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ON DECOMPOSITIONS OF A CUBE INTO CUBES AND SIMPLEXES

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Abstract. Some combinatorial results concerning finite decompositions (dissections) of a k-dimensional cube into cubes (respectively, simplexes) of the same dimension are presented in the paper. In connection with such decompositions, the notion of a decomposability number is introduced and the problem of description of all these numbers is discussed.

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Let C be an arbitrary k-dimensional cube in the Euclidean space R^k, where $k \geq 1$, and suppose that C is decomposed into finitely many cubes $\{C_i : i \in I\}$ of the same dimension. Obviously, the edges of each cube C_i of this decomposition are parallel to the corresponding edges of C. Let us denote by $N(k)$ the set of all natural numbers card(I), where $\{C_i : i \in I\}$ ranges over all possible finite decompositions of C into cubes. Clearly, the set $N(k)$ does not depend on the choice of C.

If $n \in N(k)$, then we say that n is a decomposability number for C (with respect to the family of all cubes in R^k). We are going to show that almost all natural numbers are decomposability numbers for C, i.e. there exists a natural number r such that

$$\{r+1, r+2, r+3, \dots\} \subset N(k).$$

In other words, for any dimension $k \geq 1$, the set $N(k)$ is co-finite in the set N of all natural numbers.

We need several simple lemmas.

Lemma 1. *The following two assertions are valid:*

1) $1 \in N(k)$;

2) *if a and b are strictly positive natural numbers such that $a \geq b$ and $n \in N(k)$, then $n + a^k - b^k \in N(k)$.*

Proof. Assertion 1) is trivial. Let us show the validity of assertion 2). For this purpose, decompose C into a^k smaller pairwise congruent cubes and replace some b^k cubes of them by one cube C' (this procedure is evidently possible). Since $n \in N(k)$, there exists a decomposition of C' into n cubes. These n cubes together with $a^k - b^k$ smaller cubes yield a decomposition of the original cube C into $n + a^k - b^k$ cubes. Thus, we obtain the required relation $n + a^k - b^k \in N(k)$. □

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Lemma 2. *Suppose that natural numbers*

$$a_1, a_2, \dots, a_p,\ \ b_1, b_2, \dots, b_p$$

satisfying the inequalities

$$a_1 \geq b_1,\ a_2 \geq b_2,\ \dots,\ a_p \geq b_p$$

are given and suppose that $n \in N(k)$*. Then we have*

$$n + (a_1^k - b_1^k) + (a_2^k - b_2^k) + \dots + (a_p^k - b_p^k) \in N(k).$$

Proof. This lemma easily follows from Lemma 1 by using the induction on p. □

Remark 1. Let us denote $d = 2^k - 1$. In virtue of Lemma 2, we can deduce that if $n \in N(k)$ and p is a natural number, then $n + pd \in N(k)$. Indeed, it suffices to put in the above-mentioned lemma

$$a_1 = a_2 = \dots = a_p = 2,$$
$$b_1 = b_2 = \dots = b_p = 1.$$

Lemma 3. *Let* j *be an arbitrary element of the set* $\{0, 1, \dots, d-1\}$*, where* d *is as in Remark* 1*. Then there exists a number* $n_j \in N(k)$ *such that*

$$n_j = t_j d + j$$

for some positive integer t_j*.*

Proof. Taking into account Lemmas 1 and 2, we may write

$$a_1^k + a_2^k + \dots + a_p^k - (p-1) \in N(k)$$

for any finite sequence $(a_1, a_2, \dots, a_p)$ of natural numbers satisfying the inequalities

$$1 \leq a_1,\ 1 \leq a_2,\ \dots, 1 \leq\ a_p.$$

Let us define

$$p = d - j + 1,\ a_1 = a_2 = \dots = a_p = d.$$

It can be easily verified that in this case we have

$$a_1^k + a_2^k + \dots + a_p^k - (p-1) = t_j d + j$$

for an appropriate positive integer t_j, which completes the proof. □

Theorem 1. *There exists a natural number* r *such that*

$$\{r+1, r+2, r+3, \dots\} \subset N(k).$$

Proof. According to the previous lemma, for each $j \in \{0, 1, \dots, d-1\}$, there exists a number $n_j \in N(k)$ satisfying the equality

$$n_j = t_j d + j$$

for some positive integer t_j. Let us put

$$r = \max(n_0, n_1, \dots, n_{d-1})$$

and show that r is the required natural number.

