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# ON SOME MUTUAL POSITIONS OF HYPERPLANES IN A FINITE-DIMENSIONAL AFFINE SPACE

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**Abstract.** Several combinatorial questions and facts connected with certain types of mutual positions of finitely many hyperplanes in a finite-dimensional affine space are considered. An application of one of such facts to a multi-dimensional version of the well-known Sylvester theorem is presented.

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Let $E$ be a finite-dimensional affine space and let $n = \dim(E)$. Without loss of generality, we may identify $E$ with the canonical product space $R^n$ (where $R$ denotes, as usual, the real line).

Let $X$ be a subset of $E$. We say that $X$ is admissible if, for any pairwise distinct points $x_1 \in X, x_2 \in X, \ldots, x_n \in X$, there exists a unique hyperplane in $E$ containing all these points. We denote that hyperplane by $L(x_1, x_2, \ldots, x_n)$. Also, for any two distinct points $x$ and $y$ in $E$, we denote by $l(x, y)$ the straight line passing through these points.

Let $\mathcal{L}$ be an injective family of hyperplanes in $E$. We say that this family is admissible if, for any pairwise distinct hyperplanes $L_1 \in \mathcal{L}$, $L_2 \in \mathcal{L}$, $\ldots$, $L_n \in \mathcal{L}$, the corresponding exterior normal vectors $e(L_1), e(L_2), \ldots, e(L_n)$ are linearly independent. In that case, we have

$$L_1 \cap L_2 \cap \cdots \cap L_n = \{x\}$$

for some uniquely determined point $x \in E$. We denote $x = x(L_1, L_2, \ldots, L_n)$ and define the set of points

$$X(\mathcal{L}) = \{x(L_1, L_2, \ldots, L_n) : \{L_1, L_2, \ldots, L_n\} \subset \mathcal{L}\}.$$

If $x \in X(\mathcal{L})$, then we say that $x$ is a point associated with a given family $\mathcal{L}$.

Let $S = [x_1, x_2, \ldots, x_{n+1}]$ denote the $n$-dimensional simplex in $E$ whose vertices are $x_1, x_2, \ldots, x_{n+1}$. We say that $S$ is associated with $\mathcal{L}$ if each facet (i.e., each $(n-1)$-dimensional face) of $S$ is carried by a hyperplane belonging to $\mathcal{L}$. Obviously, if $S$ is associated with $\mathcal{L}$, then all its vertices $x_1, x_2, \ldots, x_{n+1}$ are also associated with $\mathcal{L}$.

We say that a simplex $S = [x_1, x_2, \ldots, x_{n+1}]$ is admissible for $\mathcal{L}$ if the following two conditions hold:

(a) $S$ is associated with $\mathcal{L}$;

(b) there exists a point $x$ also associated with $\mathcal{L}$ and belonging to $]x_i, x_j[$ where $x_i$ and $x_j$ are some two distinct vertices of $S$.

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Suppose now that $\mathcal{L}$ is an injective finite family of hyperplanes in $E$ and $\mathrm{card}(\mathcal{L}) = k$. Obviously, there are only finitely many purely combinatorial possibilities for mutual positions (arrangements) in $E$ of elements of $\mathcal{L}$. Briefly, there are only finitely many combinatorial types of mutual positions of $k$ hyperplanes in $E$. The total number of these combinatorial types is denoted by $p_n(k)$.

The problem of finding an exact formula for $p_n(k)$ is very difficult. In this context, let us note that if $n = 1$, then $p_n(k) = 1$ for all natural numbers $k$. If $n = 2$, then for small natural numbers $k$ we have

$$p_2(0) = 1,\ p_2(1) = 1,\ p_2(2) = 2,\ ,p_2(3) = 4,\ p_2(4) = 9,\ p_2(5) = 47,\ \dots\ .$$

Let us consider more thoroughly the case $n = 2$, $k = 4$. In this case, all combinatorial types of mutual positions of four straight lines $l_1, l_2, l_3, l_4$ on the affine plane are well known and are presented below (see Fig. 1).

