=-i\cdot 2^{4k+1}d_k\sum_{n=1}^{\infty}2^{2ks}p^{2kt}\rho_{2k}(u,\chi)z^n,$$

$$E_{3,2}(\tau;2k+1,4p,\chi)=-i\cdot 2^{4k+1}p^{-(2k+1)}d_k\sum_{n=1}^{\infty}c_k\phi^t(p)2^{2ks}\rho_{2k}(u,\chi)z^n.$$

Proof. It follows from (1.9) that

$$\mathcal{L}(2k+1,\chi)=\sum_{\substack{n=1\\ p\nmid n}}^{\infty}(-1)^{n-1}(2n-1)^{-(2k+1)}$$

$$= (1-\phi(p)p^{-(2k+1)}) \sum_{n=1}^{\infty} (-1)^{n-1}(2n-1)^{-(2k+1)}$$
$$= (1-\phi(p)p^{-(2k+1)}) \cdot \frac{\pi^{2k+1}}{2^{2k+2}(2k)!} E_k \quad \text{(see [7], Ch. I)}.$$

Then the result follows from (2.11)–(2.13) and Lemmas 10–12. □

3. In what follows, let $n = 2^{2s} p_1^{2t_1} \cdots p_j^{2t_j} n_1^2 u$, $N = p_1 \cdots p_j$, $(2N, n_1) = 1$, where u is square-free, $2 \nmid k$, $M = p_1^{2t_1} \cdots p_j^{2t_j}$, $\chi = \phi\psi$ is a character mod $4N$.

Lemma 13. *Let*

$$S_{1,k}(n,\chi) = \sum_{\alpha=0}^{\infty} \left(2^{-\frac{k}{2}\alpha} \psi^{\alpha}(2) \sum_{\substack{r \bmod 2^{\alpha+2} \\ 2\nmid r}} \phi(r) \left(\frac{2}{r}\right)^{\alpha} \varepsilon_r^k \exp(2\pi i n r/2^{\alpha+2}) \right). \quad (3.1)$$

Then

$$S_{1,k}(n,\chi) = 2^{(2-k)s}\psi^s(4) \left(1 + \phi(3)\left(\frac{-1}{k}\right) i\right) \times \left(2^{k-2}\overline{\psi}(4) \cdot \frac{2^{(k-2)s}\overline{\psi}^s(4) - 1}{2^{k-2}\overline{\psi}(4) - 1} + c_k(u,\chi) \right), \quad (3.2)$$

where

$$c_k(u,\chi) = \begin{cases} -1 & \text{if } (-1)^{\frac{k-1}{2}}\phi(3)u \not\equiv 1 \pmod 4; \\ 1 + 2^{\frac{3-k}{2}}\psi(2)\left(\frac{2}{u}\right) & \text{if } (-1)^{\frac{k-1}{2}}\phi(3)u \equiv 1 \pmod 4. \end{cases}$$

Proof. It is well-known that $\left(\frac{2}{r}\right) = (-1)^{(r^2-1)/8}$, therefore $\left(\frac{2}{4r+1}\right) = (-1)^r$ and $\left(\frac{2}{4r+3}\right) = (-1)^{r+1}$; furthermore, $\varepsilon_r = 1$, $\phi(r) = 1$ when $r \equiv 1 \pmod 4$, and $\varepsilon_r = i$, $\phi(r) = \phi(3)$ when $r \equiv 3 \pmod 4$. Hence

$$S_{1,k}(n,\chi) = \sum_{\alpha=0}^{\infty} \Bigg(2^{-\frac{k}{2}\alpha}\psi^{\alpha}(2) \exp(2\pi i n/2^{\alpha+2}) \times \left(1 + (-1)^{\alpha}\phi(3) i^k \exp(2\pi i n/2^{\alpha+1})\right) \sum_{r \bmod 2^{\alpha}} (-1)^{r\alpha} \exp(2\pi i n r/2^{\alpha}) \Bigg)$$
$$= \exp(2\pi i n/4)\left(1 + \phi(3) i^k \exp(\pi i n)\right) + \sum_{\alpha=1}^{\infty} \Bigg(2^{-\frac{k}{2}\alpha}\psi^{\alpha}(2)\exp(2\pi i n/2^{\alpha+2}) \left(1 + (-1)^{\alpha}\phi(3) i^k \exp(2\pi i n/2^{\alpha+1})\right) \times (1 + (-1)^{\alpha}\exp(2\pi i n/2^{\alpha})) \sum_{r \bmod 2^{\alpha-1}} \exp(2\pi i n r/2^{\alpha-1}) \Bigg). \quad (3.3)$$

Since $2 \nmid n_1$, we have $Mn_1^2 \equiv 1 \pmod 8$ and

$$\exp(2\pi i n/2^{2s+3}) = \exp(\pi i u/4) = \frac{1}{\sqrt{2}} \cdot \left(\frac{2}{u}\right)\left(1 + \left(\frac{-1}{u}\right) i\right) \quad \text{when } 2 \nmid u; \quad (3.4)$$

$$\exp(2\pi in/2^{2s+1}) = (-1)^u, \quad \exp(2\pi in/2^{2s+2}) = i^u; \tag{3.5}$$

if $2 \nmid u$, then $i^u = \left(\frac{-1}{u}\right) i$; $2 \nmid k$, hence $i^k = \left(\frac{-1}{k}\right) i$. It is easy to verify that

$$\begin{aligned} 1 + (-1)^\alpha \exp(2\pi in/2^\alpha) = {} & 1 + (-1)^\alpha \ \text{ when } \ \alpha \le 2s; \\ & 1 - (-1)^u \ \text{ when } \ \alpha = 2s+1; \\ & 0 \ \text{ when } \ \alpha = 2s+2 \ \text{ and } \ 2 \mid u. \end{aligned}$$

Now from (1.3) and (3.3) we get

$$\begin{aligned} S_{1,k}(n,\chi) = & \sum_{\alpha=0}^{s-1} 2^{(2-k)\alpha} \psi^\alpha(4) \left(1 + \phi(3)\left(\frac{-1}{k}\right) i\right) \\ & + 2^{(2-k)s} \psi^s(4) i^u \left(1 + \phi(3)\left(\frac{-1}{k}\right) i \cdot (-1)^u\right) \\ & + 2^{(2-k)s} \cdot 2^{-\frac{k}{2}} \psi^s(4)\psi(2)\exp(\pi iu/4) \\ & \times \left(1 - \phi(3)\left(\frac{-1}{k}\right) i \cdot i^u\right)(1-(-1)^u). \end{aligned} \tag{3.6}$$