Indeed, take any natural number $n > r$. Clearly, n can be represented in the form

$$n = td + j,$$

where t is a positive integer and $j \in \{0, 1, \dots, d-1\}$. Consequently, we get

$$n - n_j = (t - t_j)d$$

or, equivalently,

$$n = n_j + (t - t_j)d.$$

Since $n_j \in N(k)$, we finally have $n \in N(k)$ (in view of Remark 1). The proof of Theorem 1 is completed. □

Remark 2. In general (i.e. for an arbitrary dimension k), a complete description of the set $N(k)$ is unknown. Some more detailed information about $N(k)$ can be obtained for small natural numbers k. Obviously,

$$N(1) = \{1, 2, 3, \dots, n, \dots\}.$$

Also, it can be easily verified that

$$N(2) = \{1, 4, 6, 7, 8, \dots, n, \dots\}$$

and

$$\{71, 72, 73, \dots, n, \dots\} \subset N(3).$$

In this context, it should also be mentioned that none of numbers n satisfying the inequalities $1 < n < 2^k$ belongs to the set $N(k)$. In addition, we have $2^k \in N(k)$ but $2^k + 1 \notin N(k)$ for $k \geq 2$.

Remark 3. A square can be decomposed into finitely many squares whose sizes are pairwise distinct. This result was first obtained by A. Stöhr and R. Sprague (see [1] and [2]). At the present time we know of the constructions which yield a decomposition of a square into 24 squares with pairwise distinct sizes. On the other hand, it is not difficult to prove that, for $k \geq 3$, none of the k-dimensional cubes can be decomposed into finitely many k-dimensional cubes whose sizes differ from each other (for more details about this topic see, e.g., [3]).

Let P be a k-dimensional rectangular parallelepiped in the Euclidean space R^k. We denote by N_P the set of all natural numbers n for which there exists at least one decomposition of P into n cubes of dimension k. Obviously, if P is a k-dimensional cube, then the set N_P coincides with the set $N(k)$ introduced earlier. If $n \in N_P$, then we say that n is a decomposability number for P (with respect to the family of all cubes in R^k).

Theorem 2. *For any rectangular parallelepiped P, the set N_P is either empty or co-finite in N.*

Proof. If $N_P = \varnothing$, then there is nothing to prove.

Suppose now that $N_P \neq \varnothing$ and consider a finite decomposition $\{C_i : i \in I\}$ of P into cubes. Fix some cube C_i of this decomposition. According to Theorem 1, there exists a natural number r such that all elements of the set

$\{r+1, r+2, r+3, \dots\}$ are decomposability numbers for C_i. Now, it is clear that all elements of the set

$$\{\mathrm{card}(I)+r, \mathrm{card}(I)+r+1, \mathrm{card}(I)+r+2, \dots\}$$

are decomposability numbers for P. This ends the proof of Theorem 2. □

Remark 4. Denote by $l_1, l_2, \dots, l_k$ the lengths of all edges of a given k-dimensional rectangular parallelepiped P, passing through one of its vertices. As shown by Dehn, the set N_P is nonempty if and only if all fractions

$$l_1/l_2,\ l_2/l_3,\ \dots,\ l_{k-1}/l_k$$

are rational numbers (in this connection, see, e.g., [4]).

Consider again a k-dimensional cube C in the space R^k, where $k \geq 1$. Let $\{T_i : i \in I\}$ be a finite decomposition of C into simplexes of the same dimension. We say that $m = \mathrm{card}(I)$ is a decomposability number for C (with respect to the family of all simplexes in R^k). The set of all such m is denoted by $M(k)$. Observe that

$$M(k) = \{s, s+1, s+2, \dots\},$$

where $s = s_C = s(k)$ is the smallest number of simplexes into which the cube C can be decomposed. Therefore the problem of description of the set $M(k)$ is equivalent to the problem of finding the precise values of the function $s(k)$ ($k = 1, 2, 3, \dots$).

Remark 5. Clearly, we have $s(1) = 1$, $s(2) = 2$ and $s(3) = 5$. By using a combinatorial argument based on some upper estimates of the volumes of the simplexes contained in a given cube, it is not difficult to show that $s(4) \geq 13$. This simple inequality will be applied below (see Theorem 3).

More generally, for a given k-dimensional convex polyhedron Q in the space R^k, denote by s_Q the smallest number of simplexes into which Q can be decomposed. The values of the function s_Q essentially depend on the combinatorial structure of Q.

Let Q and Q' be two k-dimensional convex polyhedra in R^k. We say that Q' is a primitive extension of Q if there exists a k-dimensional simplex T in R^k such that:

(1) $Q \cap T$ is a common facet of Q and T;

(2) the set of vertices of Q' is the union of the sets of vertices of Q and T.

It immediately follows from conditions (1) and (2) that $Q' = Q \cup T$.

Accordingly, we say that a k-dimensional convex polyhedron Q is primitive if there exists a finite sequence $\{Q_1, Q_2, \dots, Q_n\}$ of convex polyhedra in R^k such that:

(a) Q_1 is a k-dimensional simplex in R^k;

(b) for each integer $i \in [1, n-1]$, the polyhedron Q_{i+1} is a primitive extension of Q_i;

(c) $Q_n = Q$.

Remark 6. Any convex polygon in the plane R^2 is primitive in the sense of the above definition. Actually, an analogous fact is true for any simple polygon in R^2 (see, e.g., [5] where some closely related results are also presented).