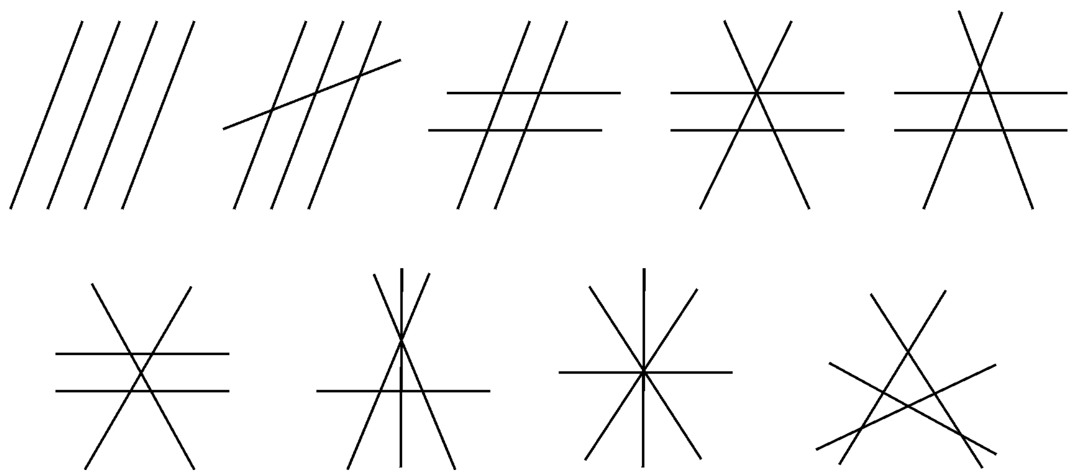

Fig. 1

We readily observe that if any two of the lines $l_1, l_2, l_3, l_4$ are not parallel and these four lines have no common point, then there is a triangle admissible for the family $\mathcal{L} = \{l_1, l_2, l_3, l_4\}$. This simple geometrical fact is shown in Figure 2 below.

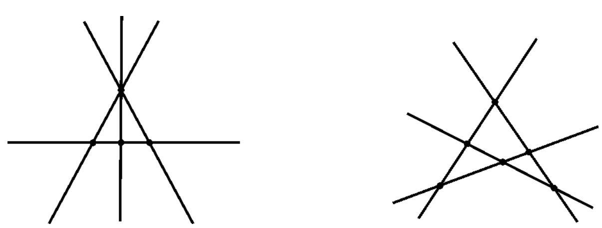

Fig. 2

The above observation is the starting point for our further constructions.

**Lemma 1.** *Let* $\dim(E) = n > 0$*, let* $\mathcal{L} = \{L_1, L_2, \dots, L_{n+2}\}$ *be an admissible family of hyperplanes in* $E$ *and suppose that* $\cap\mathcal{L} = \varnothing$*. Then there exists at least one admissible simplex for* $\mathcal{L}$*.*

*Proof.* We use the method of induction on $n$. The case $n = 1$ is trivial. Suppose that our assertion is valid for all natural numbers $m < n$ and let us establish its validity for $m = n$. Take, in a space $E$ with $\dim(E) = n$, any admissible family $\mathcal{L} = \{L_1, L_2, \dots, L_{n+2}\}$ of hyperplanes such that $\cap\mathcal{L} = \varnothing$ and define:

$$P_1 = L_1 \cap L_{n+2},\ P_2 = L_2 \cap L_{n+2},\ , \dots,\ P_{n+1} = L_{n+1} \cap L_{n+2},$$

$$\mathcal{L}' = \{P_1, P_2, \dots, P_{n+1}\}.$$

Clearly, $\mathcal{L}'$ is an admissible family of hyperplanes in the affine space $L_{n+2}$ such that