If $(-1)^{\frac{k-1}{2}}\phi(3)u \equiv 1 \,(\bmod\, 4)$, then $\left(\frac{-1}{u}\right) = \phi(3)\left(\frac{-1}{k}\right)$, and if $(-1)^{\frac{k-1}{2}}\phi(3)u \equiv 3\,(\bmod\, 4)$, then $\left(\frac{-1}{u}\right) = -\phi(3)\left(\frac{-1}{k}\right)$. Therefore, using (3.4) and (3.5), we have

$$\begin{aligned} & i^u \left(1 + \phi(3)\left(\frac{-1}{k}\right) i \cdot (-1)^u\right) + 2^{-\frac{k}{2}} \psi(2)\exp(\pi iu/4) \\ & \times \left(1 - \phi(3)\left(\frac{-1}{k}\right) i \cdot i^u\right)(1-(-1)^u) = \left(1 + \phi(3)\left(\frac{-1}{k}\right) i\right) c_k(u,\chi). \end{aligned} \tag{3.7}$$

Furthermore,

$$\begin{aligned} \sum_{\alpha=0}^{s-1} 2^{(2-k)\alpha}\psi^\alpha(4) & = \frac{1 - 2^{(2-k)s}\psi^s(4)}{1 - 2^{2-k}\psi(4)} \\ & = 2^{(2-k)s}\psi^s(4) 2^{k-2}\overline{\psi}(4) \cdot \frac{2^{(k-2)s}\overline{\psi}^s(4) - 1}{2^{k-2}\overline{\psi}(4) - 1}. \end{aligned} \tag{3.8}$$

From (3.6)–(3.8) we obtain (3.2). □

Lemma 14. *Let* $\psi_l(n)$ *be a character* mod p_l, $M_l = \prod_{l_1=1,\, l_1 \ne l}^{j} p_{l_1}^{2t_{l_1}}$, $1 \le l \le j$,

$$S_{2,k,l}(n,\psi_l) = \sum_{\substack{r \bmod p_l^{\beta+1} \\ p_l \nmid r}} \psi_l(r) \left(\frac{r}{p_l}\right)^{\beta+1} \exp(2\pi inr/p_l^{\beta+1}), \quad (1 \le l \le j). \tag{3.9}$$

Then

a) *if* ψ_l *is the main character* mod p_l, *then*

$$\begin{aligned} S_{2,k,l}(n,\psi_l) = {} & (p_l - 1)p_l^\beta \ \ \textit{if} \ \ 2 \nmid \beta, \ 1 \le \beta \le 2t_l - 1, \\ & - p_l^{2t_l+1} \ \ \textit{if} \ \ \beta = 2t_l + 1, \ p_l \mid u, \end{aligned}$$

$$\left(\frac{u}{p_l}\right)\varepsilon_{p_l}p_l^{2t_l}\sqrt{p_l}\ \ \textit{if}\ \ \beta=2t_l,\ p_l\nmid u,$$
$$0\ \ \textit{otherwise};$$

b) *if* $\psi_l=\left(\frac{\cdot}{p_l}\right)$, *then*

$$\begin{aligned}S_{2,k,l}(n,\psi_l)=&(p_l-1)p_l^{\beta}\ \ \textit{if}\ \ 2\mid\beta,\ \beta\le 2t_l-2\ \textit{or}\ \beta=2t_l,\ p_l\mid u,\\&-p_l^{2t_l}\ \ \textit{if}\ \ \beta=2t_l,\ p_l\nmid u,\\&\left(\frac{u/p_l}{p_l}\right)\varepsilon_{p_l}p_l^{2t_l+1}\sqrt{p_l}\ \ \textit{if}\ \ \beta=2t_l+1,\ p_l\mid u,\\&0\ \ \textit{otherwise};\end{aligned}$$

c) *if* $\overline{\psi}_l\neq\psi_l$, *then*

$$\begin{aligned}S_{2,k,l}(n,\psi_l)=&\sqrt{p_l}p_l^{2t_l}\overline{\psi}_l(4^sM_ln_1^2u)\left(\frac{u}{p_l}\right)g\left(\psi_l\left(\frac{\cdot}{p_l}\right)\right)\ \ \textit{if}\ \ \beta=2t_l,\ \ p_l\nmid u,\\&\sqrt{p_l}p_l^{2t_l+1}\overline{\psi}_l(4^sM_ln_1^2u/p_l)g(\psi_l)\ \ \textit{if}\ \ \beta=2t_l+1,\ \ p_l\mid u,\\&0\ \ \textit{otherwise.}\end{aligned}$$

Proof. Let $r=r_1+r_2p_l$, r_1 mod p_l, $p_l\nmid r_1$, r_2 mod p_l^{β}. Then $\psi_l(r)=\psi_l(r_1)$, $\left(\frac{r}{p_l}\right)=\left(\frac{r_1}{p_l}\right)$, $\exp(2\pi inr/p_l^{\beta+1})=\exp(2\pi inr_1/p_l^{\beta+1})\exp(2\pi inr_2/p_l^{\beta})$ and

$$\begin{aligned}S_{2,k,l}(n,\psi_l)=&\sum_{\substack{r_1\bmod p_l\\p_l\nmid r_1}}\psi_l(r_1)\left(\frac{r_1}{p_l}\right)^{\beta+1}\\&\times\exp(2\pi inr_1/p_l^{\beta+1})\sum_{r_2\bmod p_l^{\beta}}\exp(2\pi inr_2/p_l^{\beta}).\end{aligned}\tag{3.10}$$

Now the result follows from (3.10), (1.1) and (1.3). □

Lemma 15. *Let*

$$S_{3,k}(n,\chi)=\sum_{\substack{\omega=1\\(2N,\omega)=1}}^{\infty}\left(\chi(\omega)\left(\frac{-1}{\omega}\right)\varepsilon_{\omega}^{k}\omega^{-\frac{k}{2}}\sum_{r\bmod\omega}\left(\frac{r}{\omega}\right)\exp(2\pi inr/\omega)\right).\tag{3.11}$$

Then

$$\begin{aligned}S_{3,k}(n,\chi)=&n_1^{2-k}\frac{\mathcal{L}\left(\frac{k-1}{2},\chi\left(\frac{(-1)^{\frac{k-1}{2}}u}{\cdot}\right)\right)}{\mathcal{L}(k-1,\chi^2)}\sum_{\delta d=n_1}\chi^2(\delta)d^{k-2}\\&\times\prod_{q\mid d}\left(1-\left(\frac{(-1)^{\frac{k-1}{2}}u}{q}\right)\chi(q)q^{\frac{1-k}{2}}\right).\end{aligned}\tag{3.12}$$

