Georgian Mathematical Journal
Volume 13 (2006), Number 1, 109–125

THE MAXIMAL OPERATOR IN VARIABLE SPACES $L^{p(\cdot)}(\Omega,\rho)$ WITH OSCILLATING WEIGHTS

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Abstract. We study the boundedness of the maximal operator in the spaces $L^{p(\cdot)}(\Omega,\rho)$ over a bounded open set Ω in R^n with the weight $\rho(x)=\prod\limits_{k=1}^{m} w_k(|x-x_k|)$, $x_k\in\overline{\Omega}$, where w_k has the property that $r^{\frac{n}{p(x_k)}}w_k(r)$ belongs to a certain Zygmund-type class. Weight functions w_k may oscillate between two power functions with different exponents. It is assumed that the exponent $p(x)$ satisfies the Dini–Lipschitz condition. The final statement on the boundedness is given in terms of index numbers of functions w_k (similar in a certain sense to the Boyd indices for the Young functions defining Orlicz spaces).

2000 Mathematics Subject Classification: 42B25, 47B38.
Key words and phrases: Maximal functions, weighted Lebesgue spaces, variable exponent, potential operators.

1. Introduction

Nowadays there is an evident increase of interest in harmonic analysis problems and operator theory in the generalized Lebesgue spaces with variable exponent $p(x)$ and the corresponding Sobolev spaces, we refer in particular to the surveys [9], [12], [23], [7] and to [26], [13] for the basics on the spaces $L^{p(\cdot)}$.

For the results of the boundedness of maximal operators we refer to the papers by L. Diening [5] for bounded domains in $\mathbb{R}^n$ and by D. Cruz-Uribe, A. Fiorenza and C. J. Neugebauer [4] and by A. Nekvinda [19] for unbounded domains, and by V. Kokilashvili and S. Samko [11] for weighted boundedness on bounded domains.

We refer also to [6] by L. Diening and to [3] by D. Cruz-Uribe, A. Fiorenza, J. M. Martell, and C. Perez, where there are also given new insights into the problems of boundedness of singular and maximal operators in variable exponent spaces.

In [11] the power weights $|x-x_0|^{\beta}$ are considered and one of the main points of the result obtained therein is that in the condition on β (see (2.2) below) only the values of $p(x)$ at the point x_0 are essential under the usual log-condition on $p(x)$.

The problem of more general weights remains open. An explicit description of weights for which the maximal operator is bounded in the spaces $L^{p(\cdot)}$ is a challenging problem. How should one formulate the corresponding $A_{p(\cdot)}$-condition? It is natural to suppose that the Muckenhoupt condition written

ISSN 1072-947X / \$8.00 / © Heldermann Verlag *www.heldermann.de*

in natural terms of the inverse Hölder inequality may be the corresponding characterization. Whether this is so or not is an open question.

In this paper we prove the weighted boundedness for the maximal operator in the spaces $L^{p(\cdot)}(\Omega,\rho)$ for some class of general, that is, non-power weights, which are "attached" to a finite number of points $x_k \in \Omega$ (here radial type weights of the Zygmund–Bari–Stechkin class). Weights w in this class are almost increasing or almost decreasing, may oscillate between two power functions with different exponents and have non-coinciding upper and lower indices m_w and M_w (of the type of Boyd indices). As compared with the approach in [11], the main problems arising are related to the situation where the indices m_w and M_w do not coincide, in particular, when m_w is negative, while M_w is positive.

The paper is organized as follows. In Section 2 we formulate the main result – Theorem A – on the weighted boundedness of the maximal operator. In Section 3 we recall the notion of the upper and lower indices of almost increasing non-negative functions and develop some properties of weights in the Zygmund–Bari–Stechkin class, which we need to prove the main result. Section 4 presents some technical lemmas related to the variable exponent $p(x)$. Finally, Section 5 contains the proof of Theorem A.

We recall the main notation. By Ω we denote an open bounded set in R^n, $n \geq 1$, and by $p(x)$ a function on $\overline{\Omega}$ satisfying the conditions

$$1 < p_* \leq p(x) \leq p^* < \infty, \quad x \in \overline{\Omega}, \tag{1.1}$$

and

$$|p(x) - p(y)| \leq \frac{A}{\ln \frac{1}{|x-y|}}, \quad |x-y| \leq \frac{1}{2}, \quad x, y \in \overline{\Omega}. \tag{1.2}$$

We denote by $L^{p(\cdot)}(\Omega,\rho)$ the weighted Banach space of all measurable functions $f : \Omega \to \mathbb{C}$ such that

$$\|f\|_{L^{p(\cdot)}(\Omega,\rho)} := \|\rho f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int\limits_{\Omega} \left|\frac{\rho(x)f(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\} < \infty. \tag{1.3}$$

Further,

a.d. =almost decreasing $\Longleftrightarrow$ $f(s) \geq Cf(t)$ for $s \leq y$, $C > 0$;
a.i. =almost increasing $\Longleftrightarrow$ $f(s) \leq Cf(t)$ for $s \leq t$, $C > 0$;
Ω is an open bounded set in R^n;
$|\Omega|$ is the Lebesgue measure of Ω;
χ_Ω is the characteristic function of a set Ω;
$f \sim g \iff$ there exist $C_1 > 0$ and $C_2 > 0$ such that $C_1 f(x) \leq g(x) \leq C_2 f(x)$;
$B_r(x) = \{y \in R^n : \ |y-x| < r\}$;
$|B_r(x)| = \frac{r^n}{n}|S^{n-1}|$ is the volume of $B_r(x)$;
$q(x) = \frac{p(x)}{p(x)-1}$, $\quad 1 < p(x) < \infty$, $\quad \frac{1}{p(x)} + \frac{1}{q(x)} \equiv 1$;
$p_* = \inf\limits_{x\in\Omega} p(x)$, $p^* = \sup\limits_{x\in\Omega} p(x)$;
$q_* = \inf\limits_{x\in\Omega} q(x) = \frac{p^*}{p^*-1}$, $q^* = \sup\limits_{x\in\Omega} q(x) = \frac{p_*}{p_*-1}$;

C, c may denote different positive constants.