$$\mathrm{card}(\mathcal{L}') = n + 1,\ \cap\mathcal{L}' = \varnothing.$$

Since $\dim(L_{n+2}) = n - 1 < n$, we can apply the inductive assumption to $\mathcal{L}'$. Consequently, there exists an $(n-1)$-dimensional simplex

$$S' = [x_1, x_2, \dots, x_n] \subset L_{n+2}$$

which is admissible for $\mathcal{L}'$. Let $P_{i_1}, P_{i_2}, \dots, P_{i_n}$ be all those hyperplanes in $L_{n+2}$ which carry the facets of $S'$, and let

$$x = x(L_{i_1}, L_{i_2}, \dots, L_{i_n}).$$

It is obvious that the point $x$ does not belong to $L_{n+2}$. Therefore we can consider, in $E$, the $n$-dimensional simplex $S = [x, x_1, x_2, \dots, x_n]$. An easy verification shows that $S$ is admissible for the original family $\mathcal{L}$. This completes the proof of the lemma. □

*Remark* 1. Let us denote by $s(\mathcal{L})$ the number of admissible simplices for a given admissible family $\mathcal{L} = \{L_1, L_2, \dots, L_{n+2}\}$ of hyperplanes in $E$, satisfying the relation $\cap\mathcal{L} = \varnothing$. It would be interesting to find some good lower bounds for $s(\mathcal{L})$. The inequality $s(\mathcal{L}) \geq 1$ (stated by Lemma 1) is completely sufficient for our further consideration. If $\dim(E) = n = 2$, then a situation may occur, where $s(\mathcal{L}) = 1$ (see Fig. 2). Also, it is not difficult to show that if $\dim(E) = n \geq 3$, then $s(\mathcal{L}) \geq 2$.

**Lemma 2.** *Let $S = [x_1, x_2, \dots, x_{n+1}]$ be the $n$-dimensional simplex in $E$ with vertices $x_1, x_2, \dots, x_{n+1}$ and let $x'$ be a point belonging to the edge $[x_j, x_k]$ of $[x_1, x_2, \dots, x_n]$ and distinct from all vertices $x_1, x_2, \dots, x_n$. Finally, let $\Gamma$ be a hyperplane in $E$ such that:*
1) *$\Gamma$ passes through $x'$;*
2) *$\Gamma$ does not contain $x_{n+1}$;*
3) *$\Gamma$ does not contain the edge $[x_j, x_k]$.*

*Then there exist an index $i \in \{1, 2, \dots, n\}$ and a point $z$ such that*

$$z \in \Gamma \cap\ ]x_i, x_{n+1}[.$$

*In particular, we have the inequality*

$$\mathrm{dist}(z, L(x_1, x_2, \dots, x_n)) < \mathrm{dist}(x_{n+1}, L(x_1, x_2, \dots, x_n)).$$

The proof of Lemma 2 can easily be carried out by induction on $n$. Omitting details, we only note that, for $n = 2$, the formulation of this lemma is very similar to the well-known Pasch axiom from elementary geometry. The above-mentioned axiom plays an essential role in studyings the foundations of geometry (cf. [1]).

Now, let us give an application of Lemmas 1 and 2 to the problem concerning certain mutual positions of hyperplanes (or points) in a finite-dimensional affine space $E$. Problems and questions of this kind are typical in combinatorial and discrete geometry (see, for instance, [2], [3], [4]).

**Theorem 1.** *Let* $\dim(E) = n > 1$ *and let* $\mathcal{L}$ *be a finite admissible family of affine hyperplanes in* $E$*, satisfying the following conditions:*

1) $\mathrm{card}(\mathcal{L}) \geq n$*;*

2) *for any pairwise distinct hyperplanes* $L_1 \in \mathcal{L}, L_2 \in \mathcal{L}, \dots, L_n \in \mathcal{L}$*, there exists at least one hyperplane* $L \in \mathcal{L}$ *such that*

$$L \neq L_1,\ L \neq L_2,\ \dots\ ,\ L \neq L_n,\ L \cap L_1 \cap L_2 \cap \dots \cap L_n \neq \varnothing.$$

*Then we have* $\cap\mathcal{L} \neq \varnothing$*.*

*Proof.* Suppose otherwise, i.e., suppose that $\cap\mathcal{L} = \varnothing$. Then there are a point $x$ associated with $\mathcal{L}$ and a hyperplane $L \in \mathcal{L}$ for which $\mathrm{dist}(x, L) > 0$. We may assume without loss of generality that $\mathrm{dist}(x, L)$ takes a minimal possible value. According to the definition of $x$, there are some pairwise distinct hyperplanes $L_1, L_2, \dots, L_n$ from $\mathcal{L}$ such that

$$\{x\} = L_1 \cap L_2 \cap \dots \cap L_n.$$

Further, by the assumption of the theorem, there exists a hyperplane $L_{n+1} \in \mathcal{L}$ passing through $x$ and distinct from all hyperplanes $L_1, L_2, \dots, L_n$. Let us put

$$P_1 = L_1 \cap L,\ P_2 = L_2 \cap L,\ \dots\ ,\ P_{n+1} = L_{n+1} \cap L.$$

The family $\mathcal{P} = \{P_1, P_2, \dots, P_{n+1}\}$ of hyperplanes in the $(n-1)$-dimensional affine space $L$ is admissible and $\cap\mathcal{P} = \varnothing$. Applying Lemma 1 to $\mathcal{P}$, we obtain an $(n-1)$-dimensional simplex $S' = [x_1, x_2, \dots, x_n] \subset L$ admissible for $\mathcal{P}$. Denote by $x'$ a point associated with $\mathcal{P}$, belonging to some edge $[x_j, x_k]$ of $S'$ and distinct from all vertices of $S'$. We may write

$$\{x'\} = L_{i_1} \cap L_{i_2} \cap \dots \cap L_{i_{n-1}} \cap L,$$

where $i_1, i_2, \dots, i_{n-1}$ are some pairwise distinct indices from the segment $[1, n{+}1]$ of natural numbers. Again, by the assumption of the theorem, there exists a hyperplane $\Gamma \in \mathcal{L}$ passing through $x'$ and satisfying the relations

$$\Gamma \neq L_{i_1},\ \Gamma \neq L_{i_2},\ \dots,\ \Gamma \neq L_{i_{n-1}},\ \Gamma \neq L.$$

Observe that $\Gamma$ does not contain the edge $[x_j, x_k]$ and does not pass through $x$ (since our family $\mathcal{L}$ is admissible). So we may apply Lemma 2 to $\Gamma$ and to the $n$-dimensional simplex $S = [x_1, x_2, \dots, x_n, x]$. In this way we obtain that

$$\Gamma \cap\ ]x, x_i[\ \neq \varnothing$$

for some index $i \in \{1, 2, \dots, n\}$. If $z \in \Gamma \cap ]x, x_i[$, then we obviously have

$$0 < \mathrm{dist}(z, L) < \mathrm{dist}(x, L).$$

On the other hand, it can be easily verified that $z$ is a point associated with $\mathcal{L}$. So we come to a contradiction with the choice of $x$, which completes the proof of Theorem 1. □

*Remark* 2. It is not difficult to give a direct proof of Theorem 1 by applying the method of induction on $n = \dim(E)$ and using the fact that if a family $\{\Gamma_1, \Gamma_2, \dots, \Gamma_k\}$ of hyperplanes in the space $R^n$ is admissible, then the family

$$\{\Gamma_1 \cap \Gamma_k, \Gamma_2 \cap \Gamma_k, \dots, \Gamma_{k-1} \cap \Gamma_k\}$$

of hyperplanes in the $(n-1)$-dimensional affine space $\Gamma_k$ is admissible, too.

The case $n = 2$ is considered, e.g., in [2]. More precisely, suppose that $\mathcal{L}$ is a finite family of straight lines in $R^2$ satisfying the condition: for any two distinct lines $l_1 \in \mathcal{L}$ and $l_2 \in \mathcal{L}$, the relation $l_1 \cap l_2 \neq \varnothing$ implies the existence of a third line $l_3 \in \mathcal{L}$ such that $l_1 \cap l_2 \cap l_3 \neq \varnothing$. Then either $\cap\mathcal{L} \neq \varnothing$ or all lines from $\mathcal{L}$ are parallel to each other (cf. [2]).

For our further purposes, we need to recall the notion of a polarity correspondence between points and hyperplanes in the Euclidean space $R^n$.

Let $y \in R^n$ be a fixed point and let $k$ be a strictly positive real number. For each point $x \in R^n$ distinct from $y$, consider the point $x' \in l(x, y)$ such that

$$y \notin [x, x'], \ ||y - x|| \cdot ||y - x'|| = k.$$

Let $\Gamma(x)$ denote the hyperplane in $R^n$ passing through $x'$ and orthogonal to $l(x, y)$. We thus obtain the bijective mapping

$$\phi_y : x \rightarrow \Gamma(x)$$

between the set $R^n \setminus \{y\}$ and the family of all those hyperplanes in $R^n$ which do not pass through $y$. The geometric properties of $\phi_y$ are well known. In particular, points $x_1, x_2, \dots, x_k$ distinct from $y$ lie in an affine hyperplane $L$ (not containing $y$) if and only if the corresponding affine hyperplanes $\phi_y(x_1), \phi_y(x_2), \dots,$ $\phi_y(x_k)$ have a common point (which, in fact, coincides with $\phi_y^{-1}(L)$).

The polarity between points and hyperplanes in $E$ enables us to transform an admissible subset of $E$ with small cardinality into an admissible family of hyperplanes in the same $E$. More precisely, we have

**Lemma 3.** *Let* $\dim(E) = n \geq 1$ *and let* $X$ *be an admissible subset of* $E$ *with* $\mathrm{card}(X) < \mathbf{c}$*, where* $\mathbf{c}$ *denotes the cardinality of the continuum. Then there exists a point* $y \in E$ *such that the family of hyperplanes* $\{\phi_y(x) : x \in X\}$ *is admissible, too.*

*Proof.* For any pairwise distinct points $x_1, x_2, \dots, x_n$ from $X$, denote by $L(x_1, x_2, \dots, x_n)$ the unique affine hyperplane determined by these points. Also, put

$$\mathcal{L}_X = \{L(x_1, x_2, \dots, x_n) : \{x_1, x_2, \dots, x_n\} \subset X\}.$$

Then, according to our assumption, $\mathrm{card}(\mathcal{L}_X) < \mathbf{c}$. This implies that the family $\mathcal{L}_X$ does not cover the space $E$, so there exists a point $y \in E \setminus \cup\mathcal{L}_X$. It can be easily seen that $y$ is the desired point. □

Lemma 3 and Theorem 1 enable us to prove some multi-dimensional analogue of the Sylvester theorem on collinear points. This theorem was first formulated in [5]. For further extensions and generalizations, see, e.g., [6],[7] and [8].

**Theorem 2.** *Let* $\dim(E) = n > 1$ *and let* $X$ *be a finite admissible subset of* $E$ *satisfying the condition: for any pairwise distinct points* $x_1, x_2, \dots, x_n$ *from* $X$*, we have*

$$\mathrm{card}(X \cap L(x_1, x_2, \dots, x_n)) \geq n + 1.$$

*Then there exists a hyperplane in* $E$ *containing* $X$*.*

*Proof.* We may assume without loss of generality that $\mathrm{card}(X) \geq n$. According to Lemma 3, there exists a point $y \in E$ such that the family of hyperplanes $\{\phi_y(x) : x \in X\}$ is admissible. This family also satisfies the assumptions of Theorem 1. Consequently, $\cap\{\phi_y(x) : x \in X\} \neq \varnothing$. Take a point $z$ from $\cap\{\phi_y(x) : x \in X\}$ and define $L = \phi_y(z)$. Then we readily claim that $X \subset L$, which ends the proof of Theorem 2. □