Proof. 1) First consider $S_{3,k}(u,\chi)$. Let $\omega=\omega_0\omega_1^2$, ω_0 be square-free. Then by Lemma 7 we get

$$S_{3,k}(u,\chi)=\sum_{\substack{\omega_0>0\\(2N,\omega_0)=1\\q^2\nmid\omega_0}}\sum_{\substack{\omega_1|u\\(2N,\omega_1)=1}}\chi(\omega_0)\chi^2(\omega_1)\left(\frac{-1}{\omega_0}\right)\varepsilon_{\omega_0}^{k+1}\omega_0^{-\frac{k}{2}}\omega_1^{-k}\left(\frac{u}{\omega_0}\right)\sqrt{\omega_0}\omega_1\mu(\omega_1)$$

$$=\sum_{\substack{\omega_0>0\\(2N,\omega_0)=1\\q^2\nmid\omega_0}}\chi(\omega_0)\left(\frac{-u}{\omega_0}\right)\left(\frac{(-1)^{\frac{k+1}{2}}}{\omega_0}\right)\omega_0^{\frac{1-k}{2}}\sum_{\omega_1|u}\chi^2(\omega_1)\mu(\omega_1)\omega_1^{1-k}$$

$$=\prod_{q|u}(1-\chi^2(q)q^{1-k})\sum_{\substack{\omega_0>0\\q^2\nmid\omega_0}}\chi(\omega_0)\left(\frac{(-1)^{\frac{k-1}{2}}u}{\omega_0}\right)\omega_0^{\frac{1-k}{2}}. \tag{3.13}$$

By (1.9) we have

$$\mathcal{L}\left(\frac{k-1}{2},\chi\left(\frac{(-1)^{\frac{k-1}{2}}u}{\cdot}\right)\right)=\sum_{\substack{\omega_0>0\\q^2\nmid\omega_0}}\sum_{\omega_1=1}^{\infty}\chi(\omega_0)\chi^2(\omega_1)\left(\frac{(-1)^{\frac{k-1}{2}}u}{\omega_0\omega_1^2}\right)\omega_0^{\frac{1-k}{2}}\omega_1^{1-k}$$

$$=\sum_{\substack{\omega_0>0\\q^2\nmid\omega_0}}\chi(\omega_0)\left(\frac{(-1)^{\frac{k-1}{2}}u}{\omega_0}\right)\omega_0^{\frac{1-k}{2}}\sum_{\substack{\omega_1=1\\(u,\omega_1)=1}}^{\infty}\chi^2(\omega_1)\omega_1^{1-k}$$

$$=\sum_{\substack{\omega_0>0\\q^2\nmid\omega_0}}\chi(\omega_0)\left(\frac{(-1)^{\frac{k-1}{2}}u}{\omega_0}\right)\omega_0^{\frac{1-k}{2}}\prod_{q|u}(1-\chi^2(q)q^{1-k})\mathcal{L}(k-1,\chi^2). \tag{3.14}$$

(3.13) and (3.14) imply

$$S_{3,k}(u,\chi)=\frac{\mathcal{L}\left(\frac{k-1}{2},\chi\left(\frac{(-1)^{\frac{k-1}{2}}u}{\cdot}\right)\right)}{\mathcal{L}(k-1,\chi^2)}. \tag{3.15}$$

2) Since $(2N,\omega)=1$, from (3.11) we obtain

$$S_{3,k}(2^{2s}Mu,\chi)=S_{3,k}(u,\chi). \tag{3.16}$$

3) Let $n=q^{2\alpha}n_0$, $q^2\nmid n_0$, $q>2$, $q\neq p_l$, $l=1,\dots,j$, $\omega=\omega_0q^{\beta}$, $q\nmid\omega_0$, $r=r_1q^{\beta}+r_2\omega_0$, $r_1 \bmod \omega_0$, $r_2 \bmod q^{\beta}$. It is easy to verify that if $2\nmid ab$, $(a,b)=1$, then

$$\varepsilon_{ab}=\left(\frac{a}{b}\right)\left(\frac{b}{a}\right)\varepsilon_a\varepsilon_b. \tag{3.17}$$

Therefore $\varepsilon^k_{\omega_0q^{\beta}}=\left(\frac{\omega_0}{q}\right)^{\beta}\cdot\left(\frac{q}{\omega_0}\right)^{\beta}\varepsilon^k_{\omega_0}\varepsilon^k_{q^{\beta}}$.

By Lemma 4, from (3.11) we have

$$S_{3,k}(n_0,\chi)=\sum_{\substack{\omega_0=1\\(2Nq,\omega_0)=1}}^{\infty}\left(\chi(\omega_0)\left(\frac{-1}{\omega_0}\right)\varepsilon^k_{\omega_0}\omega_0^{-\frac{k}{2}}\right.$$

$$\times \sum_{\beta=0}^{\infty} \chi^{\beta}(q) \left(\frac{-1}{q}\right)^{\beta} \cdot \left(\frac{q}{\omega_0}\right)^{\beta} \left(\frac{\omega_0}{q}\right)^{\beta} \varepsilon_{q^\beta}^{k} q^{-\frac{k}{2}\beta} \left(\frac{q}{\omega_0}\right)^{\beta} \left(\frac{\omega_0}{q}\right)^{\beta}$$

$$\times \sum_{r_1 \bmod \omega_0} \left(\frac{r_1}{\omega_0}\right) \exp(2\pi i n_0 r_1/\omega_0) \sum_{r_2 \bmod q^\beta} \left(\frac{r_2}{q}\right)^{\beta} \exp(2\pi i n_0 r_2/q^\beta)\Big)$$

$$= \sum_{\substack{\omega_0=1 \\ (2Nq,\omega_0)=1}}^{\infty} \left(\chi(\omega_0) \left(\frac{-1}{\omega_0}\right) \varepsilon_{\omega_0}^{k} \omega_0^{-\frac{k}{2}} \sum_{r_1 \bmod \omega_0} \left(\frac{r_1}{\omega_0}\right) \exp(2\pi i n_0 r_1/\omega_0)\right)$$

$$\times \sum_{\beta=0}^{\infty} \left(\chi^{\beta}(q) \left(\frac{-1}{q}\right)^{\beta} q^{-\frac{k}{2}\beta} \varepsilon_{q^\beta}^{k} \sum_{r_2 \bmod q^\beta} \left(\frac{r_2}{q}\right)^{\beta} \exp(2\pi i n_0 r_2/q^\beta)\right)$$

$$= A_1(n_0) A_2(n_0). \tag{3.18}$$

Let $r_2 = r_3 + q r_4$, $r_3 \bmod q$, $r_4 \bmod q^{\beta-1}$. Then

$$A_2(n_0) = 1 + \sum_{\beta=1}^{\infty} \left(\chi^{\beta}(q) \left(\frac{-1}{q}\right)^{\beta} q^{-\frac{k}{2}\beta} \varepsilon_{q^\beta}^{k} \sum_{r_3 \bmod q} \left(\frac{r_3}{q}\right)^{\beta} \exp(2\pi i n_0 r_3/q^\beta)\right.$$

$$\left.\times \sum_{r_4 \bmod q^{\beta-1}} \exp(2\pi i n_0 r_4/q^{\beta-1})\right). \tag{3.19}$$