2. Statement of the Main Result

Let

$$\mathcal{M}^\rho f(x) = \sup_{r>0} \frac{\rho(x)}{|B_r(x)|} \int\limits_{B_r(x)\cap\Omega} \frac{|f(y)|}{\rho(y)}\, dy, \tag{2.1}$$

where

$$\rho(x) = \prod_{k=1}^{m} w_k(|x-x_k|), \quad x_k \in \overline{\Omega}.$$

We write $\mathcal{M}^\rho = \mathcal{M}$ when $\rho(x) \equiv 1$.

In [11] the boundedness of the operator $\mathcal{M}^\rho$ is proved for the power weight $\rho(x) = |x-x_0|^\beta, x_0 \in \overline{\Omega}$, under the (necessary and sufficient) condition

$$-\frac{n}{p(x_0)} < \beta < \frac{n}{p(x_0)}\,. \tag{2.2}$$

The main result of this paper, see Theorem A, deals with a certain class of weights which may oscillate between two power functions (radial Zygmund–Bari–Stechkin type weights).

The Zygmund–Bari–Stechkin class Φ_n^0 of weights and the upper and lower indices of weights (of the type of Boyd indices) used in the theorem are defined in Section 3.

Theorem A. *Let $p(x)$ satisfy conditions* (1.1), (1.2). *The operator $\mathcal{M}$ is bounded in $L^{p(x)}(\Omega,\rho)$ with the weight $\rho(x) = \prod\limits_{k=1}^{m} w_k(|x-x_k|)$, $x_k \in \Omega$, where $w_k(r)$ are functions such that $r^{\frac{n}{p(x_k)}} w_k(r) \in \Phi_n^0$, if*

$$-\frac{n}{p(x_k)} < m_{w_k} \le M_{w_k} < \frac{n}{q(x_k)}\ \ , \quad k = 1,2,\dots,m. \tag{2.3}$$

3. Preliminaries on Zygmund–Bari–Stechkin classes.

3.1. Index numbers m_w and M_w of non-negative a. i. functions.

Let

$$W = \{w \in C([0,\ell]) : w(0) = 0,\ w(t) > 0 \ \text{ for } \ t>0,\ \ w(t) \ \text{ is a.i.}\}. \tag{3.1}$$

The numbers

$$m_w = \sup_{t>1} \frac{\ln\left(\liminf\limits_{h\to 0} \frac{w(ht)}{w(h)}\right)}{\ln t} = \sup_{0<t<1} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{w(ht)}{w(h)}\right)}{\ln t}$$
$$= \lim_{t\to 0} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{w(ht)}{w(h)}\right)}{\ln t}$$

and

$$M_w = \sup_{t>1} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{w(ht)}{w(h)}\right)}{\ln t} = \lim_{t\to\infty} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{w(ht)}{w(h)}\right)}{\ln t}$$

(see [20], [22], [21]), are known as *the lower and upper indices* of the function $w(x)$ (compare these indices with the Matuszewska–Orlicz indices, see [16], p. 20; they are of the type of Boyd indices, see [14], p. 75, and [15], or [2], p. 149, about the Boyd indices themselves). We have $0 \le m_w \le M_w \le \infty$ for $w \in W$.

We call a function $w(x)$ *equilibrated* or *non-oscillating* if $M_w = m_w$.

Remark 3.1. The upper and lower indices may be also well defined for functions $w(t)$ positive for $t > 0$ which do not necessarily belong to W, for example, if a function $w(t)$ is given such that $w_a(t) := t^a w(t)$ is in W, then the indices m_{w_a} and M_{w_a} of $w_a(t)$ are well defined and there also exist the indices m_w and M_w of $w(t)$ and

$$m_{w_a} = a + m_w, \qquad M_{w_a} = a + M_w.$$

We find it convenient to introduce the following class of functions, the indices m_w and M_w of which may be negative:

$$\widetilde{W} = \{w : \quad t^a w(t) \in W \quad \text{for some} \quad a \in \mathbb{R}^1\}.$$

3.2. **The Zygmund–Bari–Stechkin class Φ^0_γ.** Let $\gamma > 0$. The class Φ^0_γ was introduced and studied in [1] (with integer γ); there also are "two-parametrical" classes $\Phi^\beta_\gamma, 0 \le \beta < \gamma < \infty$ (see [18], [17], [25] and [24], p. 253). More general classes $\Phi^{a(t)}_{b(t)}$ with limits which may oscillate" are considered in [28], [29]; the class Φ^0_γ corresponds to the case where $a(t) = t^0 = 1$ and $b(t) = t^\gamma$.

Definition 3.2 ([1])**.** The Zygmund–Bari–Stechkin type class Φ^0_γ, $0 < \gamma < \infty$, is defined as $\Phi^0_\gamma := \mathscr{Z}^0 \cap \mathscr{Z}_\gamma$, where $\mathscr{Z}^0$ is the class of functions $w \in W$ satisfying the condition

$$\int_0^h \frac{w(t)}{t}\, dt \le c w(h) \tag{$\mathbb{Z}^0$}$$

and $\mathscr{Z}_\gamma$ is the class of functions $w \in W$ satisfying the condition

$$\int_h^\ell \frac{w(t)}{t^{1+\gamma}}\, dt \le c \frac{w(h)}{h^\gamma}\,, \tag{$\mathbb{Z}_\gamma$}$$

where $c = c(w) > 0$ does not depend on $h \in (0, \ell]$.

In the sequel we refer to the above conditions as $(\mathbb{Z}^0)$- and $(\mathbb{Z}_\gamma)$-conditions.

Note that the inequalities $(\mathbb{Z}^0)$ and $(\mathbb{Z}_\gamma)$ are invertible if they both are satisfied. Namely, the following statement holds.

Lemma 3.3. *Let* $w(r) \in \Phi_\gamma^0,\ \gamma > 0$. *Then*

$$\int\limits_0^h \frac{w(r)}{r} dr \sim h^\gamma \int\limits_h^\ell \frac{w(r)}{r^{1+\gamma}} dr \sim w(h) \tag{3.2}$$

the latter equivalence being valid on any subinterval $[0, \ell - \delta],\ \delta > 0$.

The following statement is valid, see [20], [22] for $\gamma = 1$ and [10] for an arbitrary $\gamma > 0$.