*Remark* 3. Some other versions of Theorem 1 (Theorem 2) can be formulated and proved without difficulties. For example, let $\mathcal{L}$ be a finite family of hyperplanes in $E$ such that, for any two distinct and nonparallel hyperplanes $L_1$ and $L_2$ from $\mathcal{L}$, there exists a third hyperplane $L_3 \in \mathcal{L}$ satisfying

$$L_1 \neq L_3, \ L_2 \neq L_3, \ L_1 \cap L_2 \subset L_3.$$

Then either all members of $\mathcal{L}$ are parallel to each other or there exists an $(n-2)$-dimensional affine linear manifold $M \subset E$ such that $M \subset \cap\mathcal{L}$. The proof of this fact can be carried out by induction on $\dim(E)$.

It would be interesting to find the most general form of the Sylvester theorem for the space $R^n$ (formulated in terms of $k$-dimensional affine linear submanifolds of $R^n$, where $k$ ranges over the set $\{0, 1, \dots, n-1\}$).

Theorem 2 implies the following statement.

**Theorem 3.** *Let* $X$ *be a finite set of points of* $R^n$ $(n \geq 2)$ *in general position such that, for any* $(n-1)$*-dimensional sphere* $T$ *in* $R^n$*, the relation* $\mathrm{card}(T \cap X) \geq n + 1$ *implies the relation* $\mathrm{card}(T \cap X) \geq n + 2$*. Then there exists an* $(n-1)$*-dimensional sphere in* $R^n$ *containing* $X$*.*

*Proof.* Actually, Theorem 3 can easily be obtained from Theorem 2 by using the standard technique, namely, the inversion of $R^n$ whose pole coincides with one of the elements of $X$ and whose coefficient is an arbitrary strictly positive number (cf. [2]). □

*Remark* 4. We say that a set $T \subset R^2$ is a generalized circumference in $R^2$ if $T$ is either a circumference or a straight line. In other words, we treat each straight line in $R^2$ as a circumference of infinite radius. For $R^2$, we have the following analog of Theorem 3 in terms of generalized circumferences: if $X$ is a finite subset of $R^2$ such that the relation $\text{card}(X \cap T) \geq 3$ implies $\text{card}(X \cap T) \geq 4$ for every generalized circumference $T \subset R^2$, then $X$ is contained in some generalized circumference.

**Theorem 4.** *Let $\mathcal{L}$ be a family of straight lines in $R^2$ satisfying the following condition: if any three pairwise distinct lines $l_1, l_2, l_3$ from $\mathcal{L}$ are tangent to some circumference $T \subset R^2$, then there exists one more line $l_4 \in \mathcal{L}$ which is also tangent to the same $T$.*

*Then at least one of the next three relations holds:*
1) *all lines from $\mathcal{L}$ are parallel to each other;*
2) *all lines from $\mathcal{L}$ have a common point;*
3) *the family $\mathcal{L}$ is infinite.*

*Proof.* Considering all possible combinatorial types of mutual positions of three straight lines in the plane, we observe that if some three pairwise distinct lines are given in $R^2$ and are tangent to a circumference $T$ lying in $R^2$, then the radius length of $T$ can take at most four values. Starting with this observation, suppose to the contrary that none of the above relations 1) - 3) is valid and consider a circumference $T_0 \subset R^2$ of the smallest radius such that there are three pairwise distinct lines from $\mathcal{L}$ tangent to $T_0$. A simple argument shows that the existence of $T_0$ leads to a contradiction, which yields the desired result. □

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