

SEMIMARTINGALE LOCAL TIME AND THE AMERICAN PUT OPTION

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Abstract. A new result is obtained on the vanishing of the local time of a non-negative continuous semimartingale at zero. Based on this result, an early exercise premium representation of a value function of the American put option is obtained in a one-dimensional general diffusion model.

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In this paper we investigate the American put option problem in a general diffusion model and its related optimal stopping problem. The main result of the investigation is the representation of a value function of the American put option as a sum of the corresponding value function of the European type put option and the early exercise premium. In the finance literature this representation is known under the name of the early exercise premium. It was derived by many authors in terms of the well known Black–Scholes model (i.e., when the dynamics of share values is described by geometric Brownian motion). Among them were N. El Karoui and I. Karatzas [5] and S. Jacka [3]. The former authors proceeded from the results of the theory of balayage of general type random processes, while Jacka used the fact that a value function of the American put option is a solution of the problem with a free boundary (Stefan’s problem). A thorough investigation of the American put option problem in the Black–Scholes model was carried out by R. Myneni [8]. It should be however noted that the proofs of many authors somewhat lack the clearness.

Our approach is based on studying the properties of local times of general type continuous semimartingales and, as a result, we obtain a sufficient condition (Theorem 3) for the vanishing of local times at zero of non-negative continuous semimartingales. The verification of this condition for the American put option problem has turned out to be a simple matter. In particular, using this condition we derive an early exercise premium representation (Theorem 4) for a value function of the American put option in terms of a general one-dimensional diffusion model. The proof we give is quite clear. Furthermore, our result allows one to directly obtain as a particular case the early exercise premium representation for a value function of the American put option in terms of the well-known Black–Scholes model.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space with a standard Wiener process (W_t, \mathcal{F}_t^W) , $0 \leq t \leq T$, defined on it. We will consider on $(\Omega, \mathcal{F}, \mathbf{P})$ the finance market with two assets (B_t, S_t) , $0 \leq t \leq T$, where B_t is the bank account value at time t and

S_t is the share price at time t . The evolution of these assets obeys the equations

$$\begin{aligned} dB_t &= r(t)B_t dt, \quad 0 \leq t \leq T, \quad B_0 = 1, \\ dS_t &= r(t)S_t dt + \sigma(t, S_t)S_t dW_t, \quad 0 \leq t \leq T, \quad S_0 > 0, \end{aligned} \quad (1)$$

where the continuous functions $r(t)$, $\tilde{\sigma}(t, x) = \sigma(t, x)x$, $0 \leq t \leq T$, $x > 0$, are assumed to be such that

- 1) $0 \leq r(t) \leq R$,
- 2) $0 < \bar{\sigma} \leq \sigma(t, x) \leq C$,
- 3) $|\tilde{\sigma}(t, x) - \tilde{\sigma}(t, y)| \leq K|x - y|$.

Note that the random process $\frac{S_t}{B_t}$ is a martingale with respect to filtration \mathcal{F}_t^W .

Conditions (2) guarantee the existence of a unique strong solution of (1) and also of the following stochastic equation with its initial condition:

$$\begin{aligned} dS_u(t, x) &= r(u)S_u(t, x) du \\ &\quad + \sigma(u, S_u(t, x))S_u(t, x) dW_u, \quad t \leq u \leq T, \\ S_t(t, x) &= x, \quad x > 0. \end{aligned} \quad (3)$$

Let us consider the American put option for the share S_t with the payoff function $g(x) = (c - x)^+$. The corresponding payoff process \tilde{X}_t has the form $\tilde{X}_t = g(S_t)$, while the value process V_t of the American option is written as

$$V_t = \operatorname{ess\,sup}_{t \leq \tau \leq T} \mathbf{E} \left(e^{-\int_t^\tau r(u) du} g(S_\tau) \mid \mathcal{F}_t^W \right), \quad 0 \leq t \leq T.$$

It is clear that $V_t = B_t Y_t$, where

$$B_t = e^{\int_0^t r(u) du}, \quad Y_t = \operatorname{ess\,sup}_{t \leq \tau \leq T} \mathbf{E} \left(e^{-\int_0^\tau r(u) du} g(S_\tau) \mid \mathcal{F}_t^W \right). \quad (4)$$

Note that the value process V_t of the American put option majorizes the payoff process $g(S_t)$, $V_t \geq g(S_t)$, $0 \leq t \leq T$, and $Y_t \geq X_t$, where

$$X_t = e^{-\int_0^t r(u) du} g(S_t), \quad 0 \leq t \leq T. \quad (5)$$

Here X_t is the discounted payoff process while the process Y_t is its Snell envelope (a minimal supermartingale that majorizes X_t).

Along with the value process of the American put option, we define the value function of this option

$$v^T(t, x) = \sup_{t \leq \tau \leq T} \mathbf{E} \left(e^{-\int_t^\tau r(u) du} (c - S_\tau(t, x))^+ \right), \quad 0 \leq t \leq T, \quad x > 0. \quad (6)$$

It is obvious that it is also a value function of the corresponding optimal stopping problem of the nonhomogeneous Markov diffusion process S_t .