Theorem 3.4. *A function* $w \in W$ *belongs to* $\mathcal{Z}^0$ *if and only if* $m_w > 0$ *and it belongs to* $\mathcal{Z}_\gamma,\ \gamma > 0$, *if and only if* $M_w < \gamma$, *so that*

$$w \in \Phi_\gamma^0 \Longleftrightarrow 0 < m_w \le M_w < \gamma. \tag{3.3}$$

Moreover, for $w \in \Phi_\gamma^0$ *and any* $\varepsilon > 0$ *there exist constants* $c_1 = c_1(\varepsilon) > 0$ *and* $c_2 = c_2(\varepsilon) > 0$ *such that*

$$c_1 t^{M_w + \varepsilon} \le w(t) \le c_2 t^{m_w - \varepsilon}, \qquad 0 \le t \le \ell. \tag{3.4}$$

The following properties are also valid:

$$m_w = \sup\{\lambda \in (0,1) : \ t^{-\lambda} w(t) \quad \textit{is a.i.}\}, \tag{3.5}$$

$$M_w = \inf\{\mu \in (0,1) : \ t^{-\mu} w(t) \quad \textit{is a.d.}\}. \tag{3.6}$$

Corollary 3.5. *Let* $w(t), 0 < t \le \ell$, *be a function such that* $t^a w(t) \in \mathcal{Z}^0$ *for some* $a \in \mathbb{R}^1$. *Then for any* $\varepsilon > 0$ *there exists* $c_1 > 0$ *such that*

$$w(t) \le c_1 t^{m_w - \varepsilon}. \tag{3.7}$$

Similarly, if $t^a w(t) \in \mathcal{Z}_\gamma$, *then for any* $\varepsilon > 0$ *there exists* $c_2 > 0$ *such that*

$$w(t) \ge c_2 t^{M_w + \varepsilon}. \tag{3.8}$$

(*The indices* m_w *and* M_w *may be negative in this case.*)

Remark 3.6. If $w \in \widetilde{W}$ and $m_\omega > 0$, then $w \in W$.

Indeed, let $a \in \mathbb{R}^1$ be such that $w_a(t) = t^a w(t) \in W$. Then according to (3.5) the function $\frac{w_a(t)}{t^{m_{w_a} - \varepsilon}}$ is a.i. for any $\varepsilon > 0$. But $m_{w_a} = m_w + a$, so that $\frac{w(t)}{t^{m_w - \varepsilon}}$ is a.i. In particular, the function w itself is a.i., which means that it is contained in W.

Remark 3.7. Functions $w \in \mathcal{Z}_\gamma, \gamma > 0$, satisfy the doubling condition

$$w(2r) \le C w(r), \quad 0 \le r \le \ell, \tag{3.9}$$

which follows from the fact that the function $\frac{w(r)}{r^\mu}$ is a.d. for every $\mu > M_w$ according to (3.6) (observe that M_w is finite since $M_\omega < \gamma$ by Theorem 3.4).

We shall need the following lemma.

Lemma 3.8. *Let $w \in \widetilde{W}$ and $M_\omega < \gamma$. Then $\frac{t^\gamma}{[w(t)]^\lambda} \in \mathcal{Z}^0$ if $\lambda M_\omega < \gamma$, that is,*

$$\int_0^r \frac{t^{\gamma-1}dt}{[w(t)]^\lambda} \leq c\frac{r^\gamma}{[w(r)]^\lambda}, \qquad 0 < r \leq \ell. \tag{3.10}$$

Proof. For $w_1(t) = \frac{t^\gamma}{[w(t)]^\lambda}$, from the definition of the lower index we easily obtain

$$m_{w_1} = \gamma - \lambda M_w.$$

Hence $m_{w_1} > 0$. It is easily checked that $w_1 \in \widetilde{W}$, that is, there exists a number b such that $t^b w_1(t)$ is a.i. Then $w_1 \in W$ according to Remark 3.6 and hereby $w_1 \in \mathcal{Z}^0$ by Theorem 3.4. □

3.3. **On radial A_p-weights generated by oscillating functions w.** Let $\rho(x) = [w(|x-x_0|)]^\lambda$, where $\lambda \in \mathbb{R}^1$, $x \in \mathbb{R}^n$ and x_0 is a fixed point in $\mathbb{R}^n$ and $w(r)$ is a function such that $r^a w(r) \in W$ for some $a \in \mathbb{R}^1$. The following statement provides conditions in terms of the lower and upper indices m_w and M_w of the function $w(r)$, under which $\rho(x) = [w(|x-x_0|)]^\lambda$ is a Muckenhoupt weight of the class A_p. According to the definition of the weighted space given in Section 1, we define the class $A_p = A_p(\mathbb{R}^n)$, $p = const$, $1 < p < \infty$, as

$$A_p = \left\{\rho : \sup_Q \left(\frac{1}{|Q|}\int_Q \rho(x)dx\right)\left(\frac{1}{|Q|}\int_Q [\rho(x)]^{1-q}dx\right)^{p-1} < \infty\right\}, \tag{3.11}$$

where sup is taken with respect to all cubes, $\frac{1}{p} + \frac{1}{q} = 1$, see, e.g., [27] on A_p-weights.

Lemma 3.9. *Let $w \in \widetilde{W}$, $\lambda \in \mathbb{R}^1$ and Ω be a bounded domain in $\mathbb{R}^n$. Then $[w(|x-x_0|)]^\lambda \in A_p(\Omega)$ if*

$$[w(r)]^{\lambda p} r^n, \ [w(r)]^{-\lambda q} r^n \in \mathcal{Z}^0. \tag{3.12}$$

Condition (3.12) *is equivalent to the following inequalities in terms of the lower and upper indices of the function $w(r)$:*

$$-\frac{n}{\lambda p} < m_w \leq M_w < \frac{n}{\lambda q} \qquad \text{when} \qquad \lambda > 0 \tag{3.13}$$

and

$$-\frac{n}{|\lambda| q} < m_w \leq M_w < \frac{n}{|\lambda| p} \qquad \text{when} \qquad \lambda < 0. \tag{3.14}$$

Proof. For radial weights, A_p-condition (3.11) takes the form

$$\int_0^r [\rho(t)]^p t^{n-1}dt \left(\int_0^r [\rho(t)]^{-q} t^{n-1}dt\right)^{p-1} \leq Cr^{np}, \quad 0 < r \leq \ell = \operatorname{diam}\Omega, \tag{3.15}$$

where $C > 0$ does not depend on $r > 0$, see [8]. We rewrite this for $\rho(t) = [w(t)]^\lambda$ as

$$\int_0^r \frac{w_1(t)}{t}\,dt\left(\int_0^r \frac{w_2(t)}{t}\,dt\right)^{p-1} \le Cr^{np} \tag{3.16}$$

where $w_1(t) = [w(t)]^{\lambda p} t^n$, $w_2(t) = [w(t)]^{-\lambda q} t^n$. The validity of condition (3.16) is obviously related to that of the $\mathbb{Z}^0$-condition in Subsection 3.2 for the functions $w_1(t)$ and $w_2(t)$. By Theorem 3.4, the functions $w_1(t)$ and $w_2(t)$ satisfy condition (3.16) if and only if their lower indices m_{w_1} and m_{w_2} are strictly positive. To calculate these indices, suppose that $\lambda > 0$. We have

$$m_{w_1} = n + \lambda p m_w, \qquad m_{w_2} = n - \lambda p M_w$$

and the positivity of these numbers leads to condition (3.13). The case $\lambda < 0$ is considered similarly.

Under condition (3.13) we have

$$\int_0^r \frac{w_1(t)}{t}\,dt\left(\int_0^r \frac{w_2(t)}{t}\,dt\right)^{p-1} \le c w_1(r)[w_2(r)]^{p-1} = cr^{np}.$$

Thus (3.16) and, consequently, (3.15) are satisfied. □

4. PRELIMINARIES RELATED TO A VARIABLE EXPONENT SPACE

4.1. **Some basics.** We recall some basic facts for variable exponent spaces $L^{p(\cdot)}(\Omega)$ and refer, e.g., to [13] for details. The Hölder inequality holds in the form

$$\int_\Omega |f(x)g(x)|\,dx \le k\|f\|_{p(\cdot)} \cdot \|g\|_{q(\cdot)} \tag{4.1}$$

with $k = \frac{1}{p_*} + \frac{1}{q_0}$. The modular $I_p(f) = \int_\Omega |f(x)|^{p(x)}\,dx$ and the norm $\|f\|_{p(\cdot)}$ are simultaneously greater or less than 1:

$$\|f\|_{p(\cdot)}^{p^*} \le I_p(f) \le \|f\|_{p(\cdot)}^{p_*} \quad \text{if} \quad \|f\|_{p(\cdot)} \le 1 \tag{4.2}$$

and

$$\|f\|_{p(\cdot)}^{p_*} \le I_p(f) \le \|f\|_{p(\cdot)}^{p^*} \quad \text{if} \quad \|f\|_{p(\cdot)} \ge 1. \tag{4.3}$$

From (4.2) and (4.3) it follows that

$$c_1 \le \|f\|_p \le c_2 \implies c_3 \le I_\Omega^p(f) \le c_4 \tag{4.4}$$

and

$$C_1 \le I_\Omega^p(f) \le C_2 \implies C_3 \le \|f\|_p \le C_4 \tag{4.5}$$

with $c_3 = \min\left(c_1^{p_*}, c_1^{p^*}\right), c_4 = \max\left(c_2^{p_*}, c_2^{p^*}\right), C_3 = \min\left(C_1^{1/p_*}, C_1^{1/p^*}\right)$ and $C_4 = \max\left(C_2^{1/p_*}, C_2^{1/p^*}\right)$.

The imbedding

$$L^{p(x)} \subseteq L^{r(x)}, \quad 1 \le r(x) \le p(x) \le p^* < \infty$$

is valid if $|\Omega| < \infty$. In that case,

$$\|f\|_{r(\cdot)} \le m\|f\|_{p(\cdot)}, \quad m = a_2 + (1 - a_1)|\Omega|, \tag{4.6}$$

where $a_1 = \inf\limits_{x\in\Omega} \frac{r(x)}{p(x)}$ and $a_1 = \sup\limits_{x\in\Omega} \frac{r(x)}{p(x)}$.

Lemma 4.1. *Let Ω be a bounded set in $\mathbb{R}^n$, the exponent p satisfy conditions* (1.1), (1.2) *and let w be a function such that there exist exponents $a, b \in \mathbb{R}^1$ and constants $c_1 > 0$ and $c_2 > 0$ such that $c_1 r^a \le w(r) \le c_2 r^{-b}$, $0 \le r \le \ell =$* diam (Ω). *Then*

$$\frac{1}{C}[w(|x - x_0|)]^{p(x_0)} \le [w(|x - x_0|)]^{p(x)} \le C[w(|x - x_0|)]^{p(x_0)}, \tag{4.7}$$

where $C > 1$ does not depend on $x, x_0 \in \overline{\Omega}$.

Proof. Let

$$g(x, x_0) = [w(|x - x_0|)]^{p(x)-p(x_0)}.$$

To show that $\frac{1}{C} \le g(x, x_0) \le C$, that is, $|\ln g(x, x_0)| \le C_1,\ C_1 = \ln C$, we observe that $|\ln g(x, x_0)| = |p(x) - p(x_0)| \cdot |\ln w(|x - x_0|)|$. Therefore

$$|\ln g(x, x_0)| = |p(x) - p(x_0)| \cdot |\ln w(|x - x_0|)| \le A\ell \frac{|\ln w(|x - x_0|)|}{\ln \frac{2\ell}{|x-x_0|}}$$

which is bounded by the condition on w. □

4.2. **An auxiliary lemma for averages.** Let

$$\mathcal{M}_r^{w^\lambda} f(x) = \frac{[w(|x - x_0|)]^\lambda}{|B_r(x)|} \int\limits_{B_r(x)} \frac{|f(y)|}{[w(|y - x_0|)]^\lambda} \, dy, \qquad x_0 \in \Omega, \tag{4.8}$$

denote the weighted mean related to the weighted maximal operator (2.1). In (4.8) it is assumed that $f(y) = 0$ for $y \notin \Omega$. We write $\mathcal{M}_r f(x) := \mathcal{M}_r^{w^\lambda} f(x)\Big|_{\lambda=0}$.

In general, it will be admitted that λ may depend on a point $x \in \Omega$. Observe that the function $[w(t)]^{\lambda(x)}$ is also of the type of the function $w(t)$, that is, it also oscillates between two power functions.