Since $q^2 \nmid n_0$, by (1.3), (1.1) and Lemma 3 we get

$$A_2(n_0) = 1 + \chi(q) \left(\frac{-1}{q}\right) q^{-\frac{k}{2}} \varepsilon_q^k \sum_{r_3 \bmod q} \left(\frac{r_3}{q}\right) \exp(2\pi i n_0 r_3/q)$$

$$+ \chi^2(q) q^{-k} \sum_{\substack{r_3 \bmod q \\ q\nmid r_3}} \exp(2\pi i n_0 r_3/q^2) \sum_{r_4 \bmod q} \exp(2\pi i n_0 r_4/q)$$

$$= \begin{cases} 1 + \chi(q) \left(\frac{(-1)^{\frac{k-1}{2}} n_0}{q}\right) q^{\frac{1-k}{2}} & \text{if } q \nmid n_0; \\ 1 - \chi^2(q) q^{1-k} & \text{if } q \mid n_0. \end{cases} \tag{3.20}$$

Similarly to (3.18)

$$S_{3,k}(q^{2\alpha} n_0, \chi) = A_1(q^{2\alpha} n_0) A_2(q^{2\alpha} n_0) = A_1(n_0) A_2(q^{2\alpha} n_0), \tag{3.21}$$

because $q \nmid \omega_0$.

As in the case of (3.19),

$$A_2(q^{2\alpha} n_0) = 1 + \sum_{\beta=1}^{\infty} \left(\chi^{\beta}(q) \left(\frac{-1}{q}\right)^{\beta} q^{-\frac{k}{2}\beta} \varepsilon_{q^\beta}^{k} \sum_{r_3 \bmod q} \left(\frac{r_3}{q}\right)^{\beta} \exp(2\pi i q^{2\alpha} n_0 r_3/q^\beta)\right.$$

$$\left.\times \sum_{r_4 \bmod q^{\beta-1}} \exp(2\pi i q^{2\alpha} n_0 r_4/q^{\beta-1})\right). \tag{3.22}$$

By (1.3), (1.1), Lemma 3 and (3.20), from (3.22) we obtain:

a) if $q \mid n_0$, then

$$\begin{aligned}
A_2(q^{2\alpha}n_0) &= 1+\sum_{\beta=1}^{\alpha}\chi^{2\beta}(q)q^{(2-k)\beta-1}(q-1)-\chi^{2\alpha+2}(q)q^{(2-k)\alpha}q^{1-k}\\
&= (1-\chi^2(q)q^{1-k})\chi^{2\alpha}(q)q^{(2-k)\alpha}-\chi^{2\alpha}(q)q^{(2-k)\alpha}\\
&\quad+\sum_{\beta=0}^{\alpha}\chi^{2\beta}(q)q^{(2-k)\beta}-\chi^2(q)q^{1-k}\sum_{\beta=1}^{\alpha}\chi^{2(\beta-1)}(q)q^{(2-k)(\beta-1)}\\
&= (1-\chi^2(q)q^{1-k})\sum_{\beta=0}^{\alpha}\chi^{2\beta}(q)q^{(2-k)\beta}=A_2(n_0)q^{(2-k)\alpha}\sum_{\beta=0}^{\alpha}\chi^{2\beta}(q)q^{(k-2)(\alpha-\beta)}\\
&= A_2(n_0)q^{(2-k)\alpha}\sum_{\delta d=q^\alpha}\chi^2(\delta)d^{k-2}; \qquad (3.23)
\end{aligned}$$

b) if $q \nmid n_0$, then

$$\begin{aligned}
A_2(q^{2\alpha}n_0) &= 1+\sum_{\beta=1}^{\alpha}\chi^{2\beta}(q)q^{(2-k)\beta-1}(q-1)+\chi^{2\alpha+1}(q)q^{(2-k)\alpha}q^{\frac{1-k}{2}}\left(\frac{(-1)^{\frac{k-1}{2}}n_0}{q}\right)\\
&= \left(1+\chi(q)\left(\frac{(-1)^{\frac{k-1}{2}}n_0}{q}\right)q^{\frac{1-k}{2}}\right)\chi^{2\alpha}(q)q^{(2-k)\alpha}-\chi^{2\alpha}(q)q^{(2-k)\alpha}\\
&\quad+\sum_{\beta=0}^{\alpha}\chi^{2\beta}(q)q^{(2-k)\beta}-\chi^2(q)q^{1-k}\sum_{\beta=1}^{\alpha}\chi^{2(\beta-1)}(q)q^{(2-k)(\beta-1)}\\
&= \left(1+\chi(q)\left(\frac{(-1)^{\frac{k-1}{2}}n_0}{q}\right)q^{\frac{1-k}{2}}\right)\chi^{2\alpha}(q)q^{(2-k)\alpha}\\
&\quad+(1-\chi^2(q)q^{1-k})\sum_{\beta=0}^{\alpha-1}\chi^{2\beta}(q)q^{(2-k)\beta}\\
&= \left(1+\chi(q)\left(\frac{(-1)^{\frac{k-1}{2}}n_0}{q}\right)q^{\frac{1-k}{2}}\right)\Bigg(\chi^{2\alpha}(q)q^{(2-k)\alpha}\\
&\quad+\left(1-\chi(q)\left(\frac{(-1)^{\frac{k-1}{2}}n_0}{q}\right)q^{\frac{1-k}{2}}\right)q^{(2-k)\alpha}\sum_{\beta=0}^{\alpha-1}\chi^{2\beta}(q)q^{(k-2)(\alpha-\beta)}\Bigg)\\
&= A_2(n_0)q^{(2-k)\alpha}\sum_{\delta d=q^\alpha}\chi^2(\delta)d^{k-2}\prod_{q_1\mid d}\left(1-\chi(q_1)\left(\frac{(-1)^{\frac{k-1}{2}}n_0}{q_1}\right)q_1^{\frac{1-k}{2}}\right). \qquad (3.24)
\end{aligned}$$

If $q_1 \mid n_0$, then $\left(\frac{(-1)^{\frac{k-1}{2}}n_0}{q_1}\right)=0$, therefore (3.24) contains (3.23).

Taking into account that $f_1(\delta)=\chi^2(\delta)$, $f_2(\delta)=\delta^{k-2}$ and $f_3(d)=\prod_{q_1\mid d}\Big(1-\chi(q_1)\left(\frac{(-1)^{\frac{k-1}{2}}n_0}{q_1}\right)q_1^{\frac{1-k}{2}}\Big)$ are multiplicative functions, from (3.15), (3.16), (3.21) and (3.24) follows (3.12). □

Note that if $\chi = \chi_0$, then by (1.9), (1.10)

$$\mathcal{L}\left(\frac{k-1}{2}, \chi_0\left(\frac{(-1)^{\frac{k-1}{2}}u}{\cdot}\right)\right)$$

$$= \prod_{l=1}^{j}\left(1 - \left(\frac{(-1)^{\frac{k-1}{2}}u}{p_l}\right)p_l^{\frac{1-k}{2}}\right)\mathcal{L}\left(\frac{k-1}{2}, (-1)^{\frac{k-1}{2}}u\right); \tag{3.25}$$

$$\mathcal{L}(k-1, \chi_0^2) = \prod_{l=1}^{j}(1 - p_l^{1-k})(1 - 2^{1-k})\zeta(k-1); \tag{3.26}$$

if $\chi = \left(\frac{4N}{\cdot}\right)$, then $\chi^2 = \chi_0$ and by (1.9), (1.10)

$$\mathcal{L}\left(\frac{k-1}{2}, \chi\left(\frac{(-1)^{\frac{k-1}{2}}u}{\cdot}\right)\right) = \mathcal{L}\left(\frac{k-1}{2}, (-1)^{\frac{k-1}{2}}uN\right). \tag{3.27}$$

Proposition 2. *Let* $\chi(n) = \phi(n)\psi_{1,l}(n)\psi_{2,l}(n)$, *where* $\psi_{1,l}$ *is a character* mod N_l $(N_l \mid N)$ *and* $\psi_{2,l}$ *is a character* mod N/N_l, $\chi(-1) = 1$, $2 \nmid k$; *then the system of functions*

$$E_1\left(\tau; \tfrac{k}{2}, 4N, \chi\right) = \sum_{\substack{4N|m \\ n>0,\,(m,n)=1}} \overline{\chi}(n)\left(\frac{m}{n}\right)\varepsilon_n^k(m\tau+n)^{-k/2}, \tag{3.28}$$

$$E_{2,l}\left(\tau; \tfrac{k}{2}, 4N, \chi\right) = \sum_{\substack{m>0,n \\ (2N_l m, Nn/N_l)=1}} \phi(n)\overline{\psi}_{1,l}(n)\psi_{2,l}(m)\left(\frac{4N_l m}{n}\right) \times \varepsilon_n^k(4N_l m\tau+n)^{-k/2} \tag{3.29}$$

$(N_l \mid N,\ \ N_l \neq N,\ \ l = 1, \ldots, 2^j - 1,\ \ N_{l_1} \neq N_{l_2},\ \ \textit{when}\ \ l_1 \neq l_2)$,

$$E_{3,l}\left(\tau; \tfrac{k}{2}, 4N, \chi\right) = \sum_{\substack{m>0,n \\ (N_l m, 2Nn/N_l)=1}} \phi(m)\overline{\psi}_{1,l}(n)\psi_{2,l}(m)\left(\frac{-n}{mN_l}\right) \times \varepsilon_{mN_l}^k(N_l m\tau+n)^{-k/2} \tag{3.30}$$

$(N_l \mid N,\ \ l = 1, \ldots, 2^j,\ \ N_{l_1} \neq N_{l_2},\ \ \textit{when}\ \ l_1 \neq l_2)$

is the basis of the space $E_{k/2}(\widetilde{\Gamma}_0(4N), \chi)$ *for any odd* $k \geq 5$.

This proposition is proved exactly as Proposition 1.

Next, we derive Fourier expansions of functions (3.28)–(3.30).

Let $a_k = \pi^{k/2}\exp(-\pi ik/4)\Gamma\left(\frac{k}{2}\right)^{-1}$.

1) if $2 \nmid n$, $n < 0$, then by (3.17) we obtain

$$\varepsilon_{-n} = \left(\frac{-1}{n}\right)\left(\frac{n}{-1}\right)\varepsilon_n\varepsilon_{-1} = -\left(\frac{-1}{|n|}\right)(-1)\varepsilon_n \cdot i = \left(\frac{-1}{-n}\right)i\varepsilon_n. \tag{3.31}$$

It is easy to verify that if $m > 0$, $n < 0$, $\tau \in \mathbb{H}$, then

$$\sqrt{-m\tau - n} \cdot i = \sqrt{m\tau + n}\,. \tag{3.32}$$

Since $\overline{\chi}(-n)=\overline{\chi}(n)$, by (3.31) and (3.32) we have

$$\sum_{\substack{4N|m,\,n>0\\ m<0,\,(m,n)=1}} \overline{\chi}(n)\left(\frac{m}{n}\right)\varepsilon_n^k(m\tau+n)^{-k/2}=\sum_{\substack{4N|m,\,n<0\\ m>0,\,(m,n)=1}} \overline{\chi}(n)\left(\frac{m}{n}\right)\varepsilon_n^k(m\tau+n)^{-k/2};$$

therefore

$$E_1\left(\tau;\tfrac{k}{2},4N,\chi\right)=1+\sum_{\substack{4N|m,\,m>0\\ n,\,(m,n)=1}} \overline{\chi}(n)\left(\frac{m}{n}\right)\varepsilon_n^k(m\tau+n)^{-k/2}$$
$$=1+\sum_{\substack{m>0\\ n,\,(2mN,n)=1}} \overline{\chi}(n)\left(\frac{4Nm}{n}\right)\varepsilon_n^k(4Nm\tau+n)^{-k/2}.$$

Let $n=r+4Nmh$, $r \bmod 4Nm$, $h\in\mathbb{Z}$. Then $\overline{\chi}(n)=\overline{\chi}(r)$, $\varepsilon_n=\varepsilon_r$ and by Lemma 5 we get

$$E_1\left(\tau;\tfrac{k}{2},4N,\chi\right)=1+(4N)^{-\frac{k}{2}}\sum_{m=1}^{\infty}\Bigg(m^{-\frac{k}{2}}\sum_{\substack{r \bmod 4Nm\\ (2N,r)=1}}\overline{\chi}(r)\left(\frac{N}{r}\right)\left(\frac{m}{r}\right)\varepsilon_r^k$$
$$\times\sum_{h=-\infty}^{\infty}\left(\tau+\frac{r}{4Nm}+h\right)^{-\frac{k}{2}}\Bigg)$$
$$=1+(2N)^{-\frac{k}{2}}\pi^{\frac{k}{2}}\exp\left(-\frac{\pi ik}{4}\right)\Gamma\left(\frac{k}{2}\right)^{-1}\sum_{n=1}^{\infty}\Bigg(n^{\frac{k}{2}-1}\sum_{m=1}^{\infty}m^{-\frac{k}{2}}$$
$$\times\sum_{\substack{r \bmod 4Nm\\ (2N,r)=1}}\overline{\chi}(r)\left(\frac{N}{r}\right)\left(\frac{m}{r}\right)\varepsilon_r^k\exp(2\pi inr/(4Nm))\Bigg)z^n$$
$$=1+(2N)^{-\frac{k}{2}}a_k\sum_{n=1}^{\infty}\Bigg(n^{\frac{k}{2}-1}\sum_{\alpha=0}^{\infty}\sum_{\beta_1=0}^{\infty}\cdots\sum_{\beta_j=0}^{\infty}\sum_{\substack{\omega=1\\ (2N,\omega)=1}}^{\infty}(2^{\alpha}p_1^{\beta_1}\cdots p_j^{\beta_j}\omega)^{-\frac{k}{2}}$$
$$\times\sum_{\substack{r \bmod 2^{\alpha+2}p_1^{\beta_1+1}\cdots p_j^{\beta_j+1}\omega\\ (2N,r)=1}}\overline{\chi}(r)\left(\frac{2}{r}\right)^{\alpha}\left(\frac{p_1}{r}\right)^{\beta_1+1}\cdots\left(\frac{p_j}{r}\right)^{\beta_j+1}\left(\frac{\omega}{r}\right)\varepsilon_r^k$$
$$\times\exp(2\pi inr/(2^{\alpha+2}p_1^{\beta_1+1}\cdots p_j^{\beta_j+1}\omega))\Bigg)z^n. \tag{3.33}$$

Then, as in Proposition 1,

$$r=\omega\cdot 2^{\alpha+2}\sum_{h=1}^{j}r_h\prod_{\substack{h_1=1\\ h_1\neq h}}^{j}p_{h_1}^{\beta_{h_1}+1}+\omega r_{j+1}\prod_{h=1}^{j}p_h^{\beta_h+1}+2^{\alpha+2}r_{j+2}\prod_{h=1}^{j}p_h^{\beta_h+1},$$
$$0\le r_{j+2}<\omega,\quad 0\le r_h<p_h^{\beta_h+1},\quad 1\le h\le j,\quad 0\le r_{j+1}<2^{\alpha+2}.$$

$2\nmid r_{j+1}$, $p_h\nmid r_h$, since $(2N,r)=1$. After a simple calculation we obtain

$$\overline{\chi}(r)\left(\frac{2}{r}\right)^{\alpha}\left(\frac{p_1}{r}\right)^{\beta_1+1}\cdots\left(\frac{p_j}{r}\right)^{\beta_j+1}\left(\frac{\omega}{r}\right)\varepsilon_r^k$$
$$\times\exp(2\pi inr/(2^{\alpha+2}p_1^{\beta_1+1}\cdots p_j^{\beta_j+1}\omega))=\phi(N)\overline{\psi}(4)\overline{\chi}(\omega)$$
$$\times\left(\frac{-1}{\omega}\right)\varepsilon_\omega^k\left(\frac{r_{j+2}}{\omega}\right)\exp(2\pi inr_{j+2}/\omega)\overline{\psi}^{\alpha}(2)\phi(r_{j+1})\left(\frac{2}{r_{j+1}}\right)^{\alpha}\varepsilon_{r_{j+1}}^k$$
$$\times\exp(2\pi inr_{j+1}/2^{\alpha+2})\prod_{h=1}^{j}\Bigg(\phi^{\beta_h}(p_h)\left(\frac{-1}{p_h}\right)^{\beta_h+1}\varepsilon_{p_h^{\beta_h+1}}^k\overline{\psi}_h(r_h)\left(\frac{r_h}{p_h}\right)^{\beta_h+1}$$
$$\times\prod_{\substack{h_1=1\\h_1\neq h}}^{j}\left(\frac{p_{h_1}}{p_h}\right)^{(\beta_{h_1}+1)(\beta_h+1)}\overline{\psi}_h(p_{h_1}^{\beta_{h_1}+1})\exp(2\pi inr_h/p_h^{\beta_h+1})\Bigg).\quad(3.34)$$

From (3.33) and (3.34) we have

$$E_1\left(\tau;\tfrac{k}{2},4N,\chi\right)=1+(2N)^{-\frac{k}{2}}a_k\sum_{n=1}^{\infty}\Bigg(n^{\frac{k}{2}-1}\phi(N)\overline{\psi}(4)$$
$$\times\sum_{\substack{\omega=1\\(2N,\omega)=1}}^{\infty}\omega^{-\frac{k}{2}}\overline{\chi}(\omega)\left(\frac{-1}{\omega}\right)\varepsilon_\omega^k\sum_{r_{j+2}\bmod\omega}\left(\frac{r_{j+2}}{\omega}\right)\exp(2\pi inr_{j+2}/\omega)$$
$$\times\sum_{\alpha=0}^{\infty}2^{-\frac{k}{2}\alpha}\overline{\psi}^{\alpha}(2)\sum_{\substack{r_{j+1}\bmod 2^{\alpha+2}\\2\nmid r_{j+1}}}\phi(r_{j+1})\left(\frac{2}{r_{j+1}}\right)^{\alpha}\varepsilon_{r_{j+1}}^k\exp(2\pi inr_{j+1}/2^{\alpha+2})$$
$$\times\sum_{\beta_1=0}^{\infty}\cdots\sum_{\beta_j=0}^{\infty}\prod_{h=1}^{j}\Bigg(p_h^{-\frac{k}{2}\beta_h}\phi^{\beta_h}(p_h)\left(\frac{-1}{p_h}\right)^{\beta_h+1}\varepsilon_{p_h^{\beta_h+1}}^k$$
$$\times\prod_{\substack{h_1=1\\h_1\neq h}}^{j}\left(\frac{p_{h_1}}{p_h}\right)^{(\beta_{h_1}+1)(\beta_h+1)}\overline{\psi}_{h_1}(p_h^{\beta_h+1})$$
$$\times\sum_{\substack{r_h\bmod p_h^{\beta_h+1}\\p_h\nmid r_h}}\overline{\psi}_h(r_h)\left(\frac{r_h}{p_h}\right)^{\beta_h+1}\exp(2\pi inr_h/p_h^{\beta_h+1})\Bigg)\Bigg)z^n$$
$$=1+(2N)^{-k/2}a_k\phi(N)\overline{\psi}(4)\sum_{n=1}^{\infty}\Bigg(n^{\frac{k}{2}-1}S_{1,k}(n,\overline{\chi})S_{3,k}(n,\overline{\chi})$$
$$\times\sum_{\beta_1=0}^{2t_1+1}\cdots\sum_{\beta_j=0}^{2t_j+1}\prod_{h=1}^{j}\Bigg(p_h^{-\frac{k}{2}\beta_h}\phi^{\beta_h}(p_h)\left(\frac{-1}{p_h}\right)^{\beta_h+1}\varepsilon_{p_h^{\beta_h+1}}^k$$
$$\times\prod_{\substack{h_1=1\\h_1\neq h}}^{j}\left(\frac{p_{h_1}}{p_h}\right)^{(\beta_{h_1}+1)(\beta_h+1)}\overline{\psi}_{h_1}(p_h^{\beta_h+1})S_{2,k,h}(n,\overline{\psi}_h)\Bigg)\Bigg)z^n.\quad(3.35)$$

2) Let $\chi_1=\phi\overline{\psi}_{1,l}\psi_{2,l}$; if $N_l=1$, $d=0$ or $N_l=p_{\delta_1}\cdots p_{\delta_d}$, $1\le d\le j-1$, then similarly to the above we obtain

$$E_{2,l}\left(\tau;\tfrac{k}{2},4N,\chi\right)=(2N_l)^{-k/2}a_k\phi(N_l)\overline{\psi}_{1,l}(4)\sum_{n=1}^{\infty}\Bigg(n^{\frac{k}{2}-1}S_{1,k}(n,\chi_1)S_{3,k}(n,\chi_1)$$
$$\times\sum_{\beta_1=0}^{2t_{\delta_1}+1}\cdots\sum_{\beta_d=0}^{2t_{\delta_d}+1}\prod_{h=1}^{d}\Bigg(p_{\delta_h}^{-\frac{k}{2}\beta_h}\phi^{\beta_h}(p_{\delta_h})\left(\frac{-1}{p_{\delta_h}}\right)^{\beta_h+1}\varepsilon^k_{p_{\delta_h}^{\beta_h+1}}$$
$$\times\prod_{\substack{h_1=1\\h_1\ne h}}^{d}\left(\frac{p_{\delta_{h_1}}}{p_{\delta_h}}\right)^{(\beta_{h_1}+1)(\beta_h+1)}\overline{\psi}_{\delta_h}(p_{\delta_{h_1}}^{\beta_{h_1}+1})S_{2,k,\delta_h}(n,\overline{\psi}_{\delta_h})\Bigg)\Bigg)z^n,\tag{3.36}$$

and if $N_l=1$, $d=0$ or $N_l=p_{\delta_1}\cdots p_{\delta_d}$, $1\le d\le j$, then

$$E_{3,l}\left(\tau;\tfrac{k}{2},4N,\chi\right)=2^{k/2}N_l^{-k/2}a_k\sum_{n=1}^{\infty}\Bigg(n^{\frac{k}{2}-1}S_{3,k}(n,\chi_1)$$
$$\times\sum_{\beta_1=0}^{2t_{\delta_1}+1}\cdots\sum_{\beta_d=0}^{2t_{\delta_d}+1}\prod_{h=1}^{d}\Bigg(p_{\delta_h}^{-\frac{k}{2}\beta_h}\phi^{\beta_h}(p_{\delta_h})\left(\frac{-1}{p_{\delta_h}}\right)^{\beta_h+1}\varepsilon^k_{p_{\delta_h}^{\beta_h+1}}$$
$$\times\prod_{\substack{h_1=1\\h_1\ne h}}^{d}\left(\frac{p_{\delta_h}}{p_{\delta_{h_1}}}\right)^{(\beta_{h_1}+1)(\beta_h+1)}\overline{\psi}_{\delta_h}(p_{\delta_{h_1}}^{\beta_{h_1}+1})S_{2,k,\delta_h}(n,\overline{\psi}_{\delta_h})\Bigg)\Bigg)z^n.\tag{3.37}$$

For $N=p$, we obtain the following results.
Let now $n=2^{2s}p^{2t}n_1^2u$, $(2p,n_1)=1$, $q^2\nmid u$,

$$A_{1,k}(u,\chi)=\frac{\pi^{\frac{k}{2}}u^{\frac{k}{2}-1}}{\Gamma(\frac{k}{2})2^{\frac{k-1}{2}}}\left(\frac{2}{k}\right)\frac{\mathcal{L}\Big(\frac{k-1}{2},\chi\Big(\frac{(-1)^{\frac{k-1}{2}}u}{\cdot}\Big)\Big)}{\mathcal{L}(k-1,\chi^2)}\sum_{\delta d=n_1}\chi^2(\delta)d^{k-2}$$
$$\times\prod_{q\mid d}\left(1-\left(\frac{(-1)^{\frac{k-1}{2}}u}{q}\right)\chi(q)q^{\frac{1-k}{2}}\right),$$

$$d_{1,k}(u)=\begin{cases}-1 & \text{when } p\mid u,\\ \left(\frac{(-1)^{\frac{k-1}{2}}u}{p}\right)p^{\frac{k-1}{2}} & \text{when } p\nmid u,\end{cases}$$

$$d_{2,k}(u)=\begin{cases}-1 & \text{when } p\nmid u,\\ p-1+\left(\frac{(-1)^{\frac{k-1}{2}}u/p}{p}\right)p^{\frac{3-k}{2}} & \text{when } p\mid u,\end{cases}$$

$$d_{3,k}(u)=\begin{cases}\left(\frac{(-1)^{\frac{k+1}{2}}u}{p}\right)\psi(u)\varepsilon_p\sqrt{p}\,g\left(\overline{\psi}\left(\frac{\cdot}{p}\right)\right) & \text{when } p\nmid u,\\ \phi(p)\psi\left(\frac{u}{p}\right)p^{\frac{3-k}{2}}g(\overline{\psi}) & \text{when } p\mid u,\end{cases}$$

$$A_k(u) = \frac{2^{\frac{k+1}{2}}\left(\frac{k-1}{2}\right)!\,u^{\frac{k}{2}-1}}{(p^{k-1}-1)(2^{k-1}-1)\pi^{\frac{k-1}{2}}B_{\frac{k-1}{2}}}\left(\frac{2}{k}\right)\left(1-\left(\frac{(-1)^{\frac{k-1}{2}}u}{p}\right)p^{\frac{1-k}{2}}\right)$$
$$\times\mathcal{L}\left(\frac{k-1}{2},(-1)^{\frac{k-1}{2}}u\right)\sum_{d|n_1}d^{k-2}\prod_{q|d}\left(1-\left(\frac{(-1)^{\frac{k-1}{2}}u}{q}\right)q^{\frac{1-k}{2}}\right),$$

where $B_{\frac{k-1}{2}}$ are Bernoulli numbers;

$$B_{r,k}(u) = p^{k-2}(p-1)\cdot\frac{p^{(k-2)t}-1}{p^{k-2}-1}+d_{r,k}(u)\quad(r=1,2),$$
$$C_{1,k}(u,\chi) = 2^{k-2}\overline{\psi}(4)\cdot\frac{2^{(k-2)s}\overline{\psi}^s(4)-1}{2^{k-2}\overline{\psi}(4)-1}+c_k(u,\chi),$$

where $c_k(u,\chi)$ is defined by Lemma 13.

Corollary 5. *If $\chi\neq\overline{\chi}$, $\chi(-1)=1$, then*

$$E_1\left(\tau;\tfrac{k}{2},4p,\chi\right) = 1+\frac{\left(1+\phi(3)\left(\frac{-1}{k}\right)i\right)p^{-\frac{k}{2}}}{\left(1+\left(\frac{-1}{k}\right)i\right)\psi(4)}\phi(p)$$
$$\times\sum_{n=1}^{\infty}A_{1,k}(u,\overline{\chi})C_{1,k}(u,\overline{\chi})\psi^2(n_1)d_{3,k}(u)z^n,$$
$$E_2\left(\tau;\tfrac{k}{2},4p,\chi\right) = \frac{1+\phi(3)\left(\frac{-1}{k}\right)i}{1+\left(\frac{-1}{k}\right)i}\sum_{n=1}^{\infty}p^{(k-2)t}\psi^s(4)A_{1,k}(u,\chi)C_{1,k}(u,\chi)z^n,$$
$$E_{3,1}\left(\tau;\tfrac{k}{2},4p,\chi\right) = \frac{2^k}{1+\left(\frac{-1}{k}\right)i}\sum_{n=1}^{\infty}2^{(k-2)s}p^{(k-2)t}A_{1,k}(u,\chi)z^n,$$
$$E_{3,2}\left(\tau;\tfrac{k}{2},4p,\chi\right) = \frac{2^kp^{-k/2}}{1+\left(\frac{-1}{k}\right)i}\sum_{n=1}^{\infty}2^{(k-2)s}\psi^s(4)\psi^2(n_1)A_{1,k}(u,\overline{\chi})d_{3,k}(u)z^n.$$

Proof immediately follows from (3.35)–(3.37) and Lemmas 13–15, since $\exp\left(\frac{\pi ik}{4}\right)=\left(1+\left(\frac{-1}{k}\right)i\right)\left(\frac{2}{k}\right)\cdot2^{-\frac{1}{2}}$ when $2\nmid k$. □

Corollary 6. *If $\chi=\chi_0$, then*

$$E_1\left(\tau;\tfrac{k}{2},4p,\chi_0\right) = 1+\sum_{n=1}^{\infty}A_k(u)B_{1,k}(u)C_{1,k}(u,\chi_0)z^n,$$
$$E_2\left(\tau;\tfrac{k}{2},4p,\chi_0\right) = p^{k-1}\sum_{n=1}^{\infty}p^{(k-2)t}A_k(u)C_{1,k}(u,\chi_0)z^n,$$
$$E_{3,1}\left(\tau;\tfrac{k}{2},4p,\chi_0\right) = 2^kp^{k-1}\left(1+\left(\frac{-1}{k}\right)i\right)^{-1}\sum_{n=1}^{\infty}2^{(k-2)s}p^{(k-2)t}A_k(u)z^n,$$
$$E_{3,2}\left(\tau;\tfrac{k}{2},4p,\chi_0\right) = 2^k\left(1+\left(\frac{-1}{k}\right)i\right)^{-1}\sum_{n=1}^{\infty}2^{(k-2)s}A_k(u)B_{1,k}(u)z^n.$$

Proof directly follows from (3.35)–(3.37) and Lemmas 13–15, taking $\chi = \chi_0$. □

Corollary 7. *If* $\chi(n) = \left(\frac{4p}{n}\right)$, *then*

$$E_1\left(\tau; \tfrac{k}{2}, 4p, \chi\right) = 1 + \sum_{n=1}^{\infty} A_k(up) B_{2,k}(u) C_{1,k}(u,\chi) z^n,$$

$$E_2\left(\tau; \tfrac{k}{2}, 4p, \chi\right) = p^{k/2} \frac{1 + (\frac{-1}{pk})i}{1 + (\frac{-1}{k})i} \sum_{n=1}^{\infty} p^{(k-2)t} A_k(up) C_{1,k}(u,\chi) z^n,$$

$$E_{3,1}\left(\tau; \tfrac{k}{2}, 4p, \chi\right) = 2^k p^{k/2} \left(1 + \left(\frac{-1}{k}\right) i\right)^{-1} \sum_{n=1}^{\infty} 2^{(k-2)s} p^{(k-2)t} A_k(up) z^n,$$

$$E_{3,2}\left(\tau; \tfrac{k}{2}, 4p, \chi\right) = 2^k \varepsilon_p \left(\frac{-1}{p}\right)^{\frac{k+1}{2}} \left(1 + \left(\frac{-1}{k}\right) i\right)^{-1} \sum_{n=1}^{\infty} 2^{(k-2)s} A_k(up) B_{2,k}(u) z^n.$$

Proof. If $p \equiv 1 \pmod 4$, then $\chi(d) = \phi_0(d)\left(\frac{d}{p}\right)$, and if $p \equiv -1 \pmod 4$, then $\chi(d) = \left(\frac{-1}{d}\right)\left(\frac{d}{p}\right)$. Therefore $\chi(d) = \left(\frac{-1}{d}\right)^{\frac{p-1}{2}} \cdot \left(\frac{d}{p}\right)$, $\left(\frac{-1}{p}\right)^{\frac{p-1}{2}} = \left(\frac{-1}{p}\right)$ and the result follows from (3.35)–(3.37) and Lemmas 13–15. □

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