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ON THE CONVOLUTION OF FUNCTIONS OF GENERALIZED BOUNDED VARIATIONS

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Abstract. Let f and g be 2π periodic functions. If $f \in L^1[0, 2\pi]$ and g is from $\bigwedge BV^{(p)}[0, 2\pi]$ or, $\mathrm{Lip}(\alpha, p)[0, 2\pi]$ or, $r - BV[0, 2\pi]$, then f convolution g inherit the same property.

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The smoothness-increasing operator convolution is known for inheriting the best properties of each parent function. It is very well known that "$f \in L^1$ and if $g \in C^k$ or is of bounded variation (that is BV), then $f * g$ (that is the convolution of f and g) has the same property ". This concept of BV can be generalized in many ways and many interesting generalizations are obtained. In investigations on the uniform convergence of Fourier series Waterman [1] introduced the class of functions of $\bigwedge BV$. In 1980 Shiba [2] generalized this class. He introduced the class $\bigwedge BV^{(p)}$.

Definition 1. Given an interval I, a sequence of non-decreasing positive real numbers $\bigwedge = \{\lambda_m\}$ $(m = 1, 2, \dots)$ such that $\sum\limits_m (1/\lambda_m)$ diverges and $1 \le p < \infty$ we say that $f \in \bigwedge BV^{(p)}$ (that is f is a function of $p - \bigwedge$-bounded variation over (I)) if

$$V_\Lambda(f, p, I) = \sup_{\{I_m\}} \{V_\Lambda(\{I_m\}, f, p, I)\} < \infty,$$

where

$$V_\Lambda(\{I_m\}, f, p, I) = \left(\sum_m \frac{|f(b_m) - f(a_m)|^p}{\lambda_m} \right)^{1/p},$$

and $\{I_m\}$ is a sequence of non-overlapping subintervals $I_m = [a_m, b_m] \subset I = [a, b]$.

Note that for $p = 1$ one gets the class $\bigwedge BV(I)$; if $\lambda_m \equiv 1$ for all m, one gets the class $BV^{(p)}$; if $p = 1$ and $\lambda_m \equiv m$ for all m, one gets the class Harmonic $BV(I)$; if $p = 1$ and $\lambda_m \equiv 1$ for all m, one gets the class $BV(I)$.

Definition 2. For $p \ge 1$, $0 < \alpha \le 1$, we say that $f \in \mathrm{Lip}(\alpha, p)$ over I if

$$\|T_y f - f\|_{p,I} = O(|h|^\alpha) \quad \text{as} \quad h \longrightarrow 0$$

where $\|(\cdot)\|_{P,I}$ denotes the L^p norm over I and $T_h f(x) = f(x + h)$.

By considering differences of higher order the concept of bounded variation is generalized to bounded r^{th} variation which can be defined as follows.

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Definition 3. For a positive integer r, we say $f \in r - BV(I)$ (that is, f is of bounded r^{th}-variation over I) if for arbitrary $(n+1)$-points $x_0 < x_1 < \cdots < x_n$ in I, in an arithmetic progression we have

$$V^r(f, I) = \sup_n V_n^r(f, I) < \infty,$$

where

$$\sup_n V_n^r(f, I) = \sum_{i=0}^{n-r} | \triangle^r f(x_i)|,$$

in which $\triangle f(x_i) = f(x_{i+1}) - f(x_i)$ and for $k \geq 2$,

$$\triangle^k f(x_i) = \triangle^{k-1} f(x_{i+1}) - \triangle^{k-1} f(x_i)$$

so that

$$\triangle^r f(x_i) = \sum_{m=0}^{r} (-1)^m \binom{r}{m} f(x_{i+r-m}).$$

Obviously, $BV(I) \subseteq r - BV(I)$. It can be noted that the Weierstrass continuous but nowhere differentiable function $f(x) = \sum\limits_{m=4}^{\infty} b^{-m} cos(b^m x)$ $(b > 1)$ is a function of bounded r^{th} variation, but it is not a function of bounded variation as shown by Mazahar [4].

Definition 4 ([5]). For a given non-decreasing concave downward function $h(n)$ on the positive integers, we say that $f \in V[h](I)$ if there is a constant c such that $\sum\limits_{m=1}^{n} |f(I_m)| \leq ch(n)$, $n \in N$, where $\{I_m\}$ and I are as in Definition 1.

In Fourier analysis, for any two 2π periodic functions f and g, f convolution g is defined as follows.

Definition 5. For any f, $g \in L^1[0, 2\pi]$, $f * g$ is defined as

$$(f * g)(x) = \frac{1}{2\pi} \int_0^{2\pi} (f(x - y)g(y))dy.$$

These functions of generalized bounded variation share many properties of a function of bounded variation. Therefore it is interesting to know whether these properties of generalized variations are hereditary under convolution or not. We will prove the following results on the convolution of functions of generalized bounded variation.

Theorem 1. *If $f \in L^1[0, 2\pi]$ and $g \in \bigwedge BV[0, 2\pi]$, then $f * g \in \bigwedge BV[0, 2\pi]$.*

Theorem 2. *If $f \in L^1[0, 2\pi]$ and $g \in \bigwedge BV^{(p)}[0, 2\pi]$, $p \geq 1$, then $f * g \in \bigwedge BV^{(p)}[0, 2\pi]$.*

Remark. Since L^1 is a ring with respect to convolution as a ring product. By Theorem 2 the class $\bigwedge BV^{(p)}$ can be regarded as a module over the ring L^1.