Lemma 4.2. *Let $w(r) \in \mathcal{Z}_n$ and $\lambda(x) \ge 0$. Then the inequality*

$$\mathcal{M}_r^{w^\lambda}(1) = \frac{[w(|x - x_0|)]^{\lambda(x)}}{|B_r(x)|} \int\limits_{B_r(x)} \frac{dy}{[w(|y - x_0|)]^{\lambda(x)}} \le c \tag{4.9}$$

holds with $c > 0$ not depending on $r > 0$, $x_0 \in \mathbb{R}^n$ and x in any set $D \subseteq \mathbb{R}^n$ on which $\sup\limits_{x\in D} \lambda(x) < \frac{n}{M_w}$.

Proof. We consider the cases $|x - x_0| \ge 2r$ and $|x - x_0| \le 2r$.

For $|x - x_0| \ge 2r$ we obtain

$$|y - x_0| \ge |x - x_0| - |y - x| \ge |x - x_0| - r \ge \frac{1}{2}|x - x_0|.$$

Since the function $w \in \mathcal{Z}_n \subset W$ is a.i., we have $w(|y-x_0|) \geq cw\left(\frac{1}{2}|x-x_0|\right)$. Also taking into account the doubling property (3.9), we obtain

$$w(|y-x_0|) \geq cw(|x-x_0|).$$

Now estimate (4.9) becomes obvious since $\lambda(x) \geq 0$.

Let $|x-x_0| \leq 2r$. Observe that in this case

$$B(x,r) \subset B(x_0,3r)$$

since $|y-x| < r \Longrightarrow |y-x_0| \leq |y-x| + |x-x_0| < 3r$. Hence

$$\mathcal{M}_r^{w^\lambda}(1) \leq \frac{[w(|x-x_0|)]^{\lambda(x)}}{|B_r(x)|} \int\limits_{B_{3r}(x_0)} \frac{dy}{[w(|y-x_0|)]^{\lambda(x)}}$$

$$= \frac{[w(|x-x_0|)]^{\lambda(x)}}{|B_r(x)|} \int\limits_{B_{3r}(0)} \frac{dy}{[w(|y|)]^{\lambda(x)}} = c\frac{[w(|x-x_0|)]^{\lambda(x)}}{r^n} \int\limits_0^{3r} \frac{\rho^{n-1}d\rho}{[w(\rho)]^{\lambda(x)}}.$$

Then by Lemma 3.8 we get

$$\mathcal{M}_r^w(1) \leq c\left(\frac{w(|x-x_0|)}{w(3r)}\right)^{\lambda(x)} \leq c\left(\frac{w(2r)}{w(3r)}\right)^{\lambda(x)} \leq c. \qquad \Box$$

5. Proof of Theorem A

5.1. **Reduction to the case of a single weight.**

Remark 5.1. It suffices to prove Theorem A for a single weight $w(|x-x_0|)$, $x_0 \in \overline{\Omega}$, $t^{\frac{n}{p(x_0)}}w(t) \in \Phi_n^0$.

Indeed, let $\Omega = \bigcup\limits_{k=1}^{m} \Omega_k$, where Ω_k contains the point x_k in its interior and does not contain $x_j, j \neq k$, in its closure. Then

$$\|f\|_{L^{p(\cdot)}\left(\Omega, \prod\limits_{k=1}^{m} w_k(|t-t_k|)\right)} \sim \sum_{k=1}^{m} \|f\|_{L^{p(\cdot)}(\Omega_k, w_k(|t-t_k|))} \tag{5.1}$$

whenever $1 \leq p_- \leq p_+ < \infty$. This equivalence follows from the easily checked modular equivalence

$$I_\Omega^p\left(f(x)\prod_{k=1}^{m} w_k(|x-x_k|)\right) \sim \sum_{k=1}^{m} I_\Omega^p\left(f(x)w_k(|x-x_k|)\right)$$

with (4.4) and (4.5) taken into account.

Now, because of (5.1), the statement of Remark 5.1 is obtained by introducing a standard partition of unity $1 = \sum\limits_{k=1}^{m} a_k(x)$, where $a_k(x)$ are smooth functions equal to 1 in a neighborhood $B(x_k,\varepsilon)$ of the point x_k and equal to 0 outside its neighborhood $B(x_k,2\varepsilon)$ so that $a_k(x)\left[w_k(|x-x_j|)\right]^{\pm 1} \equiv 0$ in a neighborhood of the point x_k if $k \neq j$.

In what follows, Ω is an open bounded set in R^n and $x_0 \in \overline{\Omega}$.

5.2. A pointwise estimate for weighted means.

Theorem 5.2. *Let $p(x)$ satisfy conditions* (1.1) *and* (1.2) *and let $w \in W$. If*

$$0 \le m_w \le M_w < \frac{n}{q(x_0)}\,, \tag{5.2}$$

where $\frac{1}{q(x)} = 1 - \frac{1}{p(x)}$, then

$$\left[\frac{w(|x-x_0|)}{|B_r(x)|}\int\limits_{B_r(x)} \frac{|f(y)|dy}{w(|y-x_0|)}\right]^{p(x)} \le c\left(1 + \frac{1}{|B_r(x)|}\int\limits_{B_r(x)} |f(y)|^{p(y)}\,dy\right) \tag{5.3}$$

for all $f \in L^{p(\cdot)}(\Omega)$ such that $\|f\|_{p(\cdot)} \le 1$, where $c = c(p,w)$ is a constant not depending on x, r and x_0.

