Georgian Mathematical Journal
Volume 13 (2006), Number 1, 127–141

A RATIONAL MODEL FOR THE EVALUATION MAP

KATSUHIKO KURIBAYASHI

Abstract. Let X be an l-connected space and U a connected CW complex with $\dim U \leq l$. Let $\mathcal{F}(U, X)$ be the space of continuous maps from U to X. In this paper, an algebraic model for the evaluation map $\mathcal{F}(U, X) \times U \to X$ is considered in terms of the model for the function space due to Brown and Szczarba [3]. It turns out that the Brown and Szczarba model for the function space coincides with Haefliger's model.

2000 Mathematics Subject Classification: 18G30, 55P62.
Key words and phrases: Evaluation maps, function spaces, Sullivan models.

1. Introduction

Let $\mathcal{F}(U, X)$ be the space of continuous maps from a space U to a nilpotent space X. Let $ev : \mathcal{F}(U, X) \times U \to X$ be the evaluation map defined by $ev(f, x) = f(x)$ for any $f \in \mathcal{F}(U, X)$ and $x \in U$. In [9], Haefliger constructed an algebraic model for a space of sections of a fibration. As a particular case, we have a model for the function space $\mathcal{F}(U, X)$ which is a differential graded algebra describing the rational homotopy type of $\mathcal{F}(U, X)$. In his construction, the differential of the model for $\mathcal{F}(U, X)$ is defined so that some appropriate morphism of algebras becomes a model for the evaluation map $ev : \mathcal{F}(U, X) \times U \to X$; see Section 2 for more details. While the differential is determined uniquely, it is not easy to describe the form explicitly in general because of the induction step to define the differential.

Bousfield, Peterson and Smith [2] gave another model for $\mathcal{F}(U, X)$ by means of Lannes' division functor in the category of differential graded commutative algebras. It is important to mention that the work was motivated by the universal property of the evaluation map; see[2, Section 1]. Subsequently, Brown and Szczarba [3] clarified a detailed form of the division functor model for $\mathcal{F}(U, X)$. Thus one can obtain the differential of the model for $\mathcal{F}(U, X)$ explicitly by using the differential coalgebra structure of the dual to a model for U and the differential algebra structure of a minimal model for X. In contrast to the Haefliger model, a model for the evaluation map ev does not appear naturally in the framework of the study due to Brown and Szczarba.

The aim of this paper is to present a model for the map ev via the model for a function space due to Brown and Szczarba (Theorem 4.5). In consequence, we see that the Brown–Szczarba model indeed coincides with the Haefliger model mentioned above (Theorem 1.1). In what follows, we assume that a topological space is connected, nilpotent and has the homotopy type of a CW complex

ISSN 1072-947X / $8.00 /

whose rational cohomology is locally finite in the sense that the nth cohomology group is a finite dimensional vector space for any $n \geq 0$. One of our main results is stated as follows.

Theorem 1.1. *Let X be an l-connected space and U a CW complex with $\dim U \leq l$. Then there exists an isomorphism $\overline{\varphi}$ from the model for the function space $\mathcal{F}(U, X)$ due to Brown and Szczarba to that due to Haefliger such that $\widetilde{ev} = \overline{\varphi} \otimes 1 \circ \overline{ev}$. Here $\widetilde{ev}$ is the Haefliger model for the evaluation map ev (see Section 1), 1 is the identity map on a model for U and $\overline{ev}$ denotes the model for ev described in Corollary 4.6.*

For the explicit form of models for $\mathcal{F}(U, X)$, we refer the reader to Sections 2 and 3.

A general result in rational homotopy theory tells us that a minimal model for a given differential graded algebra is determined uniquely up to an isomorphism; see, for example, [8, Theorem 14.12]. However, Theorem 1.1 does not follow from this result on minimal models immediately even if the condition that $\widetilde{ev} = \varphi \otimes 1 \circ m(ev)$ is relaxed. In fact, none of the models for the function space in Theorem 1.1 is minimal in general.

This paper is set out as follows. In Sections 2 and 3, we recall the models for the function space $\mathcal{F}(U, X)$ due to Haefliger and due to Brown and Szczarba. Section 4 is devoted to studying a model for the evaluation map and to proving Theorem 1.1 completely. In Section 5, we consider a model for the evaluation map at a base point. We also remark on further works ([11], [12]) in which our model for the evaluation map is utilized effectively.

Throughout this paper, a graded algebra is defined over the rational field $\mathbb{Q}$, commutative and unital, unless stated otherwise. We say that a graded algebra is connected if $A^i = 0$ for $i < 0$ and $A^0 = \mathbb{Q}$. The free graded commutative algebra generated by a graded vector space V is denoted by $\mathbb{Q}[V]$. A differential graded algebra (DGA) is an algebra endowed with a differential of degree $+1$. Let $\mathcal{A}$ denote the category of differential graded algebras. A morphism $f : (A, d) \to (B, d')$ of DGA's is called a quasi-isomorphism if it induces an isomorphism in cohomology. We also use the terminology of [8].

2. The Model for a Function Space Due to Haefliger

For later convenience, we describe briefly the model for a function space due to Haefliger [9].

Let X be an l-connected nilpotent space and $(\mathbb{Q}[V], d)$ be a minimal model for X. Let U be a connected CW complex with $\dim U \leq l$ and B a connected *finite dimensional* model for U. We define an algebra homomorphism $(ev)' : \mathbb{Q}[V] \to \mathbb{Q}[V \otimes B_*] \otimes \mathbb{Q}[V]$ by

$$(ev)'(v) = \sum_j (v \otimes b_{j*}) \otimes b_j$$

for $v \in V$, where $\{b_j\}$ is a basis of B and B_* denotes the dual vector space to B with the dual basis $\{b_{j*}\}$. Haefliger shows that there exists a unique differential

d_H on $\mathbb{Q}[V \otimes B_*]$ such that the given map ev' is a morphism of DGA's under the differential $d_H \otimes 1 \pm 1 \otimes d$ on $\mathbb{Q}[V \otimes B_*] \otimes \mathbb{Q}[V]$; see [9, 3.5]. Let K_H be the ideal of $\mathbb{Q}[V \otimes B_*]$ generated by elements with degree less than or equal to zero and their differentials. We have the Haefliger model $(\mathbb{Q}[V \otimes B_*]/K_H, \overline{d_H})$ for the function space $\mathcal{F}(U, X)$ with the DGA map

$$\widetilde{ev} : (\mathbb{Q}[V], d) \to (\mathbb{Q}[V \otimes B_*]/K_H, \overline{d_H}) \otimes (\mathbb{Q}[V], d),$$

which is induced by $(ev)'$, as a model for the evaluation map ev. Observe that $\mathbb{Q}[V \otimes B_*]/K_H$ is a free algebra of the form $\mathbb{Q}[S]$ with $S^i = 0$ for $i \leq 0$, $S^i = (V \otimes B_*)^i$ for $i > 1$ and $S^1 \oplus d_0((V \otimes B_*)^0) = (V \otimes B_*)^1$. Here d_0 denotes the linear part of the differential d_H; that is, $d(x) - d_0(x)$ is a decomposable element in $\mathbb{Q}[V \otimes B_*]$ for any $x \in V \otimes B_*$.

3. The Model for a Function Space Due to Brown and Szczarba

In this section, we recall the model for a function space due to Brown and Szczarba [3].

Let (A, d_A) be a connected DGA in which A is a free graded algebra, say $\mathbb{Q}[V]$. Let (B, d_B) be a connected, locally finite DGA and B_* denote the differential graded coalgebra defined by $B_q = \mathrm{Hom}(B^{-q}, \mathbb{Q})$ for $q \leq 0$ together with the coproduct D and the differential d_{B*} which are dual to the multiplication of B and to the differential d_B, respectively. We denote by I the ideal of the free algebra $\mathbb{Q}[A \otimes B_*]$ generated by $1 \otimes 1 - 1$ and all elements of the form

$$a_1 a_2 \otimes \beta - \sum_i (-1)^{|a_2||\beta_i'|}(a_1 \otimes \beta_i')(a_2 \otimes \beta_i''),$$

where $a_1, a_2 \in \mathbb{Q}[V]$, $\beta \in B_*$ and $D(\beta) = \sum_i \beta_i' \otimes \beta_i''$. Observe that $\mathbb{Q}[A \otimes B_*]$ is a DGA with the differential $d := d_A \otimes 1 \pm 1 \otimes d_{B*}$.

Theorem 3.1 ([3, Theorems 3.3 and 3.5])**.** (i) $(d_A \otimes 1 \pm 1 \otimes d_{B*})(I) \subset I$.
(ii) *The composition map*

$$\rho : \mathbb{Q}[V \otimes B_*] \hookrightarrow \mathbb{Q}[A \otimes B_*] \to \mathbb{Q}[A \otimes B_*]/I$$

is an isomorphism of graded algebras.

This theorem enables us to define a differential δ on $\mathbb{Q}[V \otimes B_*]$ by $\rho^{-1}\tilde{d}\rho$, where $\tilde{d}$ is the differential on $\mathbb{Q}[A \otimes B_*]/I$ induced by d.

Let $\Delta[q]$ be the simplicial set consisting of non-decreasing maps to the ordered set $[q] = \{0, 1, ..., q\}$. As usual we can write

$$\Delta[q]_p = \{(i_0, i_1, ..., i_p) \mid 0 \leq i_0 \leq \cdots \leq i_p \leq q\}.$$

Let $\Delta\mathcal{S}$ be the category of simplicial sets. For $X, Y \in obj\Delta\mathcal{S}$, let Simpl(X, Y) denote the set of simplicial maps from X to Y. We define the function space $\mathcal{F}(X, Y) \in obj\Delta\mathcal{S}$ is defined by $\mathcal{F}(X, Y)_q = \mathrm{Simpl}(X \times \Delta[q], Y)$.

Let A_{PL} be the simplicial commutative cochain algebra of polynomial differential forms with coefficients in $\mathbb{Q}$; see [1] and [8, section 10]. For $A, B \in obj\mathcal{A}$, let $\mathrm{DGA}(A, B)$ denote the set of DGA maps from A to B. Following Bousfield and Gugenheim [1], we define functors $\Delta : \mathcal{A} \to \Delta\mathcal{S}$ and $\Omega : \Delta\mathcal{S} \to \mathcal{A}$

by $\Delta(A) = \mathrm{DGA}(A, A_{PL})$ and $\Omega(X) = \mathrm{Simpl}(X, A_{PL})$, respectively. For any objects A and B in $\mathcal{A}$, the function space $\mathcal{F}(A, B) \in \Delta\mathcal{S}$ is defined by $\mathcal{F}(A, B)_q = \mathrm{DGA}(A, (A_{PL})_q \otimes B)$.

The singular simplicial set of a topological space U is denoted by ΔU and let $|X|$ be the geometrical realization of a simplicial set X. By definition, $A_{PL}(U)$ the DGA of polynomial differential forms on U is given by $A_{PL}(U) = \Omega\Delta U$. We define a map of simplicial sets $\alpha : \mathcal{F}(X, Y) \to \Delta\mathcal{F}(|X|, |Y|)$ by $\alpha(f) = |f| : |X \times \Delta[q]| \to |Y|$ for $f \in \mathcal{F}(X, Y)_q$. For any space U, let $s : |\Delta U| \to U$ stand for the homotopy equivalence defined by $s(\sigma, f) = f(\sigma)$; see, for example, [6, (12.10)]. There exists a sequence of homotopy equivalences

$$\mathcal{F}(U, T) \simeq \mathcal{F}(|\Delta U|, |\Delta T|) \xleftarrow[\simeq]{s} |\Delta\mathcal{F}(|\Delta U|, |\Delta T|)| \xleftarrow[\simeq]{|\alpha|} |\mathcal{F}(\Delta U, \Delta T)|$$

for any topological spaces U and T; see [3, Theorem 2.1].

Let $m : \mathbb{Q}[V] = A \xrightarrow{\simeq} \Omega\Delta T$ be a minimal model for ΔT and $\beta : B \xrightarrow{\simeq} \Omega\Delta U$ a model in which B is not necessarily free. For any simplicial set K, we can define a bijection

$$\eta : \mathrm{DGA}(A, \Omega(K)) \xrightarrow[\cong]{} \mathrm{Simpl}(K, \Delta(A))$$

by $\eta : \phi \mapsto f; f(\sigma)(a) = \phi(a)(\sigma)$, where $a \in A$ and $\sigma \in K_n$. The map $m : A \xrightarrow{\simeq} \Omega\Delta T$ defines a $\mathbb{Q}$-localization $h : \Delta T \to \Delta(A)$ via the bijection η; see [8, Theorem 17.12]. It follows from [10, Theorem 3.11] that the map h induces a $\mathbb{Q}$-localization $h_* : \mathcal{F}(\Delta U, \Delta T) \to \mathcal{F}(\Delta U, \Delta A)$ if $H^i(U; \mathbb{Q})$ is finitely generated for all i and zero for i greater than some integer N.

Let $\{b_i\}$ be a basis of B and $\{\beta_i\}$ its dual basis of B_*. One can define a simplicial isomorphism $\Psi : \Delta(\mathbb{Q}[A \otimes B_*]/I, \widetilde{d}) \to \mathcal{F}(A, B)$ by

$$\Psi(w)(a) = \sum_i (-1)^{\alpha(|b_i|)} w(a \otimes \beta_i) \otimes b_i,$$

where $\alpha(n) = [(n + 1)/2]$ is the greatest integer less than or equal to $(n + 1)/2$ ([3, Corollary 3.4]). Moreover, we have a sequence consisting of the simplicial isomorphism Ψ and homotopy equivalences:

$$\mathcal{F}(\Delta U, \Delta A) \xleftarrow[\simeq]{\tilde{\eta}} \mathcal{F}(A, \Omega\Delta U) \xleftarrow[\simeq]{\beta_*} \mathcal{F}(A, B) \xleftarrow[\cong]{\Psi} \Delta(\mathbb{Q}[A \otimes B_*]/I, \widetilde{d}). \tag{3.1}$$

Observe that the homotopy equivalence $\tilde{\eta}$ is induced by the quasi-isomorphism $\Omega\Delta U \otimes (A_{PL})_q \cong \Omega\Delta U \otimes \Omega(\Delta[q]) \to \Omega(\Delta U \times \Delta[q])$ and the bijection η; see [4, Theorem 1.29]. For a simplicial set K, define $\xi_K : K \to \Delta|K|$ by $\xi_K(\sigma) = q_\sigma : \Delta^n \to \{\sigma\} \times \Delta \to |K|$. We have a sequence of DGA maps

$$\begin{array}{ccccc} \Omega\Delta|\Delta(\mathbb{Q}[A \otimes B_*]/I, \widetilde{d})| & \xrightarrow{\Omega(\xi_K)} & \Omega\Delta(\mathbb{Q}[A \otimes B_*]/I, \widetilde{d}) & \xleftarrow{\Omega\Delta\rho} & \Omega\Delta(\mathbb{Q}[V \otimes B_*], \delta) \\ \| & & & & \uparrow{\scriptstyle \eta^{-1}(id)} \\ A_{PL}(|\Delta(\mathbb{Q}[A \otimes B_*]/I, \widetilde{d})|) & & & & (\mathbb{Q}[V \otimes B_*], \delta) \end{array}$$

in which $\Omega(\xi_K)$ and $\Omega\Delta\rho$ are quasi-isomorphisms and $\eta^{-1}(id)$ denotes the adjunction map to the identity map id on the simplicial set $\Delta\mathbb{Q}[V \otimes B_*]$.

Applying the realization functor $|\ \ |$ and the functor $A_{PL}(\ \ )$ to sequence (3.1), and combining with the above sequence of DGA maps, we obtain quasi-isomorphisms which connect $A_{PL}(\mathcal{F}(U,T)) = \Omega\Delta(\mathcal{F}(U,T))$ with $\Omega\Delta(\mathbb{Q}[V \otimes B_*], \delta)$.

Theorem 3.2. *Let T be an l-connected space and U a connected nilpotent CW complex with $\dim U \leq l$. Assume further that $H^i(U;\mathbb{Q})$ is finitely generated and zero for $i >$ some N. Let $B \to A_{PL}(U)$ be a model and $A = \mathbb{Q}[V] \to A_{PL}(T)$ a minimal model. Then the adjunction map $\eta^{-1}(id) : (\mathbb{Q}[V \otimes B_*], \delta) \to \Omega\Delta(\mathbb{Q}[V \otimes B_*], \delta)$ is a quasi-isomorphism and hence the DGA $(\mathbb{Q}[V \otimes B_*], \delta)$ is an algebraic model for the function space $\mathcal{F}(U,T)$.*

Notice that the algebra $\mathbb{Q}[V \otimes B_*]$ has in general an element of negative degree.

In order to prove Theorem 3.2, we first consider a minimal model for $(\mathbb{Q}[V \otimes B_*], \delta)$. Let $\{a_k, b_k, c_j\}$ be a basis for B_* such that $d_{B_*}(a_k) = b_k$ and $d_{B_*}(c_j) = 0$. We choose a basis $\{v_i\}$ for V so that $|v_i| \leq |v_{i+1}|$ and $dv_{i+1} \in \mathbb{Q}[V_i]$, where V_i is the subspace spanned by the elements $v_1, \ldots, v_i$. The result [3, Lemma 5.1] asserts that there exist free algebra generators w_{ij}, u_{ik} and v_{ik} such that

(i) $w_{ij} = v_i \otimes c_j + x_{ij}$, where $x_{ij} \in \mathbb{Q}[V_{i-1} \otimes B_*]$,

(ii) δw_{ij} is decomposable and in $\mathbb{Q}[\{w_{sl}; s < i\}]$,

(iii) $u_{ik} = v_i \otimes a_k$ and $\delta u_{ik} = v_{ik}$.

The following theorem yields Theorem 3.2.

Theorem 3.3 ([3, Lemma 5.2, Theorem 5.3])**.** *Under the same notation as above, there exists a differential D on $(\mathbb{Q}[V \otimes H_*(B_*)])$ $(\cong \mathbb{Q}[w_{ij}])$ making it minimal DGA such that*

$$\mathbb{Q}[V \otimes B_*] \cong \mathbb{Q}[V \otimes H_*(B_*)] \otimes \mathbb{Q}[u_{ik}, v_{ik}]$$

as a DGA. Moreover, $(\mathbb{Q}[V \otimes H_*(B_*)], D)$ is a deformation retract of $(\mathbb{Q}[V \otimes B_*], \delta)$ in the category $\mathcal{A}$. In particular, the inclusion $\iota : (\mathbb{Q}[V \otimes H_*(B_*)], D) \to (\mathbb{Q}[V \otimes B_*], \delta)$ with retraction is a quasi-isomorphism.

Proof of Theorem 3.2. Put $W = V \otimes H_*(B_*)$. Using the inclusion $\iota : (\mathbb{Q}[W], D) \to (\mathbb{Q}[V \otimes B_*], \delta)$ mentioned in Theorem 3.3, we construct a commutative diagram

$$\begin{array}{ccc} \Omega\Delta(\mathbb{Q}[V \otimes B_*], \delta)) & \xleftarrow{\Omega\Delta(\iota)} & \Omega\Delta(\mathbb{Q}[W], D) \\ {\scriptstyle \eta^{-1}(id)}\uparrow & & \uparrow{\scriptstyle \eta^{-1}(id_{\Delta\mathbb{Q}[W]})} \\ (\mathbb{Q}[V \otimes B_*], \delta) & \xleftarrow[\iota]{} & (\mathbb{Q}[W], D), \end{array}$$

where id_W stands for the identity map on the DGA $(\mathbb{Q}[W], D)$. Theorem 3.3 implies that the inclusion ι is a homotopy equivalence in the category $\mathcal{A}$. Therefore we have a homotopy equivalence $\Delta\iota$ and a quasi-isomorphism $\Omega\Delta\iota$. It follows from the assumption on the spaces U and T that the algebra $\mathbb{Q}[W]$ is connected. Since the DGA $(\mathbb{Q}[W], D)$ is minimal, the result [1, 10.1 Theorem] enables us to conclude that the adjunction map $\eta^{-1}(id_{\Delta\mathbb{Q}[W]})$ is a quasi-isomorphism and hence so is $\eta^{-1}(id)$. □

Here we describe a variant of the Brown and Szczarba model for a function space. Let K be the ideal of $E := \mathbb{Q}[V \otimes B_*]$ generated by $\oplus_{i \le 0} E^i$ and $\delta(E^0)$. Then E/K is a free algebra as mentioned in the end of Section 2. The induced DGA $(E/K, \overline{\delta})$ is the Brown and Szczarba model mentioned in Theorem 1.1 if B is taken to be *a finite dimensional model* for U.

Lemma 3.4. *The projection* $\pi : (E, \delta) \to (E/K, \overline{\delta})$ *is a homotopy equivalence.*

Proof. Let $\varphi : E \to \mathbb{Q}[V \otimes H_*(B_*)] \otimes \mathbb{Q}[u_{ik}, v_{ik}]$ denote the isomorphism in Theorem 3.3. Then φ induces an isomorphism

$$\overline{\varphi} : E/K \xrightarrow{\cong} \mathbb{Q}[V \otimes H_*(B_*)] \otimes \mathbb{Q}[u_{ik}, v_{ik} \mid \ |u_{ik}| > 0, |v_{ik}| > 1].$$

It follows from the proof of [3, Theorem 5.2] that the DGA $\mathbb{Q}[V \otimes H_*(B_*)]$ in Theorem 3.3 is also a retract of $(E/K, \overline{\delta})$. We have a commutative diagram

$$\begin{array}{ccc} E/K & \xleftarrow{\quad \pi \quad} & E \\ {\scriptstyle j}\nwarrow & & \nearrow{\scriptstyle \iota} \\ & \mathbb{Q}[V \otimes H_*(B_*)] & \end{array}$$

in which ι and j are injective homotopy equivalences. □

We conclude this section with a comment on the DGA $(\mathbb{Q}[A \otimes B_*]/I, \delta)$. Let A be a minimal DGA and B a connected, locally finite DGA. As mentioned in Remark after [3, Corollary 3.4], it follows from the proof of [3, Theorem 3.3] that

$$\mathrm{DGA}(\mathbb{Q}[A \otimes B_*]/I, C) \cong \mathrm{DGA}(A, C \otimes B)$$

for a connected DGA C. Thus $(\mathbb{Q}[A \otimes B_*]/I, \delta)$ is regarded as the Lannes division functor $(A : B)$ in the category $\mathcal{A}$.

4. A Model for the Evaluation Map

In this section, we consider an algebraic model for the map $com : \mathcal{F}(U, T) \times \mathcal{F}(U', U) \to \mathcal{F}(U', T)$ defined by composing maps. In consequence, a model for the evaluation map

$$ev : \mathcal{F}(U, T) \times U \longrightarrow T$$

is given in terms of the models due to Brown and Szczarba. Henceforth, we assume that the source space of a given function space is a CW complex of finite type whose dimension is less than or equal to the connectivity of the target space. In this case, the function space is connected.

For any simplicial sets X, Y and Z, we define simplicial maps $c : \Delta\mathcal{F}(|Y|, |Z|) \times \Delta\mathcal{F}(|X|, |Y|) \to \Delta\mathcal{F}(|X|, |Z|)$ and $\widetilde{c} : \mathcal{F}(Y, Z) \times \mathcal{F}(X, Y) \to \mathcal{F}(X, Z)$ by

$$c(v, u)(|x|, |\sigma|) = v \circ u(|x|, |\sigma|) = v(u(|x|, |\sigma|), |\sigma|)$$

for $|x| \in |X|, |\sigma| \in \Delta^q$ and by

$$\widetilde{c}(g, f)(x, \sigma) = g \circ f(x, \sigma) = g(f(x, \sigma), \sigma)$$

for $x \in X, \sigma \in \Delta[q]$, respectively. It is not difficult to verify that the diagram

$$\begin{array}{ccc}
|\Delta\mathcal{F}(|Y|,|Z|)| \times |\Delta\mathcal{F}(|X|,|Y|)| & \xrightarrow[s\times s]{\simeq} & \mathcal{F}(|Y|,|Z|) \times \mathcal{F}(|X|,|Y|) \\
 & & \downarrow com \\
\cong \uparrow & & \mathcal{F}(|X|,|Z|) \\
 & & \simeq \uparrow s \\
|\Delta\mathcal{F}(|Y|,|Z|) \times \Delta\mathcal{F}(|X|,|Y|)| & \xrightarrow[|c|]{} & |\Delta\mathcal{F}(|X|,|Z|)|
\end{array}$$

is commutative. Here we have prepared some lemmas in order to represent the map *com* using c and $\widetilde{c}$.

Let $\alpha : \mathcal{F}(X,Z) \to \Delta\mathcal{F}(|X|,|Z|)$ be the simplicial map defined by sending $f \in \mathcal{F}(X,Z)_q$ to the realization $|f| : |X \times \Delta[q]| \to |Z|$. Observe that the projections ρ_X and $\rho_{\Delta[q]}$ induce an homeomorphism $|\rho_X| \times |\rho_{\Delta[q]}| : |X \times \Delta[q]| \to |X| \times |\Delta[q]|$. In what follows, $|X \times \Delta[q]|$ is identified with $|X| \times |\Delta[q]|$ by the homeomorphism. We work here in the category of compactly generated spaces.

Lemma 4.1. *The following diagram is commutative:*

$$\begin{array}{ccc}
\Delta\mathcal{F}(|Y|,|Z|) \times \Delta\mathcal{F}(|X|,|Y|) & \xrightarrow{c} & \Delta\mathcal{F}(|X|,|Z|) \\
\alpha\times\alpha \uparrow & & \uparrow \alpha \\
\mathcal{F}(Y,Z) \times \mathcal{F}(X,Y) & \xrightarrow[\widetilde{c}]{} & \mathcal{F}(X,Z).
\end{array}$$

Proof. For any element $(a,x,\tau) \in |X \times \Delta[q]| = \amalg_n \Delta^n \times (X \times \Delta[q])_n/\sim$, we see that

$$|g \circ f|(a,x,\tau) = (a, g \circ f(x,\tau)) = (a, g(f(x,\tau),\tau)).$$

On the other hand, it follows that

$$\begin{aligned}
|g| \circ |f|((a,x),(a,\tau)) &= |g|(|f|((a,x),(a,\tau)),(a,\tau)) \\
&= |g|(|f|((a,x,\tau)),(a,\tau)) = |g|((a,f(x,\tau)),(a,\tau)) \\
&= |g|(a,f(x,\tau),\tau) = (a, g(f(x,\tau),\tau)).
\end{aligned}$$

In consequence, we have $|g| \circ |f| = |g \circ f|$. □

Let $h : \Delta U \to \Delta A_1$ be the adjunction map to a minimal model $m : A_1 \xrightarrow{\simeq} \Omega\Delta U$ and $\eta_{A_1} : A_1 \to \Omega\Delta A_1$ the adjunction map to the identity map on ΔA_1. Under the same notation as in Section 3, we have the following lemma.

Lemma 4.2. *The diagram*

$$\begin{array}{ccc}
\mathcal{F}(\Delta A_1, \Delta A) & \xrightarrow{h^*} & \mathcal{F}(\Delta U, \Delta A) \\
\tilde{\eta} \uparrow \simeq & & \\
\mathcal{F}(A, \Omega\Delta A_1) & & \simeq \uparrow \tilde{\eta} \\
\eta_{A_1 *} \uparrow \simeq & & \\
\mathcal{F}(A, A_1) & \xrightarrow[m_*]{\simeq} & \mathcal{F}(A, \Omega\Delta U)
\end{array}$$

is commutative and hence h^* is a homotopy equivalence.

Proof. For $\sigma \in \Delta[q] \times \Delta U'$, $f \in \mathcal{F}(A, A_1)$ and $a \in A$, we see that

$$\tilde{\eta} m_*(f)(\sigma)(a) = m_*(f)(a)(\sigma) = m(f(a))(\sigma).$$

On the other hand, we have

$$\begin{aligned} h^*(\tilde{\eta}\eta_{A_1*}(f))(\sigma)(a) &= (\tilde{\eta}\eta_{A_1*}(f)(h(\sigma))(a) = \eta_{A_1*}(f)(a)(h(\sigma)) \\ &= \eta_{A_1}(f(a))(h(\sigma)) = h(\sigma)(f(a)) = m(f(a))(\sigma). \end{aligned}$$

This completes the proof. □

Let $h' : \Delta T \to \Delta A$ be the adjunction map to a minimal model $A \overset{\simeq}{\to} \Omega\Delta T$. It is easy to show that the diagram

$$\begin{array}{ccc} \mathcal{F}(\Delta U, \Delta T) \times \mathcal{F}(\Delta U', \Delta U) & \xrightarrow{\tilde{c}} & \mathcal{F}(\Delta U', \Delta T) \\ {\scriptstyle h'_* \times 1}\downarrow & & \downarrow{\scriptstyle h'_*} \\ \mathcal{F}(\Delta U, \Delta A) \times \mathcal{F}(\Delta U', \Delta U) & \xrightarrow{\tilde{c}} & \mathcal{F}(\Delta U', \Delta A) \\ {\scriptstyle h^* \times 1}\uparrow & & \nearrow{\scriptstyle \tilde{c}} \\ \mathcal{F}(\Delta A_1, \Delta A) \times \mathcal{F}(\Delta U', \Delta U) & & \\ {\scriptstyle 1 \times h_*}\downarrow & & \\ \mathcal{F}(\Delta A_1, \Delta A) \times \mathcal{F}(\Delta U', \Delta A_1) & & \end{array} \qquad (4.1)$$

is commutative. If $H^*(U')$ and $H^*(U)$ are of finite dimension, then h'_* and h_* are $\mathbb{Q}$-localizations.

In order to describe a candidate of an algebraic model for the map *com* : $\mathcal{F}(U, T) \times \mathcal{F}(U', U) \to \mathcal{F}(U', T)$, we introduce some terminology.

Two DGA maps $\varphi : A \to B$ and $\varphi' : A' \to B'$ are *equivalent* if there are quasi-isomorphisms $a : A \to A'$ and $b : B \to B'$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A' & \xrightarrow[\varphi']{} & B' \end{array}$$

is commutative. In this case we write $\varphi \xrightarrow{\simeq} \varphi'$. If DGA maps φ and φ' are connected via equivalent DGA maps; if that is, there exists a sequence

$$\varphi \xrightarrow[\simeq]{} \varphi_1 \xleftarrow[\simeq]{} \varphi_2 \xrightarrow[\simeq]{} \cdots \xleftarrow[\simeq]{} \varphi',$$

then we say that φ is weakly equivalent to φ'.

We here recall the ideal I of the free algebra $\mathbb{Q}[A \otimes B_*]$ defined in Section 3. For such an ideal of any other free algebra of the form $\mathbb{Q}[A' \otimes B'_*]$, the same notation I is used when no confusion is caused in the context.

Proposition 4.3. *Let (A, d), (A_1, d) and (B, d) be minimal models for T, U and U', respectively. Suppose that $H^*(U'; \mathbb{Q})$ and $H^*(U; \mathbb{Q})$ are of finite dimension. Assume further that the map*

$$m(c) : \mathbb{Q}[V \otimes B_*] \cong \mathbb{Q}[A \otimes B_*]/I \to \Big((\mathbb{Q}[A \otimes A_{1*}]/I) \big/ K \Big) \otimes \mathbb{Q}[A_1 \otimes B_*]/I$$

defined by

$$m(c)(a \otimes b_*) = \sum_j (-1)^{\alpha(|a_j|)} \pi(a \otimes a_{j*}) \otimes (a_j \otimes b_*),$$

for $a \otimes b_* \in V \otimes B_*$, *is a well-defined DGA map under an appropriate basis* $\{a_j\}$ *of* A_1. *Here* K *is the ideal of* $\mathbb{Q}[V \otimes A_{1*}] \cong \mathbb{Q}[A \otimes A_{1*}]/I$ *mentioned in Section* 3 *and* $\pi : \mathbb{Q}[A \otimes A_{1*}]/I \to (\mathbb{Q}[A \otimes A_{1*}]/I)\big/K$ *is the projection in Lemma* 3.4. *Then the map* $m(c)$ *is weakly equivalent to the map* $A_{PL}(com)$.

To prove Proposition 4.3, we have prepared a lemma.

Lemma 4.4. *Let* K *and* π *be as in Proposition 4.3. Suppose that the map*

$$\begin{aligned}\tilde{\tilde{c}} : ((\mathbb{Q}[A \otimes A_{1*}]/I)\big/K, A_{PL}) \times (\mathbb{Q}[A_1 \otimes B_*]/I, A_{PL})\\ \to (\mathbb{Q}[A \otimes B_*]/I, A_{PL}) \cong (\mathbb{Q}[V \otimes B_*], A_{PL})\end{aligned}$$

defined by

$$\tilde{\tilde{c}}(g, f)(a \otimes b_*) = \sum_{j \in J} (-1)^{\alpha(|a_j|)} g\pi(a \otimes a_{j*}) \cdot f(a_j \otimes b_*),$$

for $a \in V$, *is well defined, where* $\cdot$ *denotes the multiplication of* A_{PL}. *Then the following diagram is commutative:*

$$\begin{array}{ccc}
\mathcal{F}(\Delta A_1, \Delta A) \times \mathcal{F}(\Delta U', \Delta A_1) & \xrightarrow{\ \tilde{c}\ } & \mathcal{F}(\Delta U', \Delta A) \\
\Big\uparrow{\scriptstyle \tilde{\eta} \times \tilde{\eta}} & & \Big\uparrow{\scriptstyle \eta} \\
\mathcal{F}(A, \Omega\Delta A_1) \times \mathcal{F}(A_1, \Omega\Delta U') & & \mathcal{F}(A, \Omega\Delta U') \\
\Big\uparrow{\scriptstyle \eta_{A_1*} \times \beta_*} & & \Big\uparrow \\
\mathcal{F}(A, A_1) \times \mathcal{F}(A_1, B) & & \Big| {\scriptstyle \beta_*\Psi} \\
\Big\uparrow{\scriptstyle \Psi \times \Psi} & & \Big| \\
(\mathbb{Q}[A \otimes A_{1*}]/I, A_{PL}) \times (\mathbb{Q}[A_1 \otimes B_*]/I, A_{PL}) & & (\mathbb{Q}[A \otimes B_*]/I, A_{PL}). \\
\Big\uparrow{\scriptstyle \Delta(\pi \otimes 1)} & \nearrow{\scriptstyle \tilde{\tilde{c}}} & \\
(\mathbb{Q}[(A \otimes A_{1*}]/I))\big/K, A_{PL}) \times (\mathbb{Q}[A_1 \otimes B_*]/I, A_{PL}) & &
\end{array}$$

Proof. By the definition of $\tilde{c}$, we see that for $(x, \sigma) \in \Delta U' \times \Delta[q]$,

$$\begin{aligned}&\tilde{c} \circ (\tilde{\eta} \times \tilde{\eta}) \circ (\eta_{A_1*} \times \beta_*) \circ (\Psi \times \Psi) \circ \Delta(\pi \otimes 1)(g, f)(x, \sigma)\\ &\quad = \tilde{c}(\tilde{\eta}\eta_{A_1*}\Psi(\Delta(\pi)g), \tilde{\eta}\beta_*\Psi(f))(x, \sigma) = \tilde{\eta}\eta_{A_1*}\Psi(g\pi)(\tilde{\eta}\beta_*\Psi(f)(x, \sigma), \sigma).\end{aligned}$$

Moreover, it follows that for $a \in A$,

$$\begin{aligned}&\tilde{\eta}\eta_{A*}\Psi(g\pi)(\eta\beta_*\Psi(f)(x, \sigma), \sigma)(a) = \eta_{A*}\Psi(g\pi)(a)(\eta\beta_*\Psi(f)(x, \sigma), \sigma)\\
&\quad = \sum_j (-1)^{\alpha(|\alpha_j|)} g\pi(a \otimes a_{j*}) \otimes \eta_A(a_j)(\tilde{\eta}\beta_*\Psi(f)(x, \sigma), \sigma)\\
&\quad = \sum_j (-1)^{\alpha(|a_j|)} g\pi(a \otimes a_{j*})(\sigma) \cdot \tilde{\eta}\beta_*\Psi(f)(x, \sigma)(a_j)\\
&\quad = \sum_j (-1)^{\alpha(|a_j|)} g\pi(a \otimes a_{j*})(\sigma) \cdot \beta_*\Psi(f)(a_j)(x, \sigma)\end{aligned}$$

$$= \sum_{i,j}(-1)^{\alpha(|a_j|)} g\pi(a\otimes a_{j*})(\sigma)\cdot \sum_i (-1)^{\alpha(|b_i|)} f(a_j\beta_i)\otimes\beta(b_i)(x,\sigma)$$
$$= \sum_{i,j}(-1)^{\alpha(|a_j|)+\alpha(|b_i|)} g\pi(a\otimes a_{j*})(\sigma)\cdot f(a_j\otimes\beta_i)(\sigma)\cdot\beta b_i(x).$$

On the other hand, we see that

$$\begin{aligned}\eta\beta_*\Psi\tilde{\tilde{c}}(g,f)(x,\sigma)(a) &= \beta_*\Psi\tilde{\tilde{c}}(g,f)(a)(x,\sigma)\\ &= \sum_i(-1)^{\alpha(|b_i|)}\tilde{\tilde{c}}(g,f)(a\otimes\beta_i)(\sigma)\cdot\beta b_i(x).\end{aligned}$$

The definition of the map $\tilde{\tilde{c}}$ enables us to conclude that the diagram is commutative. □

Proof of Proposition 4.3. Consider the diagram

$$\begin{array}{ccc}((\mathbb{Q}[A\otimes A_{1*}]/I)\big/K, A_{PL})\times(\mathbb{Q}[A_1\otimes B_*]/I, A_{PL}) & \xrightarrow{\tilde{\tilde{c}}} & (\mathbb{Q}[A\otimes B_*]/I, A_{PL})\\ \mu\downarrow\cong & & \downarrow =\\ \Delta((\mathbb{Q}[A\otimes A_{1,*}]/I)\big/K\otimes\mathbb{Q}[A_1\otimes B_*]/I) & \xrightarrow[\Delta(m(c))]{} & \Delta(\mathbb{Q}[A\otimes B_*]/I)\end{array}$$

in which μ is the isomorphism defined by $\mu(g,f)=g\cdot f$. As easily seen, the diagram is commutative. By assumption, the map $m(c)$ is well defined and hence so is $\tilde{\tilde{c}}$. By virtue of Lemma 3.4, we see that $\Delta(\pi\otimes 1)$ in Lemmas 4.4 is a homotopy equivalence. Thus Lemmas 4.1 and 4.4 enable us to conclude that the DGA map

$$\begin{aligned}\Omega\Delta|\Delta(m(c))| : \Omega\Delta|\Delta((\mathbb{Q}[A\otimes A_{1*}]/I)\big/K\otimes\mathbb{Q}[A_1\otimes B_*]/I)|\\ \leftarrow \Omega\Delta|\Delta(\mathbb{Q}[A\otimes B_*]/I)|\end{aligned}$$

is weakly equivalent to $A_{PL}(com)$. It is readily seen that $\Omega\Delta|\Delta(m(c)|$ coincides with the DGA map $m(c)$ up to quasi-isomorphisms $\Omega(\xi_K)$ and $\eta^{-1}(id)$ described in Section 3. Observe that $\eta^{-1}(id) : ((\mathbb{Q}[A\otimes A_{1*}]/I)\big/K)\otimes\mathbb{Q}[A_1\otimes B_*]/I \to \Omega\Delta((\mathbb{Q}[A\otimes A_{1*}]/I)\big/K)\otimes\mathbb{Q}[A_1\otimes B_*]/I)$ is a quasi-isomorphism. This follows from Theorem 3.2 and Lemma 3.4. □

Unfortunately, we have been less successful in deriving an appropriate sufficient condition for the map $m(c)$ in Proposition 4.3 to be a DGA map. However, the above consideration allows us to construct a model for the evaluation map.

If U is the space consisting of a single point, then the map $com : \mathcal{F}(U,T)\times\mathcal{F}(U',U)\to\mathcal{F}(U',T)$ is nothing but the evaluation map $ev : \mathcal{F}(U,T)\times U\to T$. We establish our main theorem in this paper.

Theorem 4.5. *Let $(A,d_A)\to\Omega\Delta T$ and $(A_1,d_{A_1})\to\Omega\Delta U$ be minimal models for T and U, respectively. Suppose that $H^*(U;\mathbb{Q})$ is of finite dimension. Then the map $m(ev) : A\to(\mathbb{Q}[A\otimes A_{1*}]/I)\big/K\otimes A_1$ defined by*

$$m(ev)(x) = \sum_j(-1)^{\alpha(|a_j|)}\widetilde{\pi}(x\otimes a_{j*})\otimes a_j,$$

for $x \in A$, is a well-defined DGA map, where $\{a_j\}$ and $\{a_{j*}\}$ are a basis of A_1 and its dual basis of A_*, respectively, and $\widetilde{\pi} : \mathbb{Q}[A \otimes A_{1*}] \to (\mathbb{Q}[A \otimes A_{1*}]/I)\big/K$ denotes the projection. Moreover, the map $m(ev)$ is weakly equivalent to $A_{PL}(ev)$.

Proof. For $C, D \in obj\mathcal{A}$, let $(C, D)_{DGM}$ denote the set of morphism of differential graded *modules* from C to D. We define a map

$$\Psi : \mathrm{DGA}(\mathbb{Q}[A \otimes A_{1*}], (\mathbb{Q}[A \otimes A_{1*}]/I)\big/K) = (A \otimes A_{1*}, (\mathbb{Q}[A \otimes A_{1*}]/I)\big/K)_{DGM} \longrightarrow (A, ((\mathbb{Q}[A \otimes A_{1*}]/I\big)/K) \otimes A_1)_{DGM}$$

by the same formula as that used for the map $\Psi : \Delta(\mathbb{Q}[A \otimes A_{1*}]/I) \to \mathcal{F}(A, B)$ in Section 3. It is readily seen that $\Psi(\widetilde{\pi}) = m(ev)$ and $\widetilde{\pi}(\alpha) = 0$ for $\alpha \in I$. Thus it follows from [3, Theorem 3.3] that $m(ev)$ is a well-defined DGA map. By Proposition 4.3, we have the latter half of the theorem. □

In order to relate the model for ev in Theorem 4.5 with the Haefliger model for ev, we need the following corollary.

Corollary 4.6. *Let $(A, d_A) \to \Omega\Delta T$ be a minimal model for T and $(\widetilde{A_1}, d_{\widetilde{A_1}}) \to \Omega\Delta U$ a finite dimensional model for U, which is not necessarily free as an algebra. Then the map $\overline{ev} : A \to (\mathbb{Q}[A \otimes \widetilde{A_1}_*]/I) \otimes \widetilde{A_1}$ defined by*

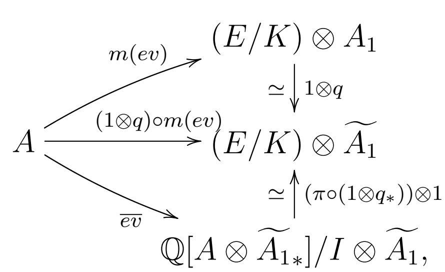
$$\overline{ev}(x) = \sum_j (-1)^{\alpha(|a_j|)}(x \otimes a_{j*}) \otimes a_j,$$

for $x \in A$, is a well-defined DGA map, where $\{a_j\}$ is a basis of $\widetilde{A_1}$ and $\{a_{*j}\}$ is the dual basis of $\widetilde{A_1}_*$. Moreover, $\overline{ev}$ is weakly equivalent to $A_{PL}(ev)$; that is, $\overline{ev}$ is a model for the evaluation map.

Proof. Let $q : (A_1, d_{A_1}) \xrightarrow{\simeq} (\widetilde{A_1}, d_{\widetilde{A_1}})$ be a minimal model for the given model $(\widetilde{A_1}, d_{\widetilde{A_1}})$. By the same argument as in the proof of Theorem 4.5, we see that $(\overline{ev})' : A \to (\mathbb{Q}[A \otimes \widetilde{A_1}_*]/I) \otimes \widetilde{A_1}_*$ defined by

$$(\overline{ev})'(x) = \sum_j (-1)^{\alpha(|a_j|)}(x \otimes a_{j*}) \otimes a_j,$$

for $x \in A$, is a well-defined DGA map. Let $\pi : E = \mathbb{Q}[A \otimes A_{1*}]/I \to E/K$ be the projection in Lemma 3.4, which is a homotopy equivalence. Then we have a commutative diagram



in which the vertical arrows are quasi-isomorphisms.

The result follows from Theorem 4.5. □

Proof of Theorem 1.1. Let $(A, d_A) = (\mathbb{Q}[V], d_A) \to \Omega\Delta T$ be a minimal model and $(\widetilde{A_1}, d_{\widetilde{A_1}}) \to \Omega\Delta U$ a finite dimensional model with a basis $\{a_j\}$. We consider the Brown and Szczarba model of the same form $(\mathbb{Q}[A \otimes \widetilde{A_1}_*]/I, \delta) = (\mathbb{Q}[V \otimes \widetilde{A_1}_*], \delta)$ as in Corollary 4.6. Put $Z = V \otimes \widetilde{A_1}_*$. We define the algebra isomorphism $\varphi : \mathbb{Q}[Z] \to \mathbb{Q}[Z]$ by $\varphi(x \otimes a_{j*}) = (-1)^{\alpha(|a_j|)}(x \otimes a_{j*})$ for $x \in Z$. Define a new differential d'_H on $\mathbb{Q}[Z]$ by $d'_H = \varphi \circ \delta \circ \varphi^{-1}$. Then it is readily seen that the algebra map $ev' : A \to \mathbb{Q}[Z] \otimes \widetilde{A_1}_*$ in Section 2 coincides with the composition $(\varphi \otimes 1) \circ (\overline{ev})'$. Therefore ev' is a DGA map with respect to the differential $d'_H \otimes 1 \pm 1 \otimes d_{\widetilde{A_1}_*}$ on $\mathbb{Q}[Z] \otimes \widetilde{A_1}_*$. Since the differential on $\mathbb{Q}[Z]$, which makes the algebra map ev' a DGA map, is determined uniquely, it follows that the induced DGA $(\mathbb{Q}[Z]/K, \overline{d'_H})$ is nothing but the Haefliger model for the function space $\mathcal{F}(U, T)$. It immediately follows that the induced map $\overline{\varphi} : (\mathbb{Q}[Z]/K, \overline{\delta}) \to (\mathbb{Q}[Z]/K, \overline{d'_H})$ is an isomorphism of DGA's with $\overline{\varphi} \otimes 1 \circ \overline{ev} = \widetilde{ev}$. This achieves the proof. □

Remark 4.7. Recently, Buijs and Murillo [5] have constructed an algebraic model for the evaluation map without assuming the connectivity of a given function space $\mathcal{F}(U, T)$. This model is also described in terms of the Brown–Szczarba model. In their construction, the model for U is taken to be of finite dimension.

5. Appendix

Let T be a connected space and U a based connected space whose rational cohomology is of finite dimension. Let $ev_* : \mathcal{F}(U, T) \to T$ be the evaluation map *at the base point* defined by $ev_*(f) = f(*)$ for $f \in \mathcal{F}(U, T)$, where $*$ is the base point of U. In this section, we construct a simplicial model and an algebraic model for ev_* by using a minimal model for T and a model for U which is not necessarily free as an algebra.

The evaluation map $ev_* : \mathcal{F}(U, T) \longrightarrow T$ is viewed as the map $i^\natural : \mathcal{F}(U, T) \to \mathcal{F}(*, T)$ which arises from the inclusion $i : * \to U$. Let $B \overset{\beta}{\underset{\simeq}{\to}} \Omega\Delta U$ be a model for U. Since the induced map $\Omega\Delta i : \Omega\Delta U \to \Omega\Delta * \cong \mathbb{Q}$ is an augmentation of the algebra $\Omega\Delta U$ (see, for example, [8, §10 Example 1]), it follows that the composition $\varepsilon : B \overset{\beta}{\to} \Omega\Delta U \to \Omega\Delta * \cong \mathbb{Q}$ is an augmentation of B. Let $(A, d) = (\mathbb{Q}[V], d) \to A_{PL}(T)$ be a minimal model for T. It is readily seen that the diagram

$$\begin{array}{ccc} \mathcal{F}(A, \Omega\Delta U) & \xrightarrow{(\Omega\Delta i)_*} & \mathcal{F}(A, \Omega\Delta *) \\ {\scriptstyle \beta_*}\uparrow \simeq & & \uparrow \cong \\ \mathcal{F}(A, B) & \xrightarrow[\varepsilon_*]{} & \mathcal{F}(A, \mathbb{Q}) \end{array}$$

is commutative. Moreover, the isomorphism Ψ in sequence (3.1) makes the diagram

$$\begin{array}{ccc}
\mathcal{F}(A,B) & \xrightarrow{\ \varepsilon_*\ } & \mathcal{F}(A,\mathbb{Q}) \\
\Psi \uparrow & & \uparrow \Psi \\
\Delta(\mathbb{Q}[A\otimes B_*]/I,d) & \xrightarrow[\Delta(1\otimes\varepsilon^\sharp)]{} & \Delta(\mathbb{Q}[A\otimes \mathbb{Q}]/I,d)
\end{array}$$

commutative, where $\varepsilon^\sharp : \mathbb{Q} \to B_*$ denotes the dual map to ε. Thus we see that $\Delta(1\otimes\varepsilon^\sharp)$ is a simplicial model for the map $\mathcal{F}(\Delta(i),1) : \mathcal{F}(\Delta U,\Delta A) \to \mathcal{F}(\Delta(*),\Delta A)$; that is, $\Delta(1\otimes\varepsilon^\sharp)$ is regarded as $\mathcal{F}(\Delta(i),1)$ up to homotopy equivalences in (3.1).

As in the preceding section, we assume further that U is a connected CW complex whose dimension is less than or equal to the connectivity of T. Then naturality of the quasi-isomorphisms connecting $A_{PL}(\mathcal{F}(U,T))$ with $(\mathbb{Q}[V\otimes B_*],\delta)$; see Section 3, allows us to conclude that the induced map $A_{PL}(i^\natural) : A_{PL}(\mathcal{F}(*,T)) \to A_{PL}(\mathcal{F}(U,T))$ is weakly equivalent to the inclusion $1\otimes\varepsilon^\sharp : (\mathbb{Q}[V],d) \to (\mathbb{Q}[V\otimes B_*],\delta)$. Thus we have an algebraic model of the form $1\otimes\varepsilon^\sharp$ for the evaluation map ev_*.

Remark 5.1. The evaluation map ev fits in the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{F}(U,T)\times U & \xrightarrow{\ ev\ } & T \\
{\scriptstyle 1\times i}\uparrow & \nearrow_{ev_*} & \\
\mathcal{F}(U,T)\times *. & &
\end{array}$$

Thus one may expect that an algebraic model for ev_* can be constructed by applying Theorem 4.5. Though the latter theorem is valid, we have to choose a minimal model for U in the construction.

Remark 5.2. Let $\mathcal{F}(U,T;f)$ be the connected component of a function space $\mathcal{F}(U,T)$ containing a given map $f : U \to T$. While we have so far confined ourselves to considering a model for a *connected* function space, an algebraic model for the evaluation map ev_* on $\mathcal{F}(U,T;f)$ can be constructed by using the simplicial model $\Delta(1\otimes\varepsilon^\sharp)$. The model for ev_* plays a crucial role in analyzing the rational Gottlieb groups; see [11] for the details.

Proposition 5.3. *The DGA* $(\mathbb{Q}[V\otimes B_*],\delta)$ *mentioned above is a relative Sullivan algebra with the base* $(\mathbb{Q}[V],d)$.

Proof. The vector space $V\otimes B_*$ is decomposed as $V\otimes 1 \oplus (V\otimes B_*^+)$, where $B_*^+ = \oplus_{i<0}(B_*)^i$. Then it follows that $\mathbb{Q}[V\otimes B_*] \cong \mathbb{Q}[V]\otimes\mathbb{Q}[V\otimes B_*^+]$. Define $B^+_{*(j)}$ by $\mathrm{Hom}(B^j,\mathbb{Q})$ for $j>0$ and by $B^+_{*(0)} = 0$. We observe that $d_{B_*}(B^+_{*(j)}) \subset B^+_{*(j-1)}$. Put $V(k) = \oplus_{i+j\le k}V_{ij}$, where $V_{ij} = V^i\otimes B^+_{*(j)}$. It is readily seen that $\cup_k V(k) = V\otimes B_*^+$ and $\delta(V(k)) \subset \mathbb{Q}[V]\otimes\mathbb{Q}[V(k-1)]$. This completes the proof. □

Remark 5.4. We can regard $(\mathbb{Q}[V\otimes B_*],\delta)$ itself as a Sullivan algebra. Indeed, the vector spaces $V(k)' = V\otimes 1 \oplus V(k)$ $(k\in\mathbb{Z})$ give the structure of a Sullivan algebra in $(\mathbb{Q}[V\otimes B_*],\delta)$.

Recall that $1 \otimes \varepsilon^\sharp$ is weakly equivalent to $A_{PL}(i^\sharp)$. Therefore, by applying [7, Lemma 3.6] repeatedly, we have a commutative diagram

$$\begin{array}{ccc} A_{PL}(\mathcal{F}(*,T)) & \xrightarrow{A_{PL}(i^\natural)} & A_{PL}(\mathcal{F}(U,T)) \\ m\uparrow & & \uparrow n \\ (\mathbb{Q}[V],d) & \underset{1\otimes\varepsilon^\sharp}{\rightarrowtail} & (\mathbb{Q}[V\otimes B_*],D) \end{array}$$

in which m and n are quasi-isomorphisms. It turns out that the inclusion $1 \otimes \varepsilon^\sharp : (\mathbb{Q}[V], d) \rightarrowtail (\mathbb{Q}[V \otimes B_*], \delta)$ is a Sullivan model for the evaluation map $ev_* : \mathcal{F}(U,T) \to T$.

We conclude this section by describing briefly an application of the model for ev_*. Let $\mathcal{F}_*(U,T)$ denote the function space consisting of base-point-preserving maps from U to T. By applying [8, Proposition 15.5] to the evaluation fibration $\mathcal{F}_*(U,T) \to \mathcal{F}(U,T) \xrightarrow{ev} T$, we have the following theorem.

Theorem 5.5. *Let T be a l-connected space and U a connected CW complex with $\dim U \leq l$. Then there exists a Sullivan model for the function space $\mathcal{F}_*(U,T)$ of the form $(\mathbb{Q}[V \otimes B_*]/(\mathbb{Q}[V]^+), \overline{\delta}) = (\mathbb{Q} \otimes_{\mathbb{Q}[V]} \mathbb{Q}[V \otimes B_*], 1 \otimes \delta)$, where $\mathbb{Q}[V]^+ = \oplus_{i>0}\mathbb{Q}[V]^i$.*

Remark 5.6. Let (L, d_L) be a Lie model for a space U; that is, there exists a quasi-isomorphism $C^*(L, d_L) = \text{dual } C_*(L, d_L) \xrightarrow{\simeq} A_{PL}(U) = \Omega\Delta U$, where $C_*(\ )$ stands for Quillen's functor from the category of connected chain Lie algebras to the category of one-connected cocommutative chain coalgebras; see [8, Section 22]. Then we have a model for the function space $\mathcal{F}_*(U,T)$ of the form $((\mathbb{Q}[V \otimes C_*(L, d_L)]/(\mathbb{Q}[V]^+), \overline{\delta})$. The model is viewed as a Quillen-Sullivan mixed type model for $\mathcal{F}_*(U,T)$. In the forthcoming paper [12], by applying the mixed type model, we shall study a sufficient condition for the fibration $\Omega^{k+1}T = \mathcal{F}_*(S^{k+1}, T) \to \mathcal{F}_*(U \cup_\alpha e^{k+1}, T) \to \mathcal{F}_*(U,T)$, which is induced by a cofibre sequence $U \to U \cup_\alpha e^{k+1} \to S^{k+1}$, to split rationally in the sense that $\mathcal{F}_*(U \cup_\alpha e^{k+1}, T)$ is homotopy equivalent to $\mathcal{F}_*(U,T) \times \Omega^{k+1}T$ after localization at zero.

Acknowledgement

The author would like to thank the referee for a comment on Proposition 4.3 without which Theorems 1.1 and 4.5 could not have been refined.

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(Received 28.04.2005)

Author's address:

Department of Mathematical Sciences
Faculty of Science
Shinshu University, Matsumoto, Nagano 390-8621
Japan
E-mail: kuri@math.shinshu-u.ac.jp