Now let us introduce the following notation:

$$D^T \equiv \left\{ (t, x) : 0 \leq t < T, \quad 0 < x, \quad v^T(t, x) > g(x) \right\},$$

where D^T is the continuation domain of stopping problem; D_t^T is the section of D^T at a point t , $0 \leq t < T$,

$$D_t^T \equiv \{x : (t, x) \in D^T\}.$$

We denote by $P(t, x; u, B)$, $u \geq t$, $x > 0$, with a Borel subset B of $(0, +\infty)$ the transition probability of the nonhomogeneous diffusion process S_t .

As is known (see [1], p. 445, formula (10)), for the random processes X_t and Y_t the following expansions are valid:

$$X_t = X_0 + m_t + A_t, \quad A_t = A_t^+ - A_t^-, \quad 0 \leq t \leq T, \quad (7)$$

$$Y_t = Y_0 + M_t - C_t, \quad 0 \leq t \leq T, \quad (8)$$

where m_t , M_t are uniformly integrable martingales, while the processes A_t^+ , A_t^- , C_t are predictable nondecreasing integrable processes, $A_0^+ = A_0^- = C_0 = 0$.

Let us now prove the lemma which will be used in proving our next theorem.

Lemma 1. *Consider the process $S_u(t, x)$ satisfying equation (3). If $0 < x < y$, then*

$$S_u(t, x) \leq S_u(t, y) \quad \text{P-a.s. for all } u, \quad t \leq u \leq T.$$

Proof. It is clear that the inequality $S_u(t, x) \leq S_u(t, y)$ P-a.s. is equivalent to the equality $(S_u(t, x) - S_u(t, y))^+ = 0$ P-a.s., which in turn is the same as

$$\mathbb{E}(S_u(t, x) - S_u(t, y))^+ = 0.$$

Due to (3) we have

$$\begin{aligned} d(S_u(t, x) - S_u(t, y)) &= r(u)(S_u(t, x) - S_u(t, y)) du \\ &\quad + (\tilde{\sigma}(u, S_u(t, x)) - \tilde{\sigma}(u, S_u(t, y))) dW_u, \quad t \leq u \leq T. \end{aligned}$$

Let us write the well-known Tanaka–Meyer formula (see, e.g., [9]) for the process $(S_u(t, x) - S_u(t, y))^+$

$$\begin{aligned} (S_u(t, x) - S_u(t, y))^+ &= (x - y)^+ \\ &\quad + \int_t^u I_{(S_v(t, x) - S_v(t, y) > 0)} d(S_v(t, x) - S_v(t, y)) + \frac{1}{2} L_u^\circ(S(t, x) - S(t, y)) \\ &= \int_t^u I_{(S_v(t, x) - S_v(t, y) > 0)} r(v)(S_v(t, x) - S_v(t, y)) dv \\ &\quad + \int_t^u I_{(S_v(t, x) - S_v(t, y) > 0)} (\tilde{\sigma}(v, S_v(t, x)) - \tilde{\sigma}(v, S_v(t, y))) dW_v \\ &\quad + \frac{1}{2} L_u^\circ(S(t, x) - S(t, y)). \end{aligned} \quad (9)$$

Since in (9) the last stochastic integral is a square integrable martingale, its mathematical expectation is equal to zero (see [7], Ch. IV, §2).

Taking the mathematical expectation of both sides of equality (9), we obtain

$$\begin{aligned} & \mathbb{E}(S_u(t, x) - S_u(t, y))^+ \\ &= \mathbb{E} \int_t^u r(v) (S_v(t, x) - S_v(t, y))^+ dv + \frac{1}{2} \mathbb{E} L_u^\circ(S(t, x) - S(t, y)). \end{aligned}$$

Using Yor's corollary [13], we have

$$\begin{aligned} & L_u^\circ(S(t, x) - S(t, y)) \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_t^u I_{(0 < S_v(t, x) - S_v(t, y) \leq \varepsilon)} (\tilde{\sigma}(v, S_v(t, x)) - \tilde{\sigma}(v, S_v(t, y)))^2 dv \\ &\leq \overline{\lim}_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_t^u I_{(0 < S_v(t, x) - S_v(t, y) \leq \varepsilon)} K^2 (S_v(t, x) - S_v(t, y))^2 dv \leq \overline{\lim}_{\varepsilon \rightarrow 0} \int_t^u K^2 \varepsilon dv = 0, \end{aligned}$$

which implies

$$\mathbb{E}(S_u(t, x) - S_u(t, y))^+ = \mathbb{E} \int_t^u r(v) (S_v(t, x) - S_v(t, y))^+ dv.$$

Denoting

$$\varphi(u) \equiv \mathbb{E}(S_u(t, x) - S_u(t, y))^+, \quad t \leq u \leq T,$$

we obtain

$$\varphi(u) \leq R \int_t^u \varphi(v) dv$$

for all u and using Gronwall's lemma, we have

$$\mathbb{E}(S_u(t, x) - S_u(t, y))^+ = 0. \quad \square$$

Theorem 1. *The section of D^T at the point t has the form*

$$D_t^T = (b^T(t), +\infty), \quad 0 \leq t < T,$$

where $b^T(t)$ is some function which satisfies the condition

$$0 < b^T(t) < c, \quad 0 \leq t < T.$$

Proof. It is clear that $0 \notin D_t^T$. Indeed, suppose that $0 \in D_t^T$, it is the same as $\{(t, 0) \in D^T : v^T(t, 0) > g(0) = c\}$, which contradicts the condition $v^T(t, x) < c$.