Proof. From (5.2) and the continuity of $p(x)$ we conclude that there exists a $d > 0$ such that

$$M_w q(x) < n \quad \text{for all} \quad |x-x_0| \le d. \tag{5.4}$$

Without loss of generality we assume that $d \le 1$. Let

$$p_r(x) = \min_{|y-x|\le r} p(y)$$

and $\frac{1}{q_r(x)} = 1 - \frac{1}{p_r(x)}$. From (5.2) it is easily seen that

$$M_w q_r(x) < n \ \text{ if } \ |x-x_0| \le \frac{d}{2} \ \text{ and } \ 0 < r \le \frac{d}{4}\,. \tag{5.5}$$

The theorem is proved. □

1^0. The case $|x - x_0| \le \frac{d}{2}$ and $0 < r \le \frac{d}{4}$ (the main case). Applying the Hölder inequality with exponents $p_r(x)$ and $q_r(x)$ to the integral on the right-hand side of the equality

$$\left|\mathcal{M}_r\Big(\frac{f(y)}{w(|y-x_0|)}\Big)\right|^{p(x)} = \frac{c}{r^{np(x)}}\left(\int\limits_{B_r(x)} \frac{|f(y)|}{w(|y-x_0|)}\,dy\right)^{p(x)},$$

we get

$$\left|\mathcal{M}_r\left(\frac{f(y)}{w(|y-x_0|)}\right)\right|^{p(x)}$$

$$\le \frac{c}{r^{np(x)}}\left(\int\limits_{B_r(x)} |f(y)|^{p_r(x)}\,dy\right)^{\frac{p(x)}{p_r(x)}} \cdot \left(\int\limits_{B_r(x)} \frac{dy}{[w(|y-x_0|)]^{q_r(x)}}\right)^{\frac{p(x)}{q_r(x)}}, \tag{5.6}$$

where the last integral converges, since for small $|y - x_0|$ one has

$$[w(|y-x_0|)]^{q_r(x)} \ge c|y-x_0|^{(M_w+\varepsilon)q_r(x)},$$

where one may choose ε sufficiently small so that according to (5.4), $|y - x_0|^{(M_w+\varepsilon)q_r(x)} \ge |y-x_0|^{n-\delta}$ for some $\delta > 0$.

We may make use of estimate (4.9) in (5.6), since $w \in W$ and $w \in \mathcal{Z}_n$ under the condition $M_w < \frac{n}{q(x_0)} < n$ according to Theorem 3.4. We obtain

$$\left|\mathcal{M}_r\left(\frac{f(y)}{w(|y-x_0|)}\right)\right|^{p(x)} \le c\,\frac{[w(|x-x_0|)]^{-p(x)}}{r^{\frac{np(x)}{p_r(x)}}}\left(\int\limits_{B_r(x)} |f(y)|^{p_r(x)}\,dy\right)^{\frac{p(x)}{p_r(x)}}.$$

Here

$$\int\limits_{B_r(x)} |f(y)|^{p_r(x)}\,dy \le \int\limits_{B_r(x)} dy + \int\limits_{\substack{B_r(x)\\ \{y\colon |f(y)|\ge 1\}}} |f(y)|^{p(y)}\,dy,$$

since $p_r(x) \le p(y)$ for $y \in B_r(x)$. Since $p(x)$ is bounded, we see that

$$\left|\mathcal{M}_r\left(\frac{f(y))}{w(|y-x_0|)}\right)\right|^{p(x)} \le c_1\,\frac{[w(|x-x_0|)]^{-p(x)}}{r^{\frac{np(x)}{p_r(x)}}}\left[r^n + \frac{1}{2}\int\limits_{B_r(x)} |f(y)|^{p(y)}\,dy\right]^{\frac{p(x)}{p_r(x)}}.$$

Since $r \le \frac{d}{2} \le \frac{1}{2}$ and the second term in the brackets is also less than or equal to $\frac{1}{2}$, we arrive at the estimate

$$\begin{aligned}|\mathcal{M}_r^w f|^{p(x)} &\le \frac{c}{r^{\frac{np(x)}{p_r(x)}}}\left[r^n + \int\limits_{B_r(x)} |f(y)|^{p(y)}\,dy\right]\\ &\le c\,r^{n\,\frac{p_r(x)-p(x)}{p_r(x)}}\left[1 + \frac{1}{r^n}\int\limits_{B_r(x)} |f(y)|^{p(y)}\,dy\right].\end{aligned}$$

This implies (4.2) since

$$r^{n\,\frac{p_r(x)-p(x)}{p_r(x)}} \le c.$$

Indeed,

$$r^{n\,\frac{p_r(x)-p(x)}{p_r(x)}} = e^{\frac{n}{p_r}\,[p(x)-p_r(x)]\ln\frac{1}{r}},$$

where

$$\left|\frac{n}{p_r}\big[p(x)-p_r(x)\big]\ln\frac{1}{r}\right| \le n\big|p(x)-p(\xi_r)\big|\ln\frac{1}{r}$$

with $\xi_r \in B_r(x)$, and then by (2.2),

$$\left|\frac{n}{p_r}\big[p(x)-p_r(x)\big]\ln\frac{1}{r}\right| \le nA\frac{\ln\frac{1}{r}}{\ln\frac{1}{|x-\xi_r|}} \le nA,$$

since $|x-\xi_r| \le r$.

2^0. The case $|\mathbf{x}-\mathbf{x_0}| \geq \frac{\mathbf{d}}{\mathbf{2}}$, $\mathbf{0} < \mathbf{r} \leq \frac{\mathbf{d}}{\mathbf{4}}$. This case is trivial since

$$|y-x_0| \geq |x-x_0| - |y-x| \geq \frac{d}{2} - \frac{d}{4} = \frac{d}{4}\,.$$

Taking into account that $\frac{w(t)}{t^2}$ is almost decreasing, we have $w(|y-x_0|) \geq Cw\left(\frac{d}{4}\right) = const.$ Since $w(|x-x_0|) \leq Cw(\operatorname{diam}\Omega)$, it follows that

$$\mathcal{M}_r^w f(x) \leq c\mathcal{M}_r f(x),$$

and one may proceed as above for the case $\beta = 0$ (where the condition $|x-x_0| \leq \frac{d}{2}$ is not needed).

3^0. The case $\mathbf{r} \geq \frac{\mathbf{d}}{\mathbf{4}}$. This case is also easy. It suffices to show that $\mathcal{M}_r^w f(x)$ is bounded. We have

$$\mathcal{M}_r^w f(x) \leq \frac{cw(\operatorname{diam}\Omega)}{\left(\frac{d}{4}\right)^n}\left[\int\limits_{|y-x_0|\leq\frac{d}{8}} \frac{|f(y)|}{w(|y-x_0|)}\,dy + \int\limits_{|y-x_0|\geq\frac{d}{8}} \frac{|f(y)|}{w(|y-x_0|)}\,dy\right].$$

Here the first integral is estimated using the Hölder inequality with the exponents

$$p_{\frac{d}{8}} = \min_{|y-x_0|\leq\frac{d}{8}} p(y) \quad \text{and} \quad q_{\frac{d}{8}} = p'_{\frac{d}{8}}$$

which is possible since $\alpha q_{\frac{d}{8}} < n$. The estimate of the second integral is trivial since $|y-x_0| \geq \frac{d}{8}$.