Theorem 3. *If* $f \in L^1[0,2\pi]$ *and* $g \in \mathrm{Lip}(\alpha,p)$ *over* $[0,2\pi]$, $0 \le \alpha \le 1$, $p \ge 1$, *then* $f * g \in \mathrm{Lip}(\alpha,p)$ *over* $[0,2\pi]$.

Theorem 4. *If* $f \in L^1[0,2\pi]$ *and* $g \in r\!-\!BV[0,2\pi]$, *then* $f*g \in r\!-\!BV[0,2\pi]$.

Theorem 5. *If* $f \in L^1[0,2\pi]$ *and* $g \in V[h]([0,2\pi])$, *then* $f*g \in V[h]([0,2\pi])$.

Lemma. *If* $f \in L^1[0,2\pi]$ *and* $g \in L^p[0,2\pi]$ $(p \ge 1)$, *then*

$$\|T_h f * g - f * g\|_{p,[0,2\pi]} \le \|f\|_1 \|T_h g - g\|_{p,[0,2\pi]}.$$

Proof. For any $h \in L^q[0,2\pi]$, where q satisfies $\frac{1}{p}+\frac{1}{q}=1$, from the Fubini–Tonelli theorem we get

$$\begin{aligned}
&\left|\frac{1}{2\pi}\int_0^{2\pi}[T_h f * g(x) - f * g(x)]h(x)dx\right| \\
&\quad\le \frac{1}{2\pi}\int_0^{2\pi}|h(x)|\left\{\frac{1}{2\pi}\int_0^{2\pi}|f(y)|\,|(T_h g - g)(x-y)|dy\right\}dx \\
&\quad= \frac{1}{2\pi}\int_0^{2\pi}|f(y)|\left\{\frac{1}{2\pi}\int_0^{2\pi}|h(x)||(T_h g - g)(x-y)|dx\right\}dy \\
&\quad\le \|f\|_1\|h\|_q\|T_h g - g\|_p, \quad \text{from Hölder's inequality.}
\end{aligned}$$

Hence the result follows from the converse of the Hölder's inequality [3, Exercise 3.6, p. 65]. □

Proof of Theorem 1. For any two real numbers a and b we have

$$|f * g(b) - f * g(a)| \le \frac{1}{2\pi}\int_0^{2\pi}|f(y)||g(b-y)-g(a-y)|dy. \tag{1}$$

Thus, for every sequence $\{I_k\}_{k=1}^{2n}$ of non-overlapping subintervals $I_k = [a_k,b_k] \subset I = [0,2\pi]$, we get

$$\begin{aligned}
\sum_{k=1}^{2n}\frac{|f*g(I_k)|}{\lambda_k} &\le \frac{1}{2\pi}\int_0^{2\pi}|f(y)|\left(\sum_{k=1}^{2n}\frac{|g(b_k-y)-g(a_k-y)|}{\lambda_k}\right)dy \\
&\le V_\wedge(g,[0,2\pi])\|f\|_1.
\end{aligned} \tag{2}$$

Hence the result follows. □

Proof of Theorem 2. Since every function of $\bigwedge BV^{(p)}$ is bounded, from the lemma we get

$$\int_0^{2\pi}|(f*g)(I_k)|^p dx \le \|f\|_1^p \int_0^{2\pi}|g(I_k)|^p dx, \tag{3}$$

for every sequence $\{I_k\}_{k=1}^{2n}$ of non-overlapping subintervals $I_k = [a_k, b_k] \subset I = [0, 2\pi]$. Dividing both the sides of the above equation by λ_k and performing summation from $k = 1$ to $2n$, we get

$$\int_0^{2\pi} \left(\sum_{k=1}^{2n} \frac{|(f * g)(I_k)|^p}{\lambda_k} \right) dx \leq \|f\|_1^p \int_0^{2\pi} \left(\sum_{k=1}^{2n} \frac{|g(I_k)|^p}{\lambda_k} \right) dx$$
$$= 2\pi V_\wedge(g, p, \{I_k\}) \|f\|_1^p.$$

Hence the result follows. □

Proof of Theorem 3. Theorem 3 easily follows from the lemma. □

Proof of Theorem 4. From the definition of convolution we get

$$\triangle (f * g)(x) = \frac{1}{2\pi} \int_0^{2\pi} f(y) \triangle g(x - y) dy,$$

$$\triangle^2 (f * g)(x) = \frac{1}{2\pi} \int_0^{2\pi} f(y) \triangle^2 g(x - y) dy.$$

Similarly, for any positive integer r we get

$$\triangle^r (f * g)(x) = \frac{1}{2\pi} \int_0^{2\pi} f(y) \triangle^r g(x - y) dy.$$

Thus, for arbitrary $(n+1)$-points $x_0 < x_1 < x_2 < \cdots < x_n$ in $I = [0, 2\pi]$, in an arithmetic progression we get

$$|\triangle^r (f * g)(x_k)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(y)| |\triangle^r g(x_k - y)| dy, \quad \forall k = \overline{0, n - r}.$$

By taking summation over $k = 0, 1, 2, \ldots, n - r$, we get

$$\sum_{k=0}^{n-r} |\triangle^r (f * g)(x_k)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(y)| \left(\sum_{k=0}^{n-r} |\triangle^r g(x_k - y)| \right) dy$$
$$\leq V^r(f, [0, 2\pi]) \|f\|_1.$$

Hence the result follows. □

Proof of Theorem 5. Theorem 5 can be easily obtained from (2) by taking $\lambda_k = 1$ for all k. □

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