

MONADIC BL -ALGEBRAS

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Abstract. We define and study monadic BL -algebras as a pair of BL -algebras one of which is a special case of relatively complete subalgebra.

2000 Mathematics Subject Classification: 03B52, 03B05.

Key words and phrases: BL -algebras, monadic algebras.

1. INTRODUCTION

The notion BL -algebra was introduced by Hajek in [1] as an algebraic counterpart of the Basic logic \mathcal{BL} . The predicate Basic logic $Q\mathcal{BL}$ is defined in a standard way: given some universe and complete BL -algebra, which generates the variety of all BL -algebras, and defining existential (universal) quantifier as supremum (infimum), the predicate calculus is defined as all formulas having value 1 for any assignment.

Monadic BL -algebras is an algebraic model for the monadic predicate calculus $q\mathcal{BL}$, in which only a single fixed individual variable occurs. We follow P. R. Halmos' study of monadic Boolean algebras.

Adapting the axiomatization of the predicate calculus $Q\mathcal{BL}$, given by Hajek in [1], we can define the modal Basic logic \mathcal{MBL} as a logic which contains the Basic logic \mathcal{BL} , formulas of which are written as the axiom schemes

- M1.** $\forall\alpha \rightarrow \alpha$
- M2.** $\alpha \rightarrow \exists\alpha$
- M3.** $\forall(\exists\alpha \rightarrow \beta) \rightarrow (\exists\alpha \rightarrow \forall\beta)$
- M4.** $\forall(\alpha \rightarrow \exists\beta) \rightarrow (\exists\alpha \rightarrow \exists\beta)$
- M5.** $\forall(\alpha \vee \exists\beta) \rightarrow (\forall\alpha \vee \exists\beta)$

and is closed under modus ponens and necessitation ($\alpha/\forall\alpha$).

Let L denote a first-order language based on $\vee, \wedge, *, \rightarrow, \exists, \forall$ and L_m denote the monadic propositional language based on $\vee, \wedge, *, \rightarrow, \exists, \forall$, and $\text{Form}(L)$ and $\text{Form}(L_m)$ be the set of all formulas of L and L_m respectively. We fix a variable x in L , associate with each propositional letter p in L_m a unique monadic predicate $p^*(x)$ in L and define by induction a translation $\Psi : \text{Form}(L_m) \rightarrow \text{Form}(L)$ by putting

- $\Psi(p) = p^*(x)$ if p is propositional variable,
- $\Psi(\alpha \circ \beta) = \Psi(\alpha) \circ \Psi(\beta)$, where $\circ = \vee, \wedge, *, \rightarrow$,
- $\Psi(\exists\alpha) = \exists x\Psi(\alpha)$,
- $\Psi(\forall\alpha) = \forall x\Psi(\alpha)$.

Through this translation Ψ we can identify the formulas of L_m with the monadic formulas of L containing the variable x . Moreover, it is a routine to check that $\Psi(\mathcal{MBL}) \subseteq \mathcal{QBL}$.

In a monadic Boolean algebra B the set of elements of B , which is closed under quantifier operations, forms a relatively complete Boolean subalgebra of B . And, conversely, any relatively complete Boolean subalgebra of a Boolean algebra defines quantifier operations on the Boolean algebra converting the latter into a monadic Boolean algebra. We have the same situation for monadic Heyting algebras [2]. But we do not have such kind of balance for monadic BL -algebras, and for monadic MV -algebras as well [3]. In other words, not every relatively complete BL -subalgebra defines a monadic operator converting a BL -algebra into a monadic BL -algebra. For this aim we introduce the notion of an m -relatively complete subalgebra of a given BL -algebra (as an m -relatively complete subalgebra of a given MV -algebra [3]). We restore thereby the balance between m -relatively complete subalgebras of a given BL -algebra A and monadic operators on A .

2. MONADIC BL -ALGEBRAS

A monadic BL -algebra (MBL -algebra for short)

$$(B, \vee, \wedge, \rightarrow, *, \exists, \forall, 0, 1)$$

is a universal algebra of type $(2, 2, 2, 1, 1, 1, 0, 0)$ such that:

- 1) $(B, \vee, \wedge, 0, 1)$ is a bounded lattice;
- 2) $(B, *, 1)$ is a commutative monoid with identity:

$$x * q = q * p,$$

$$p * (q * r) = (p * q) * r,$$

$$p * 1 = 1 * p.$$

- 3) (1) $p \wedge (q \rightarrow (p * q)) = p$,
- (2) $((p \rightarrow q) * p) \vee q = q$,
- (3) $(p \rightarrow (p \vee q)) = 1$,
- (4) $((p \rightarrow r) \rightarrow (r \rightarrow (p \vee q))) = 1$,
- (5) $(p \wedge q) * r = (p * r) \wedge (q * r)$,
- (6) $p \wedge q = p * (p \rightarrow q)$,
- (7) $p \vee q = ((p \rightarrow q) \rightarrow q) \wedge ((q \rightarrow p) \rightarrow p)$,
- (8) $(p \rightarrow q) \vee (q \rightarrow p) = 1$.
- 4) $E1. p \rightarrow \exists p = 1$.
- $A1. \forall p \rightarrow p = 1$.
- $E2. \forall(p \rightarrow \exists q) = (\exists p \rightarrow \exists q)$.
- $A2. \forall(\exists p \rightarrow q) = (\exists p \rightarrow \forall q)$.
- $A3. \forall(p \vee \exists q) = (\forall p \vee \exists q)$.

Let us note that the monoidal operation multiplication $*$, which preserves partial order \leq and binary (left-) residuation operation \rightarrow , is characterized by $x * y \leq z$ if and only if $x \leq y \rightarrow z$.

Let us consider the set M of all functions from some non-empty universe X into the unit interval *BL*-algebra $([0, 1], \vee, \wedge, \rightarrow, *, 0, 1)$. Then we can convert M into the *MBL*-algebra in the following way. $p \circ q = p(x) \circ q(x)$ for every $p, q \in M$, where $\circ \in \{\vee, \wedge, \rightarrow, *\}$ and $\exists p = \text{Sup}_X p(x)$, $\forall p = \text{Inf}_X p(x)$. We call this *MBL*-algebra the functional *MBL*-algebra.

E1 and *A1* are obvious in the functional *MBL*-algebra. Let us prove the validity of *E2*, *A2* and *A3* in the functional model.

$$E2. \quad \text{Inf}_X(p(x) \rightarrow \text{Sup}_X q(x)) = (\text{Sup}_X p(x) \rightarrow \text{Sup}_X q(x)).$$

Indeed, $\text{Sup}_X p(x) \geq p(x) \implies \text{Sup}_X p(x) \rightarrow \text{Sup}_X q(x) \leq p(x) \rightarrow \text{Sup}_X q(x) \implies \text{Sup}_X p(x) \rightarrow \text{Sup}_X q(x) \leq \text{Inf}_X p(x) \rightarrow \text{Sup}_X q(x) \implies \text{Inf}_X(\text{Sup}_X p(x) \rightarrow \text{Sup}_X q(x)) \leq \text{Inf}_X(p(x) \rightarrow \text{Sup}_X q(x))$. If $z \leq \text{Inf}_X(p(x) \rightarrow \text{Sup}_X q(x))$, then $z \leq p(x) \rightarrow \text{Sup}_X q(x)$ for all $x \in X \implies p(x) \leq (z \rightarrow \text{Sup}_X q(x))$ for all $x \in X \implies \text{Sup}_X p(x) \leq z \rightarrow \text{Sup}_X q(x) \implies z \leq \text{Sup}_X p(x) \rightarrow \text{Sup}_X q(x)$. On the other hand, $p(x) \rightarrow \text{Sup}_X q(x) \geq \text{Sup}_X p(x) \rightarrow \text{Sup}_X q(x) \implies \text{Inf}_X(p(x) \rightarrow \text{Sup}_X q(x)) \geq \text{Sup}_X p(x) \rightarrow \text{Sup}_X q(x)$. Hence we have the equality *E2*.

$$A2. \quad \text{Inf}_X(\text{Sup}_X p(x) \rightarrow q(x)) = \text{Sup}_X p(x) \rightarrow \text{Inf}_X q(x).$$

Indeed, $\text{Sup}_X p(x)$ is a constant, say, $a \in [0, 1]$. Then $\text{Inf}_X(a \rightarrow q(x)) = a \rightarrow \text{Inf}_X q(x)$. Since for each $x \in X$ $q(x) \geq \text{Inf}_X q(x)$, we have $(a \rightarrow q(x)) \geq (a \rightarrow \text{Inf}_X q(x)) \implies \text{Inf}_X(a \rightarrow q(x)) \geq a \rightarrow \text{Inf}_X q(x)$. On the other hand, if $z \leq (a \rightarrow q(x))$ for every $x \in X$, then $z * a \leq q(x)$ for every $x \in X \implies z * a \leq \text{Inf}_X q(x) \implies z \leq (a \rightarrow \text{Inf}_X q(x)) \implies a \rightarrow q(x) \leq a \rightarrow \text{Inf}_X q(x) \implies \text{Inf}_X(a \rightarrow q(x)) \leq a \rightarrow \text{Inf}_X q(x)$. Thus $\text{Inf}_X(\text{Sup}_X p(x) \rightarrow q(x)) = \text{Sup}_X p(x) \rightarrow \text{Inf}_X q(x)$.

$$A3. \quad \text{Inf}_X(p(x) \vee \text{Sup}_X q(x)) = \text{Inf}_X p(x) \vee \text{Sup}_X q(x).$$

Indeed, $\text{Sup}_X q(x) \leq \text{Sup}_X q(x) \vee p(x) \implies \text{Sup}_X q(x) \leq \text{Inf}_X(\text{Sup}_X q(x) \vee p(x))$. Similarly, $\text{Inf}_X p(x) \leq \text{Inf}_X(\text{Sup}_X q(x) \vee p(x))$. Therefore $\text{Inf}_X(p(x) \vee \text{Sup}_X q(x)) \leq \text{Inf}_X p(x) \vee \text{Sup}_X q(x)$. Conversely, let $z \leq \text{Sup}_X q(x) \vee p(x)$ for all $x \in X$. Consider two cases:

1) $\text{Sup}_X q(x) \leq \text{Inf}_X p(x) \implies z \leq p(x)$ for every $x \in X \implies z \leq \text{Inf}_X p(x)$ and $z \leq \text{Sup}_X q(x) \vee \text{Inf}_X p(x)$.

2) $\text{Sup}_X q(x) > \text{Inf}_X p(x) \implies$ for some $x_0 \in X$ $\text{Sup}_X q(x) \geq p(x_0)$. Thus $z \leq \text{Sup}_X q(x)$ and $z \leq \text{Sup}_X q(x) \vee \text{Inf}_X p(x)$. Therefore $z \leq \text{Sup}_X q(x) \vee p(x) \implies z \leq \text{Inf}_X p(x)$.

We shall denote by \mathbb{MBL} the category and the variety of all *MBL*-algebras. In the sequel, for the sake of brevity, we shall sometimes denote the *MBL*-algebra $(B, \vee, \wedge, \rightarrow, *, \exists, \forall, 0, 1)$ by (B, \exists, \forall) .

Now we shall prove some useful equalities.

Lemma 1. (1) $\forall 1 = 1, \quad \exists 0 = 0.$

- (2) $\forall \exists p = \exists p, \exists p \forall p = \forall p.$
- (3) $\exists \exists p = \exists p, \forall \forall p = \forall p.$
- (4) $\forall(\exists p \rightarrow \exists q) = \exists p \rightarrow \exists q.$
- (5) $\exists(\exists p \rightarrow q) \rightarrow (\exists p \rightarrow \exists q).$
- (6) *If $p \leq q$, then $\forall p \leq \forall q$ and $\exists p \leq \exists q$.*
- (7) $\forall(\exists p \vee \exists q) = \exists p \vee \exists q.$
- (8) *If $\forall p = p$, then $\exists p = p$.*
- (9) $\exists(p \vee q) = \exists p \vee \exists q.$
- (10) $\exists(\exists p * \exists q) = \exists p * \exists q.$
- (11) $\forall(p \rightarrow q) \rightarrow (\forall p \rightarrow \forall q) = 1.$
- (12) $\forall(p \rightarrow q) \rightarrow (\exists p \rightarrow \exists q) = 1.$
- (13) $(\forall p * \exists q) \rightarrow \exists(p * q) = 1.$
- (14) $\exists(p * \exists q) = \exists p * \exists q.$
- (15) $\exists(p * \forall q) = \exists p * \forall q.$
- (16) $\exists(\exists p \rightarrow q) \rightarrow (\exists p \rightarrow \exists q) = 1.$
- (17) $\exists(p \rightarrow \exists q) \rightarrow (\forall p \rightarrow \exists q) = 1.$
- (18) $\exists(p * p) = \exists p * \exists p.$
- (19) $\exists(\exists p \rightarrow \exists q) = \exists p \rightarrow \exists q, \exists(\forall p \rightarrow \forall q) = \forall p \rightarrow \forall q.$
- (20) $\exists(\exists p \wedge \exists q) = \exists p \wedge \exists q.$

Proof. (1) $\forall 1 = \forall(p \rightarrow \exists p)$. But, by *E2*, $\forall(p \rightarrow \exists p) = \exists p \rightarrow \exists p = 1$. The other inequality is proved analogously.

(2) $\forall \exists p \leq \exists p$. On the other hand, $\exists p \rightarrow \forall \exists p = \forall(\exists p \rightarrow \exists p) = 1$ (by *A2*) $\implies \exists p \leq \forall \exists p$. The other inequality is proved analogously.

(3) From *E1* we have $\exists \exists p \geq \exists p$. On the other hand, *E2* implies $\exists \exists p \rightarrow \exists p = \forall(\exists p \rightarrow \exists p) = \forall 1 = 1$. Therefore $\exists \exists p \leq \exists p$. The other inequality is proved analogously.

(4) $\forall(\exists p \rightarrow \exists q) = \exists p \rightarrow \forall \exists q = \exists p \rightarrow \exists q$.

5) $\exists p \rightarrow q \leq \exists p \rightarrow \exists q \implies (\exists p \rightarrow q) \rightarrow (\exists p \rightarrow \exists q) = 1 \implies \forall((\exists p \rightarrow q) \rightarrow (\exists p \rightarrow \exists q)) = \exists(\exists p \rightarrow q) \rightarrow (\exists p \rightarrow \exists q) = 1$ (by *E2*).

(6) $p \leq q \implies \forall p \leq q \implies \forall p \vee q = q \implies \forall q = \forall(\forall p \vee q) = \forall p \vee \forall q \implies \forall p \leq \forall q$. $p \leq q \implies p \leq \exists q \implies \exists p \rightarrow \exists q = \forall(p \rightarrow \exists q) = \forall 1 = 1 \implies \exists p \leq \exists q$.

(7) $\forall(\exists p \vee \exists q) = \forall \exists p \vee \forall \exists q = \exists p \vee \exists q$.

(8) $\forall p = p \implies \exists p = \exists \forall p = \forall p = p$.

(9) $p \leq \exists p, q \leq \exists q \implies p \vee q \leq \exists p \vee \exists q \implies \exists(p \vee q) \leq \exists(\exists p \vee \exists q)$. By Lemma 1, item 7 we have $\forall(\exists p \vee \exists q) = \exists p \vee \exists q$. Hence, by Lemma 1, item 8, $\exists(\exists p \vee \exists q) = \exists p \vee \exists q$. Therefore $\exists(p \vee q) \leq \exists p \vee \exists q$. On the other hand, $\exists p \leq \exists(p \vee q), \exists q \leq \exists(p \vee q) \implies \exists p \vee \exists q \leq \exists(p \vee q)$.

(10) $(\exists x * \exists y) \leq \exists(\exists x * \exists y)$. On the other hand, $(\exists x * \exists y) \rightarrow (\exists x * \exists y) =$ (by *E2*) $\forall((\exists x * \exists y) \rightarrow (\exists x * \exists y)) = 1 \implies \exists(\exists x * \exists y) \leq (\exists x * \exists y)$.

(11). $\forall(p \rightarrow q) \rightarrow (\forall p \rightarrow q) = 1 \Rightarrow \forall(\forall(p \rightarrow q) \rightarrow (\forall p \rightarrow q)) =$ (according to (A2)) $\forall(p \rightarrow q) \rightarrow \forall(\forall p \rightarrow q) = 1$. But $\forall(\forall p \rightarrow q) = \forall p \rightarrow \forall q$. Therefore $\forall(p \rightarrow q) \rightarrow (\forall p \rightarrow \forall q) = 1$.

(12). $\forall(p \rightarrow q) \rightarrow (p \rightarrow \exists q) = 1 \Rightarrow \forall(\forall(p \rightarrow q) \rightarrow (p \rightarrow \exists q)) =$ (according to (A2)) $\forall(p \rightarrow q) \rightarrow \forall(p \rightarrow \exists q) =$ (according to (E2)) $\forall(p \rightarrow q) \rightarrow (\exists p \rightarrow \exists q) = 1$.

(13). $p \rightarrow (q \rightarrow (p * q)) = 1$ according to the definition of the implication \rightarrow . Then $\forall(p \rightarrow (q \rightarrow (p * q))) =$ (according to (11)) $\forall p \rightarrow \forall(q \rightarrow (p * q)) =$ (according to (12)) $\forall p \rightarrow (\exists q \rightarrow \exists(p * q)) = (\forall p * \exists q) \rightarrow \exists(p * q) = 1$.

(14). It is obvious that $\exists(p * \exists q) \leq \exists p * \exists q$. On the other hand, $\exists q \rightarrow \forall \exists q = 1 \Rightarrow (\exists p * \exists q) \rightarrow (\exists p * \forall \exists q) = 1$. According to (13) $(\exists p * \exists q) \rightarrow (\exists(p * \exists q)) = 1$. Therefore $\exists p * \exists q \leq \exists(p * \exists q)$.

(15). $\exists(p * \forall q) = \exists(p * \exists \forall q) =$ (according to (14)) $\exists p * \exists \forall q = \exists p * \forall q$.

(16). $\exists p \rightarrow q \rightarrow (\exists p \rightarrow \exists q) = 1$. Hence $\forall((\exists p \rightarrow q) \rightarrow (\exists p \rightarrow \exists q)) = 1 =$ (according to (E2)) $\exists(\exists p \rightarrow q) \rightarrow (\exists p \rightarrow \exists q)$.

(17). $(p \rightarrow \exists q) \rightarrow (\forall p \rightarrow \exists q) = 1$. Therefore $\forall((p \rightarrow \exists q) \rightarrow (\forall p \rightarrow \exists q)) = \exists(p \rightarrow \exists q) \rightarrow (\forall p \rightarrow \exists q) = 1$.

(18). $p * p \leq \exists p * \exists p \Rightarrow \exists(p * p) \leq \exists(\exists p * \exists p) = \exists p * \exists p$. On the other hand, $p * p \rightarrow \exists(p * p) = 1 = \forall(p * p \rightarrow \exists(p * p)) = \forall(p \rightarrow (p \rightarrow \exists(p * p))) = \exists p \rightarrow \forall(p \rightarrow \exists(p * p)) = \exists p \rightarrow (\exists p \rightarrow \exists(p * p)) = (\exists p * \exists p) \rightarrow \exists(p * p)$. Hence $(\exists p * \exists p) \leq \exists(p * p)$.

(19). $\exists(\exists p \rightarrow \exists q) \leq$ (according to (16)) $\exists p \rightarrow \exists \exists q = \exists p \rightarrow \exists q$. The converse inequality is obvious. The identity $\exists(\forall p \rightarrow \forall q) = \forall p \rightarrow \forall q$ is proved analogously.

(20). $\exists(\exists p \wedge \exists q) \leq \exists \exists p = \exists p$, $\exists(\exists p \wedge \exists q) \leq \exists \exists q = \exists q$. Hence $\exists(\exists p \wedge \exists q) \leq \exists p \wedge \exists q \leq \exists(\exists p \wedge \exists q)$. \square

If (A, \exists, \forall) is a monadic BL -algebra, then we denote by $\exists A$ the subset of A such that $x \in \exists A$ if and only if $x = \exists y$ for some $y \in A$.

Lemma 2. $x \in \exists A \Leftrightarrow \exists x = x = \forall x$.

Proof. $x \in \exists A \Rightarrow x = \exists y$ for some $y \in A$. $\exists x = \exists \exists y = \exists y = x$ by (2). $\forall x = \forall \exists y = \exists y = x$ by (2). And $x = \exists x \Rightarrow x \in \exists A$. \square

Lemma 3. $\exists A$ is a BL -subalgebra of (A, \exists, \forall) .

Proof. The proof immediately follows from Lemma 1. \square

3. m -RELATIVELY COMPLETE BL -ALGEBRAS

A subalgebra A_0 of the BL -algebra A is said to be *relatively complete* if and only if for every $a \in A$ the sets $\{b \in A_0 : a \leq b\}$, $\{b \in A_0 : a \geq b\}$ have respectively the smallest and the largest elements: $\inf\{b \in A_0 : a \leq b\}$ or $\bigwedge_{a \leq b \in A_0} b$, $\sup\{b \in A_0 : a \geq b\}$ or $\bigvee_{a \geq b \in A_0} b$.

Theorem 4. *Let A be a monadic BL -algebra. Then its subset $\exists A$ defined above is a relatively complete subalgebra of the BL -algebra A and $\exists p = \inf\{q : q \in \exists A \ \& \ p \leq q\}$ and $\forall p = \inf\{q : q \in \exists A \ \& \ p \geq q\}$ for each $p \in A$.*

Proof. By Lemma 3, $\exists A$ is a subalgebra of A . Let $\exists A(p) = \{q : q \in \exists A \ \& \ p \leq q\}$. By (7) $q \in \exists A(p) \Rightarrow \exists p \leq \exists q = q$. But $\exists p \in \exists A(p)$, and $q \in \exists A(p) \Rightarrow \exists p \leq q$. So $\exists p = \inf\{q : q \in \exists A \ \& \ p \leq q\}$. Analogously, $\forall p = \sup\{q : q \in \exists A \ \& \ p \geq q\}$. \square

Let us consider a function $h : A_0 \rightarrow A$. The functions $\forall_h, \exists_h : A \rightarrow A_0$ are called, respectively, *right and left adjoint* to h if $a \leq \forall_h b \Leftrightarrow h(a) \leq b$ and $\exists_h(b) \leq a \Leftrightarrow b \leq h(a)$ for any $a \in A_0$ and $b \in A$. If, in addition, $\exists_h(p * p) = \exists_h(p) * \exists_h(p)$, $\forall_h(p * p) = \forall_h(p) * \forall_h(p)$, then \forall_h and \exists_h are respectively called *right and left m -adjoint* to h .

A subalgebra A_0 of the BL -algebra A is said to be *m -relatively complete* if A_0 is relatively complete and, in addition, there holds

$$(\#) \ (\forall a \in A)(\forall x \in A_0)(\exists v \in A_0)(x \geq a * a \Rightarrow v \geq a \ \& \ v * v \leq x).$$

Let us remark that not every relatively complete subalgebra A_0 of the BL -algebra A defines the monadic operator \exists converting A into a monadic BL -algebra. For illustration, let us consider BL -algebra $([0, 1], \vee, \wedge, *, \rightarrow, 0, 1)$, where $*$ and \rightarrow are MV -operations: $x * y = \max(0, x + y - 1)$, $x \rightarrow y = \min(1, 1 - x + y)$. The two-element Boolean subalgebra $(\{0, 1\}, \vee, \wedge, *, \rightarrow, 0, 1)$ is relatively complete. We define $\exists a$ as $\inf\{x \in \{0, 1\} : x \geq a\}$. If $1 \neq a \neq 0$, then $\exists a = 1$ and $\exists a * \exists a = 1$. But this operation does not satisfy the condition $\exists(a * a) = \exists a * \exists a$ (see (18)). The subalgebra $(\{0, 1\}, \vee, \wedge, *, \rightarrow, 0, 1)$ is not m -relatively complete. Indeed, let $a = 1/2$ and $x = 0$. Then $x * x \geq a * a$. But the only element $v \in \{0, 1\}$ such that $v \geq a$ is 1, and thus $v * v \not\leq x * x$.

Theorem 5. *Let $(A, \vee, \wedge, *, \rightarrow, \forall, \exists, 0, 1)$ be a monadic BL -algebra. Then the BL -subalgebra $\exists A$ of the BL -algebra $(A, \vee, \wedge, *, \rightarrow, 0, 1)$ is m -relatively complete.*

Proof. According to Theorem 4, the algebra $\exists A$ is relatively complete. Now it is enough to check the validity of the condition $(\#)$. Let $a \in A$, $x \in \exists A$ be any elements of A and $\exists A$, respectively. Suppose that $x \geq a * a$. Then $x = \exists x \geq \exists(a * a) = \exists a * \exists a$. Hence in the case where $v = \exists a$ the condition $(\#)$ is satisfied. \square

Denote by \mathbb{BL}^2 the category whose objects are pairs (A, A_0) of BL -algebras, where A_0 is an m -relatively complete subalgebra and the injective BL -algebra homomorphism $h : A_0 \hookrightarrow A$ has left and right adjoint functions \exists_h and \forall_h respectively, and whose morphisms are pairs of functions $(f, f_0) : (A, A_0) \rightarrow (A', A'_0)$ such that the following conditions are satisfied:

- (1) $f : A \rightarrow A'$ is a BL -algebra homomorphism;
- (2) $f \circ h = h' \circ f_0$, where $h' : A'_0 \rightarrow A'$ is an injective BL -algebra homomorphism;
- (3) $f_0 \circ \exists_h = \exists_{h'} \circ f$, where $\exists_{h'}$ is the left adjoint of h' ;
- (4) $f_0 \circ \forall_h = \forall_{h'} \circ f$, where $\forall_{h'}$ is the right adjoint of h' .

(1) implies that f_0 is also a BL -algebra homomorphism.

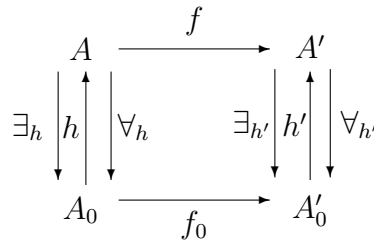


Fig. 1

Theorem 6. *There exists a one-to-one correspondence between:*

- 1) *monadic BL-algebras;*
- 2) *the pairs (A, A_0) , where A_0 is an m -relatively complete subalgebra of A ;*
- 3) *the pairs (A, A_0) , where A_0 is a subalgebra of A and the canonical embedding $h : A_0 \hookrightarrow A$ has left and right m -adjoint functions.*

Proof. For a given $A \in \text{MIBL}$ let us put $A_0 = \{\exists a : a \in A\} = \exists A$. Then, according to Theorem 5, A_0 is an m -relatively complete BL -subalgebra of A . Vice versa, for a given pair (A, A_0) , where A_0 is an m -relatively complete BL -subalgebra of A , let us define on A the operators \forall and \exists by putting

$$\forall p = \sup\{q \in A_0 : q \leq p\} = \bigvee_{p \geq q \in A_0} q,$$

$$\exists p = \inf\{q \in A_0 : q \geq p\} = \bigwedge_{p \leq q \in A_0} q.$$

It is obvious that $A1$ and $E1$ are satisfied.

$E2$. Show the validity of identity $E2 : \forall(p \rightarrow \exists q) = \exists p \rightarrow \exists q$.

$$\forall(p \rightarrow \exists q) = \bigvee_{p \rightarrow \exists q \geq r \in A_0} r.$$

$r \leq p \rightarrow \exists q \Rightarrow r * p \leq \exists q \Rightarrow \exists(r * p) \leq \exists q$ (since $\exists r = r$, according to (10))
 $\Rightarrow r * \exists p \leq \exists q \Rightarrow r \leq \exists p \rightarrow \exists q \Rightarrow r * \exists p \leq \exists q \Rightarrow r * p \leq \exists q \Rightarrow r \leq p \rightarrow \exists q$.
 Thus $r \leq p \rightarrow \exists q \Leftrightarrow r \leq \exists p \rightarrow \exists q$. So

$$\forall(p \rightarrow \exists q) = \bigvee_{p \rightarrow \exists q \geq r \in A_0} r = \bigvee_{\exists p \rightarrow \exists q \geq r \in A_0} r = \exists p \rightarrow \exists q.$$

$A2$. $\forall(\exists p \rightarrow q) = \exists p \rightarrow \forall q$.

$$\forall(\exists p \rightarrow q) = \bigvee_{\exists p \rightarrow q \geq r \in A_0} r.$$

$r \leq \exists p \rightarrow q \Rightarrow r * \exists p \leq q \Rightarrow r * \exists p \leq \forall q \Rightarrow r \leq \exists p \rightarrow \forall q \Rightarrow r * \exists p \leq \forall q \Rightarrow r * \exists p \leq q \Rightarrow r \leq \exists p \rightarrow q$. Thus $r \leq \exists p \rightarrow q \Leftrightarrow r \leq \exists p \rightarrow \forall q$. So

$$\forall(\exists p \rightarrow q) = \bigvee_{\exists p \rightarrow q \geq r \in A_0} r = \bigvee_{\exists p \rightarrow \forall q \geq r \in A_0} r = \exists p \rightarrow \forall q.$$

$$A3. \forall(p \vee \exists q) = \forall p \vee \exists q.$$

$$\forall p \vee \exists q = \bigvee_{A_0 \ni r \leq p} r \vee \exists q = \bigvee_{A_0 \ni r \leq p} (r \vee \exists q) = \bigvee_{A_0 \ni r \vee \exists q \leq p \vee \exists q} (r \vee \exists q) = \forall(p \vee \exists q).$$

Hence there exists a one-to-one correspondence between BL -algebras and pairs (A, A_0) , where A_0 is m -relatively complete subalgebra of A . Let us prove that 3) implies 2). Indeed, for a given pair (A, A_0) , where A_0 is a subalgebra of A and the canonical embedding $h : A_0 \rightarrow A$ has right and left m -adjoint functions, let us define $\forall a = h \circ \forall_h(a)$ and $\exists a = h \circ \exists_h(a)$ for every $a \in A$. It is easy to check that the left m -adjoint is order preserving, i.e. if $x \leq y$, then $\exists_h(x) \leq \exists_h(y)$. Let x be any element of A_0 and $x \geq a * a$, where $a \in A$. Let $v = \exists_h(a)$. Then $v \geq a, \exists_h(x) = x \geq \exists_h(a * a) = \exists_h(a) * \exists_h(a) = v * v$. This means that A_0 is m -relatively complete. Conversely, for a given pair (A, A_0) , where A_0 is an m -relatively complete BL -subalgebra of A , let us define $\forall_h(a) = \text{Sup}\{b \in A_0 : b \leq a\}$ and $\exists_h(a) = \text{Inf}\{b \in A_0 : b \geq a\}$. It is obvious that these functions are right and left m -adjoint to the canonical embedding $h : A_0 \rightarrow A$. Indeed, let us show that \forall_h is right m -adjoint to the canonical embedding $h : A_0 \rightarrow A$, where $h(a) = a$. Then it obvious that if $a \leq \forall_h b = \text{Sup}\{x \in A_0 : x \leq b\}$, then $h(a) = a \leq \forall_h b \leq b$. If $h(a) = a \leq b$, then $a = \forall_h a \leq \forall_h b$. So, $a \leq \forall_h b$. Moreover, $\forall_h(a * b) = \forall_h a * \forall_h b$. We can show analogously that \exists_h is left m -adjoint to the canonical embedding h . Hence 2) implies 3). \square

Theorem 7. *The category MBL is equivalent to the category BL^2 .*

Proof. Define the functors $\Phi : \text{MBL} \rightarrow \text{BL}^2$ and $\Psi : \text{BL}^2 \rightarrow \text{MBL}$ by assuming $\Phi(A) = (A, A_0)$, where $A_0 = \{\exists a : a \in A\}$ and $\Phi(f) = (f, f|_{A_0})$, and $\Psi((A, A_0)) = (A, \exists, \forall)$, where $\forall = h \circ \forall_h$ and $\exists = h \circ \exists_h$, and $\Psi((f, f_0)) = f$. It is easy to check that the functors Φ and Ψ are defined correctly. Using Theorem 6 it easy to check that $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are natural isomorphisms. Hence the category MBL is equivalent to the category BL^2 . \square

4. SUBDIRECT REPRESENTATION

Let $(A, \vee, \wedge, *, \rightarrow, \exists, \forall, 0, 1)$ be a monadic BL -algebra. A subset $F \subset A$ is called a *monadic filter* if F is a filter, i.e. for each $a, b \in A$

- $a, b \in F$ implies $a * b \in F$,
- $a \in F$ and $a \leq b$ implies $b \in F$,

and, in addition, for every $a \in A$ $a \in F$ implies $\forall a \in F$.

For any set $X \subseteq A$, let $[X]$ denote the filter generated by X , i.e. $[X] = \{a \in A : a \geq x_1 * \dots * x_n, x_1, \dots, x_n \in X\}$. Note that if X is closed under the operation $*$, then $[X] = \{a \in A : a \geq x \text{ for some } x \in X\}$.

Lemma 8. *If $\Delta = [\{a\} \cup F]$, where $a \in A_0 (= \exists A)$ and F is a monadic filter, then Δ is also a monadic filter.*

Proof. If $c \in \Delta$, then there exists $b \in F$ such that $c \geq a * \forall b$. Hence $\forall c \geq \forall(a * b) \geq \forall a * \forall b$ and since $\forall b \in F$, we get $\forall c \in \Delta$. \square

Theorem 9. *A filter $F \subset A$ is monadic if and only if $F = [F \cap A_0]$, and there exists a lattice isomorphism between the lattice of all monadic filters and the lattice of all filters of A_0 .*

Proof. Let (A, \exists, \forall) be a monadic BL -algebra and F be a monadic filter of A . Let $a \in F$. Then $\forall a \in F$ and $\forall a \in F \cap A_0$. Therefore, since $a \geq \forall a$, we have $a \in [F \cap A_0]$. Consequently, if $a \in [F \cap A_0]$, then $a \geq x$ for some $x \in F \cap A_0$. Therefore, since $\forall a \geq \forall x = x$ and $a \geq \forall a$, we have $a \in F$. Note that A_0 is a BL -algebra. The correspondence $F \mapsto [F]$ maps any filter F of A_0 to the monadic filter $[F]$ of (A, \exists, \forall) , since F is closed under $*$. It is clear that $F = [F] \cap A_0$. On the other hand, let M be a monadic filter of A . Then $M = [M \cap A_0]$. Since $M \cap A_0$ is a filter of A_0 , we have showed the required 1 – 1 correspondence between the set of monadic filters of A and the set of filters of A_0 . It is obvious from the definition that if $F_1 \subset F_2$ ($M_1 \subset M_2$), then $[F_1] \subset [F_2]$ ($M_1 \cap A_0 \subset M_2 \cap A_0$). This completes the proof of the theorem. \square

Theorem 10. *If F is a monadic filter of a monadic BL -algebra (A, \exists, \forall) , then the relation R on A , defined by xRy if and only if $(x \rightarrow y) * (y \rightarrow x) \in F$, is a congruence relation.*

Proof. Since F is a filter of a BL -algebra A , the relation R preserves the operations $\vee, \wedge, *, \rightarrow$ [1]. Let xRy , i.e. $(x \rightarrow y) * (y \rightarrow x) \in F$. Then $(x \rightarrow y), (y \rightarrow x) \in F$ and $\forall(x \rightarrow y), \forall(y \rightarrow x), \exists(x \rightarrow y), \exists(y \rightarrow x) \in F$. But $\forall(x \rightarrow y) \leq \forall x \rightarrow \forall y, \forall(y \rightarrow x) \leq \forall y \rightarrow \forall x$ (by (11)). Consequently, $\forall x \rightarrow \forall y, \forall y \rightarrow \forall x \in F$. By (17) $\forall(x \rightarrow y) \leq \exists x \rightarrow \exists y$ and $\forall(y \rightarrow x) \leq \exists y \rightarrow \exists x$. Hence $\exists x \rightarrow \exists y, \exists y \rightarrow \exists x \in F$. Therefore $\forall xR\forall y$ and $\exists xR\exists y$. \square

From Theorems 9 and 10 we obtain

Corollary 11. *For every monadic BL -algebra (A, \forall, \exists) , there exists a lattice isomorphism between the lattice of all monadic filters and the lattice of all congruences of (A, \forall, \exists) .*

A filter F of a monadic BL -algebra (A, \forall, \exists) is called *prime* provided that $F \neq A$ and $a \vee b \in F$ implies $a \in F$ or $b \in F$.

Remark. This definition is equivalent to the one given by Hájek in [1] : F is prime if and only if for each $x, y \in A$ either $x \rightarrow y \in F$ or $y \rightarrow x \in F$.

Theorem 12. *Any monadic BL -algebra (A, \forall, \exists) is isomorphic to a subdirect product of monadic BL -algebras (A_i, \forall, \exists) such that $\exists A_i$ is totally ordered.*

Proof. Let S be a family of all prime filters of $\exists A$ such that $\bigcap S = \{1\}$, which exists by Hájek’s theorem on the representation of any BL -algebra as a subdirect product of totally ordered BL -algebras [1]. Then $\bigcap S' = \{1\}$, where $S' = \{[F] : F \in S\}$ is the family of the monadic filters of A . \square

Remark. The results of the paper were reported at the International Workshop “Issues in Non-Classical Logics”, March 14–16, 2001, University of Milan, Italy.

ACKNOWLEDGMENT

The author was supported by the Cariplo Foundation for the academic year 2000/2001.

Let me express my gratitude to the referee for her/his suggestions to improve the readability of the paper.

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(Received 24.12.2005)

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