Corollary 5.3. *Let* $0 \leq m_w \leq M_w < \frac{n}{q(x_0)}$. *If conditions* (1.1), (1.2) *are satisfied, then*

$$|\mathcal{M}^w f(x)|^{p(x)} \leq c\left(1 + \mathcal{M}\left[|f(\cdot)|^{p(\cdot)}\right](x)\right) \tag{5.7}$$

for all $f \in L^{p(\cdot)}(\Omega)$ *such that* $\|f\|_{p(\cdot)} \leq 1$.

Remark 5.4. In the non-weighted case $\omega(x) \equiv 1$, estimate (5.7) is valid if condition (1.1) is replaced by the condition $1 \leq p(x) \leq p^* < \infty$, see [5].

5.3. **Proof of Theorem A itself.** To prove Theorem A, we have to show that

$$\left\|\mathcal{M}^w f\right\|_{p(\cdot)} \leq c \tag{5.8}$$

in some ball $\|f\|_{p(\cdot)} \leq R$, which is equivalent to the inequality

$$I_p(\mathcal{M}^w f) \leq c \quad \text{for} \quad \left\|f\right\|_{p(\cdot)} \leq R.$$

By (4.7) we obtain

$$I_p(\mathcal{M}^w f) \leq c\int\limits_{\Omega} w(|x-x_0|)^{p(x_0)}\left|\mathcal{M}\left(\frac{f(y)}{w(|y-x_0|)}\right)(x)\right|^{p(x)} dx.$$

Let us prove (5.8) first for

$$-\frac{n}{p(x_0)} < m_w \leq M_w < \frac{n}{q_0}, \tag{5.9}$$

where $\frac{1}{q_0} = \frac{p_*-1}{p(x_0)}$. Observe that $\frac{1}{q_0} \leq \frac{1}{q(x_0)}$ so that interval (5.9) for m_w, M_w is somewhat narrower than the whole interval $\left(-\frac{n}{p(x_0)}, \frac{n}{q(x_0)}\right)$. After that we treat the remaining case.

1^0. The case $-\frac{n}{p(x_0)} < m_w \leq M_w < \frac{n}{q_0}$.

Following the idea in [5], we represent this as

$$I_p(\mathcal{M}^w f) \leq c \int\limits_\Omega \left([w(|x-x_0|)]^{p_1(x_0)} \left| \mathcal{M}\left(\frac{f(y)}{w(|y-x_0|)} \right)(x) \right|^{p_1(x)} \right)^{p_*} dx, \quad (5.10)$$

where

$$p_1(x) = \frac{p(x)}{p_*}.$$

Estimate (5.7) with $w \equiv 1$ implies that

$$|\mathcal{M}\psi(x)|^{p_1(x)} \leq c\Big(1 + \mathcal{M}\big[\psi^{p_1(\cdot)}\big](x)\Big) \quad (5.11)$$

(see Remark 5.4) for all $\psi \in L^{p_1(\cdot)}(\Omega)$ with $\|\psi\|_{p_1(\cdot)} \leq 1$ (or equivalently, for all ψ with $\|\psi\|_{p_1} \leq C$ with fixed $C < \infty$).

Choosing $\psi(x) = \frac{f(x)}{w(|x-x_0|)}$ with $f \in L^{p(\cdot)}$ in (5.11) let us show that

$$\|\psi\|_{p_1} = \left\| \frac{f(x)}{w(|x-x_0|)} \right\|_{p_1} \leq C \quad (5.12)$$

for all $f \in L^{p(\cdot)}$ with $\|f\|_p \leq c$. Since $r^{\frac{n}{p(x_0)}} w(r) \in \Phi_n^0$, by Corollary 3.5 we have $w(|x-x_0|) \geq c|x-x_0|^{M_w+\varepsilon}$, $\varepsilon > 0$, so that

$$\int\limits_\Omega |\psi(x)|^{p_1(x)}\, dx \leq c \int\limits_\Omega \frac{|f(x)|^{\frac{p(x)}{p_*}}}{|x-x_0|^{(M_w+\varepsilon)p_1(x_0)}}\, dx. \quad (5.13)$$

To obtain (5.12), it remains to use the Hölder inequality in (5.13) with the exponents p_* and $q_* = \frac{p_*}{p_*-1}$ and take into account that

$$I_{q_*}\left(\frac{1}{|x-x_0|^{(M_w+\varepsilon)p_1(x_0)}} \right) = \int\limits_\Omega \frac{dx}{|x-x_0|^{(M_w+\varepsilon)q_0}}$$

where $(M_w+\varepsilon)q_0 < n$ for small $\varepsilon < \frac{n}{q_0} - M_w$.

By (5.12), we may apply estimate (5.11). Then (5.10) implies

$$I_p(\mathcal{M}^w f) \leq c \int\limits_\Omega \left([w(|x-x_0|)]^{p_1(x_0)} \left[1 + \mathcal{M}\left(\left| \frac{f(y)}{w(|y-x_0|)} \right|^{p_1(y)} \right) \right] \right)^{p_*} dx.$$

By property (4.7), this yields

$$
\begin{aligned}
&I_p(\mathcal{M}^w f)\\
&\le c\int\limits_{\Omega}\left\{[w(|x-x_0|)]^{p(x_0)}+\left[[w(|x-x_0|)]^{p_1(x_0)}\mathcal{M}\left(\frac{|f(y)|^{p_1(y)}}{[w(|y-x_0|)]^{p_1(x_0)}}\right)\right]^{p_*}\right\}dx.
\end{aligned}
$$

Here the integral of the first term is finite since $w(|x-x_0|)\le|x-x_0|^{m_\omega-\varepsilon}$ by (3.4) and $m_w p(x_0)>-n$. Hence

$$
I_p(\mathcal{M}^w f)\le c+c\int\limits_{\Omega}\left[\mathcal{M}^{w^{p_1(x_0)}}(|f(\cdot)|^{p_1(\cdot)})(x)\right]^{p_*}dx \tag{5.14}
$$

in terms of (2.1).