We have

$$\begin{aligned} dS_u(t, x) &= r(u) S_u(t, x) du \\ &\quad + \sigma(u, S_u(t, x)) S_u(t, x) dW_u, \quad t \leq u \leq T, \\ S_t(t, x) &= x, \quad x > 0, \\ dS_u(t, y) &= r(u) S_u(t, y) du \\ &\quad + \sigma(u, S_u(t, y)) S_u(t, y) dW_u, \quad t \leq u \leq T, \end{aligned}$$

$$S_t(t, y) = y, \quad y > 0.$$

We are to prove that if $x \in D_t^T$ and $y > 0$, then $y \in D_t^T$.

Note that for the stopping problem for the diffusion process $S_u(t, x)$ there exists an optimal stopping time $\tau^*(t, x)$ (see [6], Theorems 3.1.8–3.1.10)

Using the preceding lemma, for the value function $v^T(t, x)$ defined in (6) we have

$$\begin{aligned} v^T(t, y) - v^T(t, x) &= v^T(t, y) - \mathbb{E} \left[e^{-\int_t^{\tau^*(t, x)} r(u) du} (c - S_{\tau^*(t, x)}(t, x))^+ \right] \\ &\geq \mathbb{E} \left[e^{-\int_t^{\tau^*(t, x)} r(u) du} (c - S_{\tau^*(t, x)}(t, y))^+ - e^{-\int_t^{\tau^*(t, x)} r(u) du} (c - S_{\tau^*(t, x)}(t, x))^+ \right] \\ &= \mathbb{E} \left[e^{-\int_t^{\tau^*(t, x)} r(u) du} (c - S_{\tau^*(t, x)}(t, y)) - e^{-\int_t^{\tau^*(t, x)} r(u) du} (c - S_{\tau^*(t, x)}(t, x)) \right] \\ &\quad + \mathbb{E} \left[e^{-\int_t^{\tau^*(t, x)} r(u) du} (c - S_{\tau^*(t, x)}(t, y))^- - e^{-\int_t^{\tau^*(t, x)} r(u) du} (c - S_{\tau^*(t, x)}(t, x))^- \right] \\ &\geq \mathbb{E} \left[e^{-\int_t^{\tau^*(t, x)} r(u) du} (c - S_{\tau^*(t, x)}(t, y)) - e^{-\int_t^{\tau^*(t, x)} r(u) du} (c - S_{\tau^*(t, x)}(t, x)) \right] \\ &= \mathbb{E} \left[e^{-\int_t^{\tau^*(t, x)} r(u) du} (S_{\tau^*(t, x)}(t, x) - S_{\tau^*(t, x)}(t, y)) \right] \\ &= \mathbb{E} [S_t(t, x) - S_t(t, y)] = x - y, \end{aligned}$$

i.e., $v^T(t, y) \geq (x - y) + v^T(t, x)$. Since $v^T(t, x) > (c - x)^+$, we have

$$v^T(t, y) > x - y + (c - x)^+ > x - y + c - x = c - y,$$

where $v^T(t, y) > 0$, i.e., $v^T(t, y) > (c - y)^+$. □

To derive the early exercise premium representation of the American put option, we need to investigate the properties of a local time for general continuous semimartingales. To this end, let us consider some continuous semimartingale Z_t , $0 \leq t \leq T$, and its set of zeros

$$H = \{(t, w) : Z_t(w) = 0\}. \quad (10)$$

Denote by $L_t^\circ(Z)$ the right local time of the semimartingale at zero. We also need to use the following stopping times

$$D_t(w) = \inf \{s > t : Z_s(w) \neq 0\} \wedge T, \quad 0 \leq t \leq T, \quad (11)$$

and to consider the corresponding predictable random set

$$\tilde{H}_0 = \bigcup_r (r, D_r], \quad (12)$$

where $(r, D_r]$ denotes the stochastic intervals and the union is taken over all rationals r , such that $0 \leq r \leq T$.

Theorem 2. *Let us consider some continuous semimartingale Z_t , $0 \leq t \leq T$, and its related set of zeros $H = \{(t, w) : Z_t(w) = 0\}$. Let us consider its arbitrary, progressively measurable open subset $H_0 \subseteq H$. Then the following relations are valid:*

$$\begin{aligned} \text{I.} \quad & \int_0^t I_{\tilde{H}_0} dL_s^\circ(Z) = 0, & \int_0^t I_{\tilde{H}_0} dL_s^\circ(-Z) = 0, \\ \text{II.} \quad & \int_0^t I_{H_0} dL_s^\circ(Z) = 0, & \int_0^t I_{H_0} dL_s^\circ(-Z) = 0. \end{aligned}$$

In this case, if H_0 is also a predictable random set, then

$$\text{III.} \quad \int_0^t I_{H_0} dZ_s(w) = 0, \quad 0 \leq t \leq T.$$

Proof. We have

$$Z_t = Z_0 + \int_0^t (I_{(Z_s > 0)} + I_{(Z_s = 0)} + I_{(Z_s < 0)}) dZ_s, \quad (13)$$

$$Z_t^+ = Z_0^+ + \int_0^t (I_{(Z_s > 0)} dZ_s + \frac{1}{2} L_t^\circ(Z)), \quad (14)$$

$$(-Z_t)^+ = (-Z_0)^+ + \int_0^t I_{(-Z_s > 0)} d(-Z_s) + \frac{1}{2} L_t^\circ(-Z). \quad (15)$$

From equality (15) we obtain

$$Z_t^- = Z_0^- - \int_0^t I_{(Z_s < 0)} dZ_s + \frac{1}{2} L_t^\circ(-Z). \quad (16)$$

Subtracting (16) from (14) gives

$$Z_t = Z_0 + \int_0^t (I_{(Z_s > 0)} + I_{(Z_s < 0)}) dZ_s + \frac{1}{2} (L_t^\circ(Z) - L_t^\circ(-Z)). \quad (17)$$

If we equate equalities (17) and (13) to each other, then we have

$$\int_0^t I_{(Z_s = 0)} dZ_s = \frac{1}{2} (L_t^\circ(Z) - L_t^\circ(-Z)). \quad (18)$$

Consider the stochastic interval $(t, D_t(w)]$. Then, using the above equalities, we obtain

$$\begin{aligned} Z_{D_t}^+ - Z_t^+ &= \int_t^{D_t} I_{(Z_s > 0)} dZ_s + \frac{1}{2} (L_{D_t}^\circ(Z) - L_t^\circ(Z)), \\ Z_{D_t}^- - Z_t^- &= - \int_0^{D_t} I_{(Z_s < 0)} dZ_s + \frac{1}{2} (L_{D_t}^\circ(-Z) - L_t^\circ(-Z)). \end{aligned}$$

The latter two equalities are the same as

$$\begin{aligned} Z_{D_t}^+ - Z_t^+ &= \int_t^T I_{(t < s \leq D_t, Z_s > 0)} dZ_s + \frac{1}{2} (L_{D_t}^\circ(Z) - L_t^\circ(Z)), \\ Z_{D_t}^- - Z_t^- &= - \int_t^T I_{(t < s \leq D_t, Z_s < 0)} dZ_s + \frac{1}{2} (L_{D_t}^\circ(-Z) - L_t^\circ(-Z)). \end{aligned}$$

Let us consider the random variables

$$Z_{D_t}^+ - Z_t^+, \quad Z_{D_t}^- - Z_t^-, \quad I_{(t < s \leq D_t, Z_s > 0)}, \quad I_{(t < s \leq D_t, Z_s < 0)}$$

and for each $\omega \in \Omega$ let us distinguish two cases:

- 1) $D_t(\omega) = t$;
- 2) $D_t(\omega) > t$.

In case 1) the above-given random variables are equal to zero. In case 2), we observe that if s is such that $t < s \leq D_t(\omega)$, then $Z_s = 0$ and thus, by the continuity of the process Z_t , we have $Z_{D_t} = 0$, $Z_t = 0$, which means that in case 2) also all four random variables are equal to zero. Hence it follows that

$$L_{D_t}^\circ(Z) = L_t^\circ(Z), \quad L_{D_t}^\circ(-Z) = L_t^\circ(-Z)$$

and therefore

$$\int_{(r, D_r]} dL_s^\circ(Z) = 0, \quad \int_{(r, D_r]} dL_s^\circ(-Z) = 0,$$

By the definition of \tilde{H}_0 ,

$$\int_{\tilde{H}_0} dL_s^\circ(Z) = 0, \quad \int_{\tilde{H}_0} dL_s^\circ(-Z) = 0$$

which is the same as

$$\int_0^t I_{\tilde{H}_0} dL_s^\circ(Z) = 0, \quad \int_0^t I_{\tilde{H}_0} dL_s^\circ(-Z) = 0.$$

The first part is thereby proved.

Since $H_0 \subseteq \bigcup_r (r, D_r] = \tilde{H}_0$, the following relation is valid:

$$\int_0^t I_{H_0} dL_s^\circ(Z) = 0, \quad \int_0^t I_{H_0} dL_s^\circ(-Z) = 0.$$

From (18) we obtain

$$I_{(Z_s=0)} dZ_s = \frac{1}{2} d(L_t^\circ(Z) - L_t^\circ(-Z)).$$

When H_0 is predictable, we can write

$$\begin{aligned} I_{H_0} I_{(Z_s=0)} dZ_s &= \frac{1}{2} I_{H_0} d(L_t^\circ(Z) - L_t^\circ(-Z)), \\ \int_0^t I_{H_0} dZ_s &= \frac{1}{2} \int_0^t I_{H_0} dL_s^\circ(Z) - \frac{1}{2} \int_0^t I_{H_0} dL_s^\circ(-Z) = 0. \end{aligned} \quad \square$$

Theorem 3. *Let Z_t be a continuous non-negative semimartingale with its decomposition*

$$Z_t = Z_0 + M_t^z + G_t^+ - G_t^-, \quad 0 \leq t \leq T, \quad (19)$$

where M_t^z is a local martingale, $G = G_t^+ - G_t^-$ is a process of finite variation, and G_t^+, G_t^- are some nondecreasing processes.

If $\int_0^T I_{(H \setminus \tilde{H}_0)} dG_s^+ = 0$, then $L_t^\circ(Z) = 0$ and $\int_0^t I_{(Z_s=0)} dZ_s = 0$.

Proof. It is clear that the right local time $L_t^\circ(-Z) = 0$, $t \geq 0$.