A three-dimensional cube C is primitive (this fact is closely connected with the equality $s(3) = 5$ and is of some interest from the purely geometrical viewpoint because no facet of C is a triangle).

But the most convex polyhedra in the space R^3 are not primitive. In particular, if a convex three-dimensional polyhedron Q has no trihedral angle, then Q cannot be primitive (cf. also Example 1 below).

Lemma 4. *Let Q be a three-dimensional convex polyhedron in R^3 and let $v = v(Q)$ denote the number of vertices of Q. The following assertions are valid:*

1) $s_Q \geq v - 3$;

2) $s_Q = v - 3$ *if and only if Q is primitive.*

The proof of this lemma is based on the classical Euler formula

$$v + f = e + 2$$

and on the elementary fact that, for every convex polygon $P \subset R^2$ with n sides, the minimal number of triangles into which P can be decomposed is equal to $n - 2$ (this fact does not hold for nonconvex polygons).

It directly follows from Lemma 4 that if Q is a three-dimensional convex polyhedron and $s_Q \geq v - 2$, then Q is not primitive. We also claim that s_Q cannot be represented as a function of a single variable $v = v(Q)$.

Note that the convexity of Q is essential in the formulation of Lemma 4. Indeed, if Q is an arbitrary three-dimensional polyhedron in R^3, then the inequality $v(Q) \leq 4s_Q$ holds true. At the same time, for any natural number $n \geq 1$, there exists a three-dimensional simple polyhedron Q such that $v(Q) = 4n$ and $s_Q = n$.

Obviously, analogous facts are valid for k-dimensional polyhedra in the space R^k ($k \geq 2$), where we have the inequality $v(Q) \leq (k+1)s_Q$ and, for any natural number $n \geq 1$, there exists a k-dimensional simple polyhedron Q such that $v(Q) = (k+1)n$ and $s_Q = n$.

Example 1. In the space R^3 consider a convex bipyramid Q with $2n+2$ vertices, where $n \geq 2$. It is easy to see that there are some $2n$ faces $F_1, F_2, \dots, F_{2n}$ of Q such that the intersection $F_i \cap F_j$ either is empty or is a singleton for any two distinct integers i and j from the set $\{1, 2, \dots, 2n\}$. This fact implies that every decomposition of Q into tetrahedra needs at least $2n$ members. Consequently, we come to the inequality

$$s_Q \geq 2n = (2n + 2) - 2.$$

In particular, this inequality shows that the bipyramid Q is not primitive. The latter fact is trivial, since Q has no trihedral angles. Let Q' denote a primitive extension of Q. Then Q' has a trihedral angle but is not primitive either.

Note that a simplicial decomposition of Q into exactly $2n$ tetrahedra can be constructed without any difficulty.

Example 2. According to the preceding example, a convex hexagonal bipyramid Q in the space R^3 needs at least 6 tetrahedra for its decomposition, i.e. $s_Q = 6$. The number of vertices of Q is equal to 8. A three-dimensional cube C has the same number of vertices. However, $s_C = s(3) = 5$ (cf. Remarks 5 and 6). We see again that, for a general three-dimensional convex polyhedron P in the space R^3, the value s_P cannot be represented as a function of a single variable $v = v(P)$.

Theorem 3. *A four-dimensional cube C is not primitive.*

Proof. From the definition of a primitive k-dimensional convex polyhedron Q it follows (by easy induction) that $s_Q \leq v(Q) - k$, where $v(Q)$ denotes the number of vertices of Q.

Now, suppose to the contrary that C is primitive. Then we must have

$$s_C \leq v(C) - 4 = 16 - 4 = 12.$$

But, as mentioned in Remark 5, $s_C = s(4) \geq 13$. The contradiction obtained ends the proof of Theorem 3. □

It readily follows from Theorem 3 that, for each natural number $k \geq 4$, the k-dimensional unit cube $C \subset R^k$ is not primitive. To show this, it suffices to apply induction on k taking into account the fact that the volume of any simplex contained in C does not exceed $1/k$.

References

1. A. Stöhr, Über Zerlegungen von Rechtecken in inkongruente Quadrate. *Schr. Math. Inst. Inst. Angew. Math. Univ. Berlin* **4**(1939), 119–140.
2. R. Sprague, Beispiel einer Zerlegung des Quadrats in lauter verschiedene Quadrate. (German) *Math. Z.* **45**(1939), 607–608.
3. R. L. Brooks, C. A. B. Smith, A. H. Stone, and W. T. Tutte, The dissection of rectangles into squares. *Duke Math. J.* **7**(1940), 312–340.
4. H. Hadwiger, Vorlesungen ber Inhalt, Oberfläche und Isoperimetrie. (German) *Springer-Verlag, Berlin-Göttingen-Heidelberg,* 1957.
5. A. B. Kharazishvili, Simple polyhedra. (Russian) *Sem. Inst. Prