Georgian Mathematical Journal
Volume 13 (2006), Number 1, 79–84

ON THE FULL TRANSITIVITY OF A COTORSION HULL

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Abstract. A cotorsion hull of the separable p-group T is considered when T is a direct sum of torsion-complete groups. It is proved that in the considered case its cotorsion hull is fully transitive if and only if T is a direct sum of cyclic groups or is a torsion-complete group.

2000 Mathematics Subject Classification: 20K10, 20K21.
Key words and phrases: Cotorsion hull, torsion-complete group, full transitivity of a group.

In the paper, under the word "group" is meant the additively written Abelian group. We use the notation and terminology from the monographs [1], [2].

A group A is called a *cotorsion* one if its extension by means of any torsion-free group C is decomposed as follows: $\operatorname{Ext}(C, A) = 0$. The importance of the class of cotorsion groups in the class of Abelian groups depends on two factors:

1. For any groups A, B, the group $\operatorname{Ext}(A, B)$ is a cotorsion one.

2. Any reduced group G is isomorphically imbeddable into the group $G^* = \operatorname{Ext}(Q/Z, G)$ called a *cotorsion hull* of the group G and, in addition, G^*/G is a torsion-free divisible group. Any reduced cotorsion group C is decomposed into a direct sum $C = A \oplus G$, where A a torsion-free algebraic compact group and $G \cong \operatorname{Ext}(Q/Z, tG)$, where tG is the torsion part of the group G. If $tG = \bigoplus_p T_p$ is the decomposition into a direct sum of primary components, then

$$\operatorname{Ext}(Q/Z, tG) \cong \prod_p \operatorname{Ext}(Z(p^\infty), T_p).$$

Thus the investigation of cotorsion groups reduces essentially to the consideration of groups of the form $\operatorname{Ext}(Z(p^\infty), T)$, where T is a p-primary group.

Let p be a fixed prime number. *The indicator or the Ulm sequence* of an element a of the group A is the increasing sequence of natural numbers and symbols ∞,

$$H_A(a) \equiv H(a) = (h(a), h(pa), \dots, h(p^n a), \dots),$$

where h denotes a generalized p-height of the element, i.e., $h(a) = \sigma$ if $a \in p^\sigma A \setminus p^{\sigma+1} A$ and $h(0) = \infty$. In the set of indicators we define the order as follows:

$$H(a) \le H(b) \iff h(p^i a) \le h(p^i b), \quad i = 0, 1, 2, \dots .$$

The reduced group A is called fully transitive if for its arbitrary elements a and b, when $H(a) \le H(b)$ for any prime p, there exists an endomorphism of the group φ such that $\varphi a = b$.

ISSN 1072-947X / $8.00 /

In the p-adic topology, a complement of the p-group T is denoted by $\widehat{T}$. Its torsion part $\overline{T}$ is a torsion-complete group. In papers [3] and [4] it is respectively shown that if T is a torsion-complete group or a direct sum of cyclic p-groups, then its cotorsion hull $T^* = \mathrm{Ext}(Z(p^\infty), T)$ is fully transitive. In fully transitive groups, the lattice of characteristic subgroups is described by means of indicators.

In particular, I. Kaplansky gave a description of fully invariant subgroups of the fully transitive p-group A. Any such subgroup has the form

$$A(u) = \{a \in A \mid H(a) \geq u\},$$

where $u = (\sigma_0, \sigma_1, \dots, \sigma_n, \dots)$ is an increasing sequence of natural numbers and symbols ∞, which satisfies the condition: if between σ_n and σ_{n+1} there is a gap, then in A there exists an element of order p and height σ_n (see [2, Theorem 67.1]).

Having considered I. Kaplansky's proof, A. Mader [3] formulated the following theorem:

Let A be a module over the commutative ring R, M be the lattice of all fully characteristic submodules, S be some sublattice (with respect to intersections) and $U : A \to S$ be a mapping with the following properties:

1. *U is surjective;*
2. *$U(fa) \geq U(a)$ for any $a \in A$ and any endomorphism f of the module A;*
3. *$U(a + b) \geq U(a) \wedge U(b)$;*
4. *if $U(a) \geq U(b)$, then there exists an endomorphism f of the module A such that $f(b) = a$;*
5. *if $C \in M$, then for any $a, b \in C$ there exists $c \in C$ such that $U(c) = U(a) \wedge U(b)$.*

Then the set S^* of all dual ideals S, ordered with respect to the inclusion, is a lattice and the mapping $\alpha : S^* \to M$ defined by the rule $\alpha(D) = \{a \in A \mid U(a) \in D\}$ is a lattice isomorphism.

When T is a torsion-complete group or a direct sum of cyclic p-groups, A. Mader [3] and A. I. Moskalenko [4] also used the indicator function $H = U$ to describe a lattice of fully invariant subgroups of the cotorsion hull T^*, since in such cases, as has been noted above, T^* is fully transitive and all the conditions of A. Mader's theorem are fulfilled.

In this paper the question of full transitivity of the group T^* is studied when T is a direct sum of torsion-complete groups:

$$T = \underset{j\in J}{\oplus} \overline{B}_j, \tag{1}$$

where J is any fixed set of indices and B_j is the basic subgroup of the group $\overline{B}_j$: $B_j = \overset{\infty}{\underset{n=1}{\oplus}} B_{nj}$, $B_{nj} = \oplus Z(p^n)$, $B = \underset{j\in J}{\oplus} B_j = \underset{j\in J}{\oplus} \overset{\infty}{\underset{n=1}{\oplus}} B_{nj} = \overset{\infty}{\underset{n=1}{\oplus}} \underset{j\in J}{\oplus} B_{nj}$. As is known, here the basic subgroup of the group T is B.

The case of the finite J is considered in [3]. In what follows, J is assumed infinite.

Any element of a p-adic complement $\widehat{T}$ of the group T can be written in the form of the infinite vector $b = (b_1, b_2, \dots, b_n, \dots)$, where $b_n \in \bigoplus_{j\in J} B_{nj}$, $h(b_n) \to \infty$ as $n \to \infty$. We write this element also in the form

$$b = b_1 + b_2 + \dots = \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\sum_{j\in J} b_{nj}, \tag{2}$$

where for each n, the sum $\sum_{j\in J} b_{nj}$ with $b_{nj} \in B_{nj}$, $j \in J$, contains only a finite number of nonzero elements; $h(b_{nj}) \to \infty$ as $n \to \infty$. Note that in notation (2), when $b \in \overline{T}$ and the order $\mathcal{O}(b) = p^m$, each summand is p^m-bounded, $p^m\big(\sum_{j\in J} b_{nj}\big) = 0$, $n = 1, 2, \dots$. If however $b \in T$, the index j takes values only from a finite subset of indices.

For the separable p-group T, A. I. Moskalenko [4] represented elements T^* in the form of countable sequences

$$T^* = \Big\{(a_0, a_1 + T, \dots, a_i + T, \dots) \mid a_i \in \widehat{T},\ \ pa_{i+1} - a_i \in T;\ \ i = 0, 1, \dots\Big\}. \tag{3}$$

Writing elements in this form, it is easy to calculate the height and the indicator. In particular, if $a = (a_0, a_1 + T, \dots)$, then

$$H_{T^*}(a) = \begin{cases} H_{\widehat{T}}(a_0) \quad \text{if } \ \mathcal{O}(a_0) = \infty; \\ (h_{\widehat{T}}(a_0), h_{\widehat{T}}(pa_0), \dots, h_{\widehat{T}}(p^{n-1}a_0), \omega + m, \omega + m + 1, \dots) \\ \qquad \text{if } \ a_0 \in \widehat{T} \setminus T,\ \ \mathcal{O}(a_0) = p^n,\ \ \mathcal{O}(a_0 + T) = p^{n-m}; \\ (h_T(a_0), h_T(pa_0), \dots, h_T(p^{n-1}a_0), \omega + n + k, \omega + n + k + 1, \dots) \\ \qquad \text{if } \ \mathcal{O}(a_0) = p^n,\ \ a_0, a_1, \dots, a_k \in T,\ \ a_{k+1} \notin T; \\ H_T(a_0) \quad \text{if } \ a_i \in T \ \text{ for any } \ i, \end{cases} \tag{4}$$

where ω is the smallest infinite ordinal number.

Theorem. *If T is a direct sum of torsion-complete groups, then its cotorsion hull is fully transitive if and only if T is a direct sum of cyclic p-groups or a torsion-complete group.*

Proof. The sufficiency follows from [4] and [3]. We will prove the necessity.

Let (1) not be a direct sum of cyclic p-groups or a torsion-complete group. Then at least one of its summands $\overline{B}_1$ is not a direct sum of cyclic groups and for any finite subset of indices $\mathcal{J} \ni 1$ the group $\bigoplus_{j\notin\mathcal{J}} \overline{B}_j$ is unbounded. Hence it follows that in the basic subgroup $\bigoplus_{j\neq 1} B_j$ of the group $\bigoplus_{j\neq 1} \overline{B}_j$ there exists a direct summand

$$C = \bigoplus_{i=1}^{\infty} \langle c_{j_i} \rangle,$$

where $\langle c_{j_i} \rangle$ is a cyclic direct summand of the group B_{j_i}, $i = 1, 2, \dots$, $\mathcal{O}(c_{j_i}) \to \infty$ as $i \to \infty$; $C \cap B_1 = 0$. By the choice of C, an element of finite order of the group $\widehat{C}$, not belonging to C, does not lie in T either, since in notation (2) j takes values from an infinite subset of indices.

B_1 is an unbounded direct sum of cyclic groups. In the groups $\widehat{B}_1$ and $\widehat{C}$ let us choose elements b_0 and c_0 as follows:

$$b_0 = b_1 + b_2 + \cdots, \quad c_0 = c_1 + c_2 + \cdots,$$

where b_i belongs to the cyclic direct summand $\langle b_{1_i} \rangle$ of the group B_1, while $c_i \in \langle c_{j_i} \rangle$ and

$$0 \le e(b_i) - e(c_i) \le m, \quad h(b_i) \le h(c_i),$$
$$e(b_i) < e(c_{i+1}), \quad h(b_{i+1}) > h(c_i) + e(c_i),$$

where m is the fixed natural number, e is the exponent; $i = 1, 2, \dots$. Since B_1 and C are unbounded groups, such a choice of elements b_0 and c_0 is possible. It is obvious that $h(b_i)$, $h(c_i) \to \infty$ as $i \to \infty$; $b_0 \in \widehat{B}_1$, $c_0 \in \widehat{C}$; $\mathcal{O}(b_0) = \mathcal{O}(c_0) = \infty$, i.e. b_0 and c_0 are elements of infinite order of the group $\widehat{T}$. It is easy to verify that

$$H_{\widehat{B}_1}(b_0) = H_{\widehat{T}}(b_0) \le H_{\widehat{T}}(c_0) = H_{\widehat{C}}(c_0).$$

Let us use representation (3) of the group T^* and consider its elements

$$b = (b'_0, b'_1 + T, \dots), \quad c_0 = (c'_0, c'_1 + T, \dots),$$

where $b'_0 = b_0$, $c'_0 = c_0$. By virtue of equality (4), $H_{\widehat{T}}(b_0) = H_{T^*}(b)$ and $H_{\widehat{T}}(c_0) = H_{T^*}(c)$. Therefore $H_{T^*}(b) \le H_{T^*}(c)$.

We will show that there does not exist an endomorphism of the group T^* which maps b on c. This is equivalent to proving that there does not exist an endomorphism of the group $\widehat{T}$ which induces an endomorphism on the subgroup T and maps b_0 on c_0.

Since $H_{\widehat{T}}(b_0) \le H_{\widehat{T}}(c_0)$ and the algebraically compact group $\widehat{T}$ is fully transitive (see [3]), there exists an endomorphism φ of the group $\widehat{T}$ such that $\varphi b_0 = c_0$. Let us show that there exists an element $t \in T$ for which $\varphi t \notin T$.

By virtue of the properties of the group $\widehat{T}$,

$$\varphi b_0 = \varphi(b_1 + b_2 + \cdots) = \varphi b_1 + \varphi b_2 + \cdots = c_0 = c_1 + c_2 + \cdots.$$

Denote the projection of the group $\widehat{T}$ on the direct summand $\widehat{C}$ by π, and on $\langle c_{j_i} \rangle$ by π_i. Then

$$\varphi b_0 = \pi\varphi b_0 = \pi\varphi b_1 + \pi\varphi b_2 + \cdots = c_1 + c_2 + \cdots = c_0.$$

The element $b_i \in B_1 \subset T$. Therefore if for some i, $\pi\varphi b_i$, lying in $\overline{C}$, has an infinite support, then $\pi\varphi b_i \notin T$. Hence $\varphi b_i \notin T$ and thus φ does not induce an endomorphism on the subgroup T.

Let us assume that for each i, $\pi\varphi b_i$ has a finite support, i.e. lies in C. Since $h(b_{i+1}) > h(c_i) + e(c_i)$ and $c_i \in \langle c_{j_i} \rangle$, c_i is obtained from a finite number of summands

$$c_i = \pi_i\varphi b_1 + \pi_i\varphi b_2 + \cdots + \pi_i\varphi b_n, \quad n \le i. \tag{5}$$

If $n < i$, then $\mathcal{O}(b_n) < \mathcal{O}(c_i)$. Therefore the element $\pi_i\varphi b_n \in \langle c_{j_i} \rangle$ has a larger height than c_i. Hence the right-hand side of equality (5) has a larger height than c_i, which is impossible. Let $n = i$. Then we have three possible cases:

1. $h(\pi_i\varphi b_i) > h(c_i)$; 2. $h(\pi_i\varphi b_i) < h(c_i)$; 3. $h(\pi_i\varphi b_i) = h(c_i)$.

In the first case, by virtue of the above reasoning we have

$$h\left(\pi_i\varphi b_1 + \pi_i\varphi b_2 + \cdots + \pi_i\varphi b_i\right) > h(c_i).$$

Therefore

$$c_i \neq \pi_i\varphi b_1 + \pi_i\varphi b_2 + \cdots + \pi_i\varphi b_i.$$

In the second case, each summand on the right-hand side of equality (5), except the last one, has a height larger than c_i, while the height of the last summand is smaller than $h(c_i)$. Therefore $h(c_i) = h(\pi_i\varphi b_i) < h(c_i)$, which again gives a contradiction. Thus only the third case is fulfilled:

$$h(\pi_i\varphi b_i) = h(c_i), \quad i = 1, 2, \dots . \tag{6}$$

Hence

$$\mathcal{O}(\pi_i\varphi b_i) = \mathcal{O}(c_i), \quad e(c_i) \leq e(\pi\varphi b_i) \leq e(b_i). \tag{7}$$

By the definition of elements b_0 and c_0, we have $0 \leq e(b_i) - e(c_i) \leq m$. Then

$$0 \leq e(b_i) - e(\pi\varphi b_i) \leq m, \quad i = 1, 2, \dots . \tag{8}$$

Fix a natural number $n > m$ and for $e(b_{i_1}) > n$ consider an element $p^{e(b_{i_1})-n}b_{i_1}$ of order p^n. Then

$$\pi\varphi(p^{e(b_{i_1})-n}b_{i_1}) = p^{e(b_{i_1})-n}\pi\varphi b_{i_1} \neq 0,$$

since by (6), $\mathcal{O}(\pi\varphi b_{i_1}) \geq \mathcal{O}(c_i)$ and by (8),

$$e(b_{i_1}) - n < e(b_{i_1}) - m \leq e(\pi\varphi b_{i_1}).$$

The images $\pi\varphi b_i$, $i = 1, 2, \dots$, are elements with finite supports from C and, in view of (6), (7), $h(\pi\varphi b_i) \to \infty$, $\mathcal{O}(\pi\varphi b_i) \to \infty$ as $i \to \infty$. Therefore there exists an index $i_2 > i_1$ such that the supports of elements $\pi\varphi b_{i_1}$ and $\pi\varphi b_{i_2}$ in the group $\widehat{C}$ do not intersect. Note that like for b_{i_1} we have $p^{e(b_{i_2})-n}\pi\varphi b_{i_2} \neq 0$. We continue this process and consider the element

$$t = p^{e(b_{i_1})-n}b_{i_1} + p^{e(b_{i_2})-n}b_{i_2} + \cdots .$$

It is obvious that $t \in \widehat{B}_1$, $\mathcal{O}(t) = p^n$, $t \in \overline{B}_1$, $t \in T$,

$$\pi\varphi t = p^{e(b_{i_1})-n}\pi\varphi b_{i_1} + p^{e(b_{i_2})-n}\pi\varphi b_{i_2} + \cdots . \tag{9}$$

The right-hand side of equality (9) is an element of the group $\overline{C}$ with an infinite support. Therefore $\pi\varphi t \notin C$. Then $\pi\varphi t \notin T$ and $\varphi t \notin T$, i.e., the endomorphism φ on T does not induce an endomorphism. The theorem is proved. □

From this theorem it follows that in order to describe a lattice of fully characteristic subgroups of the group T^* when T is a direct sum of torsion-complete groups, we cannot use, in the general case, the indicator function H from condition 4 of A. Mader's theorem. Thus, simultaneously, we have to look for another function possessing the required properties.

The proven theorem also gives rise to the following question: do torsion-complete groups and direct sums of cyclic groups exhaust the classes of separable p-groups, where the cotorsion hull of a group is fully transitive?

The questions in the paper can be studied with some modifications for modules over the ring of principal ideals.

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(Received 18.04.2005)

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