We have

$$\begin{aligned} \frac{1}{2} L_t^\circ(Z) &= \int_0^t I_{(Z_s=0)} dZ_s \\ &= \int_0^t I_{(Z_s=0)} dM_s^z + \int_0^t I_{(Z_s=0)} dG_s^+ - \int_0^t I_{(Z_s=0)} dG_s^-. \end{aligned} \quad (20)$$

Using occupation time formula (see, e.g., [9], p. 216, Corollary 1) we obtain

$$\int_0^t I_{(Z_s=0)} d\langle M^z \rangle_s = 0,$$

so we have

$$\int_0^t I_{(Z_s=0)} dM_s^z = 0. \quad (21)$$

Using (21), from (20) we obtain

$$\frac{1}{2}L_t^\circ(Z) + \int_0^t I_{(Z_s=0)} dG_s^- = \int_0^t I_{(Z_s=0)} dG_s^+,$$

from which we conclude that the random processes $\frac{1}{2}L_t^\circ(Z)$ and $\int_0^t I_{(Z_s=0)} dG_s^-$ are absolutely continuous with respect to the process $\int_0^t I_{(Z_s=0)} dG_s^+$. Thus for these random processes we have the representations

$$\frac{1}{2}L_t^\circ(Z) = \int_0^t \mu_s^1 I_{(Z_s=0)} dG_s^+, \quad (22)$$

$$\int_0^t I_{(Z_s=0)} dG_s^- = \int_0^t \mu_s^2 I_{(Z_s=0)} dG_s^+, \quad (23)$$

where the processes μ_s^1 and μ_s^2 are both vary in $[0, 1]$.

From equality (22) we obtain

$$\frac{1}{2}dL_t^\circ(Z) = \mu_t^1 I_{(Z_t=0)} dG_t^+.$$

By virtue of Theorem 2, $\int_0^t I_{\tilde{H}_0} dL_s^\circ(Z) = 0$ and therefore $L_t^\circ(Z) = \int_0^t I_{H \setminus \tilde{H}_0} dL_s^\circ(Z)$. Hence

$$L_t^\circ(Z) = 2 \int_0^t I_{(H \setminus \tilde{H}_0)} \mu_s^1 I_{(Z_s=0)} dG_s^+ \leq 2 \int_0^t I_{(H \setminus \tilde{H}_0)} dG_s^+ = 0$$

and thus we obtain $L_t^\circ(Z) = 0$, $\int_0^t I_{(Z_s=0)} dZ_s = 0$. □

Before formulating the main result of this paper on the early exercise premium representation of the American put option, we need to prove one more lemma.

Lemma 2. *Consider a random process $X_t(s, x) = \ln S_t(s, x)$, $s \leq t \leq T$, where $S_t(s, x)$ is the random process defined by equation (3). Then $X_t(s, x)$ satisfies the stochastic differential equation*

$$dX_t(s, x) = \hat{\mu}(t, X_t(s, x)) dt + \hat{\sigma}(t, X_t(s, x)) dW_t, \quad s \leq t \leq T, \quad (24)$$

where

$$\hat{\mu}(t, x) = r(t) - \frac{1}{2} \sigma^2(t, e^x), \quad \hat{\sigma}(t, x) = \sigma(t, e^x),$$

and is its unique solution. Moreover, if $c(t)$, $0 \leq t \leq T$, is some measurable function, then the relations

$$\mathbf{P}\{X_t(s, x) = c(t)\} = 0$$

are valid for all $t \geq s$.

Proof. By virtue of equality (3) we have and using Ito's formula for $X_t(s, x) = \ln S_t(s, x)$, $s \leq t \leq T$, we obtain

$$d \ln S_t(s, x) = \left(r(t) - \frac{1}{2} \sigma^2(t, S_t(s, x)) \right) dt + \sigma(t, S_t(s, x)) dW_t, \quad s \leq t \leq T,$$

by which come to the desirable result (24), i.e., $X_t(s, x)$ satisfies equation (24) and is its unique solution. Indeed, suppose equation (24) has two solutions $X_t^1(s, x)$ and $X_t^2(s, x)$. Then $e^{X_t^1(s, x)}$ and $e^{X_t^2(s, x)}$ are solutions of equation (3), but this equation has only one solution and therefore $X_t^1(s, x) = X_t^2(s, x)$ for almost all $s \geq t$. If we follow [12], Theorem 9.1.9, then we have

$$\mathbf{P}\{X_t(s, x) = c(t)\} = \int_{\{c(t)\}} \hat{P}(s, x; t, y) dy = 0,$$

where $\hat{P}(s, x; t, y)$ is the transition density of the nonhomogeneous Markov diffusion process $X_t(s, x)$. \square

We will now formulate the theorem on representation of the value function of the American put option.

Theorem 4. *Let $b^T(t)$ be the boundary function of the continuation domain defined in Theorem 1. Assume that it has left and right limits $b^T(t+)$, $b^T(t-)$, $0 < t < T$. Then for the value function of the American put option the following early exercise premium representation is valid*

$$\begin{aligned} v^T(t, x) = & \mathbf{E} \left(e^{-\int_t^T r(u) du} (c - S_T(t, x))^+ \right) \\ & + \int_t^T cr(u) e^{-\int_t^u r(s) ds} P(t, x; u, (0, b^T(u)]) du, \quad 0 \leq t \leq T, \quad x > 0, \end{aligned}$$

where $P(t, x; u, \cdot)$ is the transition probability for the nonhomogeneous Markov diffusion process $S_t(s, x)$.

