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RATE OF APPROXIMATION FOR CERTAIN DURRMEYER OPERATORS

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Abstract. In the present note, we study a certain Durrmeyer type integral modification of Bernstein polynomials. We investigate simultaneous approximation and estimate the rate of convergence in simultaneous approximation.

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1. Introduction

Durrmeyer [3] introduced the integral modification of Bernstein polynomials to approximate Lebesgue integrable functions on the interval $[0,1]$. The operators introduced by Durrmeyer are defined by

$$D_n(f,x)=(n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)f(t)\,dt, \quad x\in[0,1], \tag{1}$$

where $p_{n,k}=\binom{n}{k}x^k(1-x)^{n-k}$.

Gupta [5] introduced a different Durrmeyer type modification of Bernstein polynomials and estimated the rate of convergence for functions of bounded variation. The operators introduced in [5] are defined by

$$B_n(f,x)=\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 b_{n,k}(t)f(t)\,dt, \quad x\in[0,1], \tag{2}$$

where

$$p_{n,k}(x)=(-1)^k\frac{x^k}{k!}\,\phi_n^{(k)}(x), \quad b_{n,k}(x)=(-1)^{k+1}\frac{x^k}{k!}\,\phi_n^{(k+1)}(x)$$

and $\phi_n(x)=(1-x)^n$. It is easily verified that the values of $p_{n,k}(x)$ used in (1) and (2) are the same. Also, $\sum\limits_{k=0}^{n} p_{n,k}(x)=1$, $\int\limits_0^1 b_{n,k}(t)\,dt=1$ and $b_{n,n}=0$. Guo [4] estimated the rate of convergence for bounded variation functions for the usual Bernstein–Durrmeyer operators defined by (1). By considering the integral modification of Bernstein polynomials in form (2) some approximation properties become simpler in the analysis. Therefore it is important to carry

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out a further study of different integral modifications of Bernstein polynomials $B_n(f,x)$. Recently, Agratini [1] also estimated the rate of convergence for functions of bounded variation for some integral operators which include operators (2). He obtained the results on ordinary approximation. In the present paper, we estimate the rate of convergence in simultaneous approximation for operators $B_n(f,x)$.

2. Auxiliary Results

In this section we give the results which are necessary to prove the main result.

Lemma 1 ([7]). *For $m, r \in N^0 \equiv N \cup \{0\}$ (the set of non-negative integers), $r \leq n$, if we define*

$$V_{r,n,m}(x) = \sum_{k=0}^{n-r} p_{n-r,x}(x) \int_0^1 b_{n+r,k+r}(t)(t-x)^m dt,$$

then

$$V_{r,n,0}(x) = 1, \quad V_{r,n,1}(x) = \frac{(1+r) - x(1+2r)}{n+r+1},$$

$$V_{r,n,2}(x) = \frac{(r^2+3r+2) + 2x(n-2r^2-5r-2) - 2x^2(n-2r^2-4r-1)}{n+r+1},$$

and we have the recurrence relation

$$[n+r+m+1]V_{r,n,m+1}(x) = x(1-x)\left[V^{(1)}_{r,n,m}(x) + 2mV_{r,n,m-1}(x)\right] + [(1-2x)(m+r+1)+x]V_{r,n,m}(x).$$

Remark 2. If n is sufficiently large, then by Lemma 1 it is easy to verift that

$$\frac{x(1-x)}{n} \leq V_{r,n,2}(x) \leq \frac{2x(1-x)}{n}.$$

Lemma 3 ([8]). *For every $0 \leq k \leq n$, $x \in (0,1)$ and for all $n \in N$, we have*

$$p_{n,k}(x) \leq \frac{1}{\sqrt{2enx(1-x)}}.$$

Lemma 4 ([7]). *If $f \in L_1[0,1]$, $f^{(r-1)} \in A.C._{loc}$, $f^{(r)} \in L_1[0,1]$ and $1 \leq r < n$, then*

$$B_n^{(r)}(f,x) = \frac{(n!)^2}{(n-r)!(n+r)!} \sum_{k=0}^{n-r} p_{n-r,k}(x) \int_0^1 b_{n+r,k+r}(t) f^{(r)}(t)\, dt.$$

Lemma 5. *Let $x \in (0,1)$ and $K_{n,r}(x,t) = \sum\limits_{k=0}^{n-r} p_{n-r,k}(x) b_{n+r,k+r}(t)$, then for n sufficiently large, we have*

$$\lambda_{n,\gamma}(x,y) := \int_0^y K_{n,\gamma}(x,t)\, dt \leq \frac{2x(1-x)}{n(x-y)^2}, \quad 0 \leq y < x, \tag{3}$$

$$1-\lambda_{n,\gamma}(x,z) := \int_z^1 K_{n,\gamma}(x,t)\,dt \le \frac{2x(1-x)}{n(z-x)^2}\,, \quad x<z<1. \tag{4}$$

Proof. We first prove (3) by Lemma 1 as follows:

$$\begin{aligned}\int_0^y K_{n,\gamma}(x,t)\,dt &\le \int_0^y K_{n,\gamma}(x,t)\,\frac{(x-t)^2}{(x-y)^2}\,dt \\ &\le \frac{1}{(x-y)^2}\int_0^1 K_{n,\gamma}(x,t)(t-x)^2\,dt \le \frac{V_{r,n,2}(x)}{(x-y)^2} \le \frac{2x(1-x)}{n(x-y)^2}\,.\end{aligned}$$

The proof of (4) is similar. □

3. Main Result

Theorem. *Let $f\in H_r$, $r\in N^0$. Then for every $x\in(0,1)$ and n sufficiently large, we have*

$$\begin{aligned}\left|B_n^{(r)}(f,x)-\frac12\left[f_+^{(r)}(x)+f_-^{(r)}(x)\right]\right| &\le \left(2+2r+\frac{1}{\sqrt{8e}}\right)\frac{|f_+^{(r)}(x)-f_-^{(r)}(x)|}{\sqrt{(n-r)x(1-x)}} \\ &+\frac{5}{nx(1-x)}\sum_{k=1}^n\left(\omega_x\left(g_{x,r},\frac{x}{\sqrt{k}}\right)+\omega_x\left(g_{x,r},\frac{1-x}{\sqrt{k}}\right)\right), \end{aligned} \tag{5}$$

where $H_r=\{f : f^{(r-1)}\in C[0,1],\ f_\pm^{(r)}(x)\in[0,1],\ r>0\}$, $H_0=\{f : f_\pm(x)\in[0,1]\}$, $\omega_x(f,t)=\sup_u\{|f(x+u-f(x)| : |u|\le t,\ x,x+u\in[0,1]\}$ *and*

$$g_{x,y}=\begin{cases} f_-^{(r)}(t)-f_-^{(r)}(x), & 0\le t<x,\\ 0, & t=x,\\ f_+^{(r)}(t)-f_+^{(r)}(x), & x<t\le 1.\end{cases}$$

Proof. Using Lemma 3, we can write

$$\begin{aligned}&\left|B_n^{(r)}(f,x)-\frac12\{f_+^{(r)}(t)+f_+^{(r)}(x)\}\right| \\ &\le\left|\frac{(n!)^2}{(n-r)!(n+r)!}\sum_{k=0}^{n-r}p_{n-r,k}(x)\int_0^x b_{n+r,k+r}(t)f_-^{(r)}(t)\,dt\right.\\ &\quad\left.-\sum_{k=0}^{n-r}p_{n-r,k}(x)\int_0^x b_{n+r,k+r}(t)f_-^{(r)}(t)\,dt\right|\\ &\quad+\left|\frac{(n!)^2}{(n-r)!(n+r)!}\sum_{k=0}^{n-r}p_{n-r,k}(x)\int_x^1 b_{n+r,k+r}(t)f_+^{(r)}(t)\,dt\right.\end{aligned}$$

$$
\begin{aligned}
&- \sum_{k=0}^{n-r} p_{n-r,k}(x) \int_0^x b_{n+r,k+r}(t) f_+^{(r)}(t)\,dt \Bigg| \\
&+ \frac{1}{2}\left|B_n^{(r)}(\operatorname{sign}(t-x),x)\right| \left|f_+(r)(x) - f_-^{(r)}(x)\right| \\
= {} & I_1 + I_1 + I_3, \quad \text{say.}
\end{aligned} \tag{6}
$$

Using Lemma 4, we have

$$
\begin{aligned}
I_1 \le {} & \frac{(n!)^2}{(n-r)!(n+r)!} \sum_{k=0}^{n-r} p_{n-r,k}(x) \int_0^x b_{n+r,k+r}(t) \left|f_-^{(r)}(t) - f_-^{(r)}(x)\right| dt \\
& + \left|f_-^{(r)}(x)\right| \left|\frac{(n+r)!(n-r)!}{(n!)^2} - 1\right| \\
\le {} & \sum_{k=0}^{n-r} p_{n-r,k}(x) \int_0^x b_{n+r,k+r}(t)\omega_x(x-t)\,dt + O(n^{-1}) \\
\le {} & \sum_{k=0}^{n-r} p_{n-r,k}(x) \left(\int_0^{x-\delta} + \int_{x-\delta}^x\right) b_{n+r,k+r}(t)\omega_x(x-t)\,dt + O(n^{-1}) \\
\le {} & \sum_{k=0}^{n-r} p_{n-r,k}(x) \int_0^{x-\delta} \omega_x(x-t)[b_{n+r,k+r}(t)\,dt] \\
& + \omega_x(\delta) \sum_{k=0}^{n-r} p_{n-r,k}(x) \int_{x-\delta}^x b_{n+r,k+r}(t)\,dt + O(n^{-1}) \\
\le {} & \omega_x(\delta) \sum_{k=0}^{n-r} p_{n-r,k}(x) \int_0^x b_{n+r,k+r}(t)\,dt \\
& + \int_0^{x-\delta} \frac{2x(1-x)}{n(t-x)^2}\, d(\omega_x(x-t)) + O(n^{-1}).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\int_0^{x-\delta} \frac{2x(1-x)}{n(t-x)^2}\, d(\omega_x(x-t)) \le {} & \frac{2x(1-x)}{n}\left[\frac{-\omega_x(\delta)}{\delta^2} + \frac{\omega_x(x)}{x^2}\right] \\
& + \frac{2x(1-x)}{n} \int_0^x \omega_x(t) t^{-3}\,dt \\
\le {} & \frac{2x(1-x)}{n}\left[\omega_x(\delta) + \sum_{k=1}^{n} \omega_x\left(\frac{x}{\sqrt{k}}\right)\right]
\end{aligned}
$$

$$\leq \frac{4}{nx(1-x)}\sum_{k=1}^{n}\omega_x\left(\frac{x}{\sqrt{k}}\right)$$

and

$$\omega_x(\delta)\sum_{k=0}^{n-r}p_{n-r,k}(x)\int_0^x b_{n+r,k+r}(t)\,dt \leq \omega_x\left(\frac{x}{\sqrt{n}}\right) \leq \frac{1}{nx(1-x)}\sum_{k=1}^{n}\omega_x\left(\frac{x}{\sqrt{k}}\right).$$

Thus

$$I_1 \leq \frac{5}{nx(1-x)}\sum_{k=1}^{n}\omega_x\left(g_{x,r},\frac{x}{\sqrt{k}}\right)+O(n^{-1}). \tag{7}$$

Choosing $\delta_1 = \frac{1-x}{\sqrt{n}}$ and proceeding similar to I_1, we get

$$\begin{aligned} I_2 &\leq \sum_{k=0}^{n-r}p_{n-r,k}(x)\left(\int_0^{x-\delta_1}+\int_{x-\delta_1}^{x}\right)b_{n+r,k+r}(t)\omega_x(g_{x,t},t-x)\,dt+O(n^{-1}) \\ &\leq \frac{5}{nx(1-x)}\sum_{k=1}^{n}\omega_x\left(g_{x,r},\frac{1-x}{\sqrt{k}}\right)+O(n^{-1}). \end{aligned} \tag{8}$$

Next, we estimate I_3 as follows:

$$\begin{aligned} B_n^{(r)}(\operatorname{sign}(t-x),x) &= \int_x^1 K_{n,\gamma}(x,t)\,dt - \int_0^x K_{n,\gamma}(x,t)\,dt \\ &= A_{n,\gamma} - B_{n,\gamma}, \quad \text{say.} \end{aligned}$$

It is easy to verify that $A_{n,\gamma}(x)+B_{n,\gamma}(x)=1$. Thus $B_n^{(r)}(\operatorname{sign}(t-x),x) = 1 - 2A_{n,\gamma}(x)$. Using Lemma 1, Lemma 2 and the fact that $\sum_{j=0}^{k}p_{n,j}(x)=\int_x^1 b_{n,k}(t)\,dt$, we have

$$\begin{aligned} A_{n,\gamma}(x) &= \sum_{k=0}^{n-r}p_{n-r,k}(x)\sum_{j=0}^{k+r}p_{n+r,j}(x) \\ &= \sum_{k=0}^{n-r}p_{n-r,k}(x)\sum_{j=0}^{k}p_{n+r,j}(x)+\sum_{k=0}^{n-r}p_{n-r,k}(x)\sum_{j=k+1}^{k+r}p_{n+r,j}(x) \\ &\leq \sum_{k=0}^{n-r}p_{n-r,k}(x)\sum_{j=0}^{k}p_{n+r,j}(x)+\frac{r}{\sqrt{2e(n+r)x(1-x)}}. \end{aligned}$$

By the Berry–Esseen theorem [2] we readily obtain

$$\left|\sum_{k=0}^{j}p_{n-r,k}(x)-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{(j-(n-r)x/\sqrt{(n-r)x(1-x)}}e^{-t^2/2}dt\right| < \frac{1}{\sqrt{(n-r)x(1-x)}} \tag{9}$$

and

$$\left|\sum_{k=0}^{j} p_{n+r,k}(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(j-(n+r)x/\sqrt{(n+r)x(1-x)}} e^{-t^2/2} dt\right| < \frac{1}{\sqrt{(n+r)x(1-x)}}. \quad (10)$$

Hence by (9) and (10), we have

$$\left|\sum_{k=0}^{j} p_{n-r,k}(x) - \sum_{k=0}^{j} p_{n+r,k}(x)\right| < \frac{2+r}{\sqrt{(n-r)x(1-x)}}. \quad (11)$$

By (11) it follows that

$$\left|\sum_{k=0}^{n-r} p_{n-r,k}(x)\left(\sum_{j=0}^{k} p_{n+r,j}(x) - \sum_{j=0}^{k} p_{n-r,j}(x)\right)\right| < \frac{2+r}{\sqrt{(n-r)x(1-x)}}. \quad (12)$$

Let

$$S = \sum_{k=0}^{n-r} p_{n-r,k}(x) \sum_{j=0}^{k} p_{n-r,j}(x).$$

If ξ and η are independent random variables with the same distribution $\mathcal{P}$ assigning the probability p_k, $\sum\limits_{k=1}^{\infty} p_k = 1$, to the number b_k, $k = 1, 2, \ldots$, such that $b_1 < b + 2 < \cdots$, then

$$P(\xi \leq \eta) = \sum_{k=1}^{\infty} P(\eta = b_k) P(\xi \leq b_k) = \sum_{k=1}^{\infty} p_k \sum_{i=1}^{k} p_i,$$
$$P(\xi = \eta) = \sum_{k=1}^{\infty} P(\eta = b_k) P(\xi = b_k) = \sum_{k=1}^{\infty} p_k^2$$

and we obtain

$$0 < \sum_{k=1}^{\infty} p_k \sum_{t=1}^{k} p_t - \frac{1}{2} = \frac{1}{2} \sum_{k=1}^{\infty} p_k^2. \quad (13)$$

If now $\mathcal{P}$ is the binomial distribution with the parameters $n - r$ (number of independent trials) and x (success probability in each trial), then we have $S = P(\xi \leq \eta)$ and due to (13) and Lemma 2, we have

$$0 < S - \frac{1}{2} = \frac{1}{2} \sum_{k=0}^{n-r} p_{n-r,k}^2(x) \leq \frac{1}{2\sqrt{2e(n-r)x(1-x)}} \sum_{k=0}^{n-r} p_{n-r,k}(x)$$
$$= \frac{1}{2\sqrt{2e(n-r)x(1-x)}}. \quad (14)$$

Combining (12) and (14), we obtain

$$\left|\sum_{k=0}^{n-r} p_{n-r,k}(x) \sum_{j=0}^{k} p_{n+r,j}(x) - \frac{1}{2}\right| \leq \left(2 + 2r + \frac{1}{\sqrt{8e}}\right) \frac{1}{\sqrt{(n-r)x(1-x)}}\,,$$

$$\begin{aligned} B_n^{(r)}(\operatorname{sign}(t-x), x) &= |2A_{n,\gamma}(x) - 1| \\ &\leq 2\left(2 + 2r + \frac{1}{\sqrt{8e}}\right) \frac{1}{\sqrt{(n-r)x(1-x)}}\,. \end{aligned} \tag{15}$$

Combining estimates of (6), (7), (8) and (15), our theorem follows. □

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