As is known [27], p. 201, the weighted maximal operator $\mathcal{M}^{w^{p_1}}$ is bounded in L^{p_*} with a constant $p_*>1$ if the weight $[w(|x-x_0|)]^{p_1(x_0)}$ is in A_{p_*}. By condition (3.13) of Lemma 3.9 this follows from the condition $-\frac{n}{p(x_0)}<m_w\le M_w<\frac{n}{q_0}$ which is satisfied in the case under consideration.

Therefore, by the boundedness of the weighted operator $\mathcal{M}^{w^{p_1(x_0)}}$ in L_{p_*}, from (5.14) we get

$$
I_p(\mathcal{M}^w f)\le c+c\int\limits_{\Omega}|f(y)|^{p_1(y)\cdot p_*}\,dy=c+c\int\limits_{\Omega}|f(y)|^{p(y)}\,dy<\infty. \tag{5.15}
$$

2^0. The remaining case.

To get rid of the bound on the right-hand side of (5.9), we must split the integration over Ω into two parts, one over a small neighborhood $B_\delta=B_\delta(x_0)$ of the point x_0, and the other over its exterior $\Omega\backslash B_\delta$, and choose δ sufficiently small so that the number $\frac{p_*(B_\delta)-1}{p(x_0)}$ is arbitrarily close to $\frac{p(x_0)-1}{p(x_0)}=\frac{1}{q(x_0)}$. To this end, we put

$$
\begin{aligned}
\mathcal{M}^w&=\chi_{B_\delta}\mathcal{M}^w\chi_{B_\delta}+\chi_{B_\delta}\mathcal{M}^w\chi_{\Omega\backslash B_\delta}+\chi_{\Omega\backslash B_\delta}\mathcal{M}^w\chi_{B_\delta}+\chi_{\Omega\backslash B_\delta}\mathcal{M}^w\chi_{\Omega\backslash B_\delta}\\
&=:\mathcal{M}_1^w+\mathcal{M}_2^w+\mathcal{M}_3^w+\mathcal{M}_4^w.
\end{aligned} \tag{5.16}
$$

Since the weight is strictly positive outside any neighborhood of the point x_0, we obtain

$$
\mathcal{M}_4^w f(x)\le C\mathcal{M}f(x). \tag{5.17}
$$

For $\mathcal{M}_3^\omega$ we have

$$
\mathcal{M}_3^w f(x)=\sup_{r>0}\frac{\chi_{\Omega\backslash B_\delta(x_0)}(x)}{|B_r(x)|}\int\limits_{B_r(x)\cap B_\delta(x_0)\cap\Omega}\frac{w(|x-x_0|)}{w(|y-x_0|)}|f(y)|\,dy.
$$

Here $|x-x_0|>r>|y-x_0|$. Note that the function $w_\varepsilon(t)=\frac{w(t)}{t^{M_w+\varepsilon}}$ is a.d. for any $\varepsilon>0$ according to (3.6). Therefore

$$
\frac{w(|x-x_0|)}{w(|y-x_0|)}=\frac{w_\varepsilon(|x-x_0|)}{w_\varepsilon(|y-x_0|)}\cdot\frac{|x-x_0|^{M_w+\varepsilon}}{|y-x_0|M_w+\varepsilon)}\le C\frac{|x-x_0|^{M_w+\varepsilon}}{|y-x_0|M_w+\varepsilon}.
$$

Hence

$$\mathcal{M}_3^w f(x) \leq C\mathcal{M}^{M_w+\varepsilon} f(x), \tag{5.18}$$

where $\mathcal{M}^{M_w+\varepsilon} f(x)$ is a weighted maximal function with the power weight $|x - x_0|^{M_w+\varepsilon}$. Similarly, we conclude that

$$\mathcal{M}_2^w f(x) \leq C\mathcal{M}^{m_w-\varepsilon} f(x). \tag{5.19}$$

Thus, using (5.17), (5.18) and (5.19) from (5.16) we have

$$\mathcal{M}^w f(x) \leq \chi_{B_\delta} \mathcal{M}^w \chi_{B_\delta} f(x) + \mathcal{M} f(x) + \mathcal{M}^{M_w+\varepsilon} f(x) + \mathcal{M}^{m_w-\varepsilon} f(x). \tag{5.20}$$

Here the operators $\mathcal{M}, \mathcal{M}^{M_w+\varepsilon}$ and $\mathcal{M}^{m_w-\varepsilon}$ are bounded in the space $L^{p(\cdot)}(\Omega)$ since the boundedness condition (2.2) is satisfied for $\beta = M_w+\varepsilon$ and $\beta = m_\omega-\varepsilon$, where ε is sufficiently small.

It remains to prove the boundedness of the first term on the right-hand side of (5.20). This is nothing else but the boundedness of the same operator $\mathcal{M}^w$ over a small set $\Omega_\delta = B_\delta(x_0) \cap \Omega$. By virtue of the preceding case, this boundedness holds if

$$-\frac{n}{p(x_0)} < m_w \leq M_w < \frac{n}{q_\delta}, \tag{5.21}$$

where $q_\delta = \frac{p_*(\Omega_\delta)-1}{p(x_0)}$ and $p_*(\Omega_\delta) = \min\limits_{x\in\Omega_\delta} p(x)$. Let us show that, given the condition $-\frac{n}{p(x_0)} < m_w \leq M_w < \frac{n}{q(x_0)}$, one can always choose δ sufficiently small such that (5.21) holds. Given $M_w < \frac{n}{q(x_0)}$, we have to choose δ such that $M_w < \frac{n}{q_\delta} \leq \frac{n}{q(x_0)}$. We have

$$\frac{n}{q_\delta} = \frac{n}{q(x_0)} - a(\delta), \qquad \text{where} \qquad a(\delta) = \frac{n}{p(x_0)} \left[p(x_0) - p_*(\Omega_\delta)\right].$$

By the continuity of $p(x)$ we can choose δ such that $a(\delta) < \frac{n}{q(x_0)} - M_w$. Then $\frac{n}{q_\delta} > M_w$ and condition (5.21) is fulfilled. Then the operator $\mathcal{M}^w$ is bounded in the space $L^{p(\cdot)}(B_\delta)$, which completes the proof.

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(Received 8.07.2005)

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