Proof. Consider the random process defined by equality (5):

$$X_t = e^{-\int_0^t r(u) du} (c - S_t)^+.$$

For the process $(c - S_t)^+$ we write the Tanaka–Meyer formula

$$(c - S_t)^+ = (c - S_0)^+ + \int_0^t I_{(c - S_t > 0)} d(c - S_t) + \frac{1}{2} L_t^\circ(c - S_t),$$

we have

$$\begin{aligned}
 d\left(e^{-\int_0^t r(u) du} (c - S_t)^+\right) &= -r(t)e^{-\int_0^t r(u) du} (c - S_t)^+ dt + e^{-\int_0^t r(u) du} d(c - S_t)^+ \\
 &= -r(t)e^{-\int_0^t r(u) du} dt (c - S_t)^+ + e^{-\int_0^t r(u) du} \left(-I_{(S_t < c)} dS_t + \frac{1}{2} L_t^\circ(c - S_t)\right) \\
 &= -r(t)e^{-\int_0^t r(u) du} dt (c - S_t)^+ - e^{-\int_0^t r(u) du} I_{(S_t < c)} dS_t + \frac{1}{2} e^{-\int_0^t r(u) du} dL_t^\circ(c - S_t).
 \end{aligned}$$

Now, using (1) we obtain

$$\begin{aligned}
 d\left(e^{-\int_0^t r(u) du} (c - S_t)^+\right) &= -r(t)e^{-\int_0^t r(u) du} dt (c - S_t)^+ - e^{-\int_0^t r(u) du} I_{(S_t < c)} r(t) S_t dt \\
 &\quad - e^{-\int_0^t r(u) du} I_{(S_t < c)} \sigma(t, S_t) S_t dW_t + \frac{1}{2} e^{-\int_0^t r(u) du} dL_t^\circ(c - S_t) \\
 &= -r(t)e^{-\int_0^t r(u) du} ((c - S_t)^+ + I_{(S_t < c)} S_t) dt \\
 &\quad + \frac{1}{2} e^{-\int_0^t r(u) du} dL_t^\circ(c - S_t) - e^{-\int_0^t r(u) du} I_{(S_t < c)} \sigma(t, S_t) S_t dW_t.
 \end{aligned}$$

Note that $(c - S_t)^+ + I_{(S_t < c)} S_t = cI_{(S_t < c)}$ and therefore we have

$$\begin{aligned}
 dX_t &= -r(t)e^{-\int_0^t r(u) du} cI_{(S_t < c)} dt + \frac{1}{2} e^{-\int_0^t r(u) du} dL_t^\circ(c - S_t) \\
 &\quad - e^{-\int_0^t r(u) du} I_{(S_t < c)} \sigma(t, S_t) S_t dW_t,
 \end{aligned}$$

i.e., for the process X_t we have obtained the representation

$$X_t = m_t + A_t^+ - A_t^-,$$

where

$$A_t^- = \int_0^t cr(u) I_{(S_u < c)} e^{-\int_0^u r(s) ds} du.$$

Let us consider the nonnegative process $Z_t = Y_t - X_t$, $0 \leq t \leq T$, where Y_t and X_t are the processes defined by equalities (4) and (5), respectively. Using decompositions (7) and (8) for Y_t and X_t , we find that

$$Z_t = Z_0 + M_t^z - C_t - A_t, \quad M_t^z = M_t - m_t, \quad A_t = A_t^+ - A_t^-, \quad 0 \leq t \leq T,$$

where m_t and M_t are uniformly integrable martingales, while A_t^+ , A_t^- and C_t are predictable nondecreasing integrable processes, $A_0^+ = A_0^- = C_0 = 0$.

We have

$$\frac{1}{2} L_t^\circ(Z) = \int_0^t I_{(Z_s=0)} dZ_s = \int_0^t I_{(Z_s=0)} dM_s^z - \int_0^t I_{(Z_s=0)} dC_s - \int_0^t I_{(Z_s=0)} dA_s.$$

It is known that $\int_0^t I_{(Z_s=0)} dM_s^z = 0$.

It is also known (see [10], Lemma 1) that $\int_0^t I_{(Z_s=0)} dC_s = C_t$. Therefore we have

$$\frac{1}{2} L_t^\circ(Z) + C_t + \int_0^t I_{(Z_s=0)} dA_s^+ = \int_0^t I_{(Z_s=0)} dA_s^-.$$

Taking the latter equality into account, we conclude that $\frac{1}{2} L_t^\circ(Z)$, C_t , and $\int_0^t I_{(Z_s=0)} dA_s^+$ are absolutely continuous processes with respect to the process $\int_0^t I_{(Z_s=0)} dA_s^-$.

Suppose that

$$\int_0^T I_{H \setminus \tilde{H}_0} dA_s^- = 0 \quad \text{P-a.s.} \quad (25)$$

If instead of the component G_t^+ used in Theorem 3, we take A_t^- , and in the role of the component G_t^- we take $C_t + A_t^+$, then using Theorem 3 we obtain

$$\begin{aligned} L_t^\circ(Z) &= 0, \quad \int_0^t I_{(Z_s=0)} dZ_s = 0, \\ C_t &= - \int_0^t I_{(Z_s=0)} dA_s. \end{aligned} \quad (26)$$

Let us now prove that (25) is valid. Besides of H and \tilde{H}_0 introduced above by relations (10) and (12), respectively, let us introduce the following sets:

$$\begin{aligned} \overline{H}_0 &= \left\{ (t, w) : 0 < t \leq T, S_t(w) - b^T(t-) < 0 \right\}, \\ \underline{H}_0 &= \left\{ (t, w) : 0 < t \leq T, S_t(w) - b^T(t) < 0 \right\}, \\ N &= \left\{ (t, w) : 0 < t \leq T, b^T(t-) = b^T(t) \right\} \\ &= \left\{ t : 0 < t \leq T, b^T(t-) = b^T(t) \right\} \times \Omega. \end{aligned}$$

Note that

$$H = \left\{ (t, w) : 0 \leq t \leq T, S_t(w) \leq b^T(t) \right\}.$$

We have the inclusions:

$$\overline{H}_0 \subseteq \tilde{H}_0, \quad \overline{H}_0 \subseteq H, \quad H \setminus \tilde{H}_0 \subseteq H \setminus \overline{H}_0.$$

In view of the latter inclusion for the validity of (25) it is sufficient to verify that $\int_0^T I_{(H \setminus \overline{H}_0)} dA_s^- = 0$ holds. To prove this relation let us first verify that $H \setminus \overline{H}_0 \subseteq (H \setminus \underline{H}_0) \cup N^c$.

Indeed, $\overline{H}_0 \cap N = \underline{H}_0 \cap N$ and

$$\begin{aligned} H \setminus \overline{H}_0 &\subseteq H \setminus (\overline{H}_0 \cap N) \\ &= H \setminus (\underline{H}_0 \cap N) = (H \setminus \underline{H}_0) \cup (H \setminus N) \subseteq (H \setminus \underline{H}_0) \cup N^c. \end{aligned}$$

Therefore we have

$$\begin{aligned} \int_0^T I_{H \setminus \overline{H}_0} dA_s^- &\leq \int_0^T I_{(H \setminus \underline{H}_0) \cup N^c} dA_s^- \leq \int_0^T I_{(S_s(w)=b^T(s))} dA_s^- + \int_0^T I_{(b^T(s-) \neq b^T(s))} dA_s^- \\ &= \int_0^T I_{(S_u(w)=b^T(u))} cR(u) I_{(S_u < c)} e^{-\int_0^u r(s) ds} du \leq cR \int_0^T I_{(S_u(w)=b^T(u))} du. \end{aligned}$$

By virtue of Lemma 2, for $c(t) = \ln b^T(t)$ we have $\mathbf{P}\{S_u = b^T(u)\} = 0$, $\int_0^T I_{(H \setminus \overline{H}_0)} dA_s^- = 0$, which in turn implies (25) and because of (26) we obtain

$$\begin{aligned} C_t &= - \int_0^t I_{(Z_s=0)} dA_s = - \int_0^t I_{(S_u \leq b^T(u))} dA_u \\ &= - \int_0^t I_{(S_u \leq b^T(u))} dA_u^+ + \int_0^t I_{(S_u \leq b^T(u))} dA_u^- \\ &= - \frac{1}{2} \int_0^t I_{(S_u \leq b^T(u))} e^{-\int_0^u r(s) ds} I_{(S_u=c)} dL_u^\circ(c - S_t) \\ &\quad + \int_0^t I_{(S_u \leq b^T(u))} cR(u) e^{-\int_0^u r(s) ds} I_{(S_u < c)} du. \end{aligned}$$

Note that $I_{(S_u \leq b^T(u))} I_{(S_u=c)} = 0$, $I_{(S_u \leq b^T(u))} I_{(S_u < c)} = I_{(S_u \leq b^T(u))}$. Therefore eventually we obtain

$$C_t = \int_0^t cR(u) e^{-\int_0^u r(s) ds} I_{(S_u \leq b^T(u))} du.$$

As we know, $Y_t = M_t - C_t$, $Y_T = M_T - C_T$, $Y_T = X_T$. Since

$$Y_t - Y_T = Y_t - X_T = M_t - M_T + (C_T - C_t),$$

we have

$$Y_t - \mathbf{E}(X_T | \mathcal{F}_t^W) = \mathbf{E}(C_T - C_t | \mathcal{F}_t^W), \quad (27)$$

where (\mathcal{F}_t^W) is the filtration generated by W . Multiplying both sides of equality

(27) by $e^{\int_0^t r(u) du}$, we obtain

$$\begin{aligned} V_t &= e^{\int_0^t r(u) du} \mathbb{E}(X_T | \mathcal{F}_t^W) + e^{\int_0^t r(u) du} \mathbb{E}(C_T - C_t | \mathcal{F}_t^W), \\ V_t &= \mathbb{E}\left(e^{-\int_t^T r(u) du} (c - S_T)^+ | \mathcal{F}_t^W\right) + e^{\int_0^t r(u) du} \mathbb{E}(C_T - C_t | \mathcal{F}_t^W), \\ V_t &= \mathbb{E}\left(e^{-\int_t^T r(u) du} (c - S_T)^+ | \mathcal{F}_t^W\right) + \mathbb{E}\left(\int_t^T cr(u) e^{-\int_t^u r(s) ds} I_{(S_u \leq b^T(u))} du | \mathcal{F}_t^W\right). \end{aligned}$$

We know that $V_t = v^T(t, S_t)$, where V_t is the value process of the American put option, and $v^T(s, x)$ is the value function.

Since $\mathcal{F}_t^W = \mathcal{F}_t^S$, $S_t(t, x) = x$, $x > 0$, we have

$$\begin{aligned} v^T(t, x) &= \mathbb{E}\left(e^{-\int_t^T r(u) du} (c - S_T(t, x))^+\right) \\ &\quad + \mathbb{E}\left(\int_t^T cr(u) e^{-\int_t^u r(s) ds} I_{(S_u(t, x) \leq b^T(u))} du\right), \\ v^T(t, x) &= \mathbb{E}\left(e^{-\int_t^T r(u) du} (c - S_T(t, x))^+\right) \\ &\quad + \int_t^T cr(u) e^{-\int_t^u r(s) ds} P(t, x; u, (0, b^T(u))) du. \end{aligned} \quad \square$$

Corollary 1. *Let the functions $r(t)$ and $\sigma(t, x)$ be independent of the time, i.e., $r(t) = r$, $\sigma(t, x) = \sigma(x)$. Then for the American put option, the early exercise premium representation holds*

$$v^T(t, x) = \mathbb{E}(e^{-r(T-t)} (c - S_T(t, x))^+) + \int_t^T cre^{-r(u-t)} P(t, x; u, (0, b^T(u))) du.$$

Proof. To prove this statement, it is sufficient to show that the boundary $b^T(t)$ is a monotone, in particular nondecreasing function, which automatically implies the fulfillment of the condition of Theorem 4.

We first show that the function $v^T(t, x)$ is nonincreasing with respect to t . Indeed, we have

$$\begin{aligned} v^T(t, x) &= \sup_{t \leq \tau \leq T} \mathbb{E}(e^{-r(\tau-t)} (c - S_\tau(t, x))^+), \quad 0 \leq s < t \leq T, \\ v^T(s, x) &= \sup_{s \leq \tau \leq T} \mathbb{E}(e^{-r(\tau-s)} (c - S_\tau(s, x))^+). \end{aligned}$$

Since S_t is a homogeneous Markov diffusion process, we have

$$\begin{aligned} v^T(t, x) &= \sup_{0 \leq \tau \leq T-t} \mathbf{E}(e^{-r\tau} (c - S_\tau(0, x))^+), \\ v^T(s, x) &= \sup_{0 \leq \tau \leq T-s} \mathbf{E}(e^{-r\tau} (c - S_\tau(0, x))^+). \end{aligned}$$

From the latter equalities we find that $v^T(s, x) \geq v^T(t, x)$. We have

$$v^T(s, b^T(t) + \varepsilon) \geq v^T(t, b^T(t) + \varepsilon) > g(b^T(t) + \varepsilon).$$

This inequality is true by virtue of the definition of the boundary. For each $\varepsilon > 0$ we have $b^T(t) + \varepsilon \in D_s$ or, which is the same, $b^T(t) + \varepsilon > b^T(s)$. Finally, we obtain $b^T(s) \leq b^T(t)$, which in turn implies that the boundary $b^T(t)$ is a nondecreasing function. \square

Corollary 2. *Let us consider the Black-Scholes model $r(t) = r$, $\sigma(t, x) = \sigma$. Then the value function of the American put option is written as*

$$\begin{aligned} v^T(t, x) &= \mathbf{E}(e^{-r(T-t)} (c - S_T(t, x))^+) \\ &\quad + \int_t^T c r e^{-r(u-t)} \Phi\left(\frac{\ln \frac{b^T(u)}{x} - (r - \frac{\sigma^2}{2})(u-t)}{\sigma \sqrt{u-t}}\right) du. \end{aligned}$$

where Φ is the standard normal distribution function.

Proof. By Corollary 1, the value function of the American put option has the form

$$v^T(t, x) = \mathbf{E}(e^{-r(T-t)} (c - S_T(t, x))^+) + \int_t^T c r e^{-r(u-t)} P(t, x; u, (0, b^T(u)]) du.$$

By virtue of the definition of transition probability we have

$$P(t, x; u, (0, b^T(u)]) = \mathbf{P}\{S_u(t, x) \leq b^T(u)\}.$$

Using the condition $S_t = S_0 \exp(\sigma W_t + (r - \frac{\sigma^2}{2})t)$ we obtain

$$\begin{aligned} S_u(t, x) &= x \exp\left(\sigma(W_u - W_t) + \left(r - \frac{\sigma^2}{2}\right)(u-t)\right), \quad t \leq u \leq T, \\ P(t, x; u, (0, b^T(u)]) &= \mathbf{P}\left\{\exp\left(\sigma(W_u - W_t) + \left(r - \frac{\sigma^2}{2}\right)(u-t)\right) \leq \frac{b^T(u)}{x}\right\} \\ &= \Phi\left(\frac{\frac{1}{\sigma} \ln \frac{b^T(u)}{x} - \frac{1}{\sigma} \left(r - \frac{\sigma^2}{2}\right)(u-t)}{\sqrt{u-t}}\right), \end{aligned}$$

i.e., we have

$$\begin{aligned} v^T(t, x) &= \mathbf{E}(e^{-r(T-t)} (c - S_t(t, x))^+) \\ &\quad + \int_t^T c r e^{-r(u-t)} \Phi\left(\frac{\ln \frac{b^T(u)}{x} - (r - \frac{\sigma^2}{2})(u-t)}{\sigma \sqrt{u-t}}\right) du. \end{aligned} \quad \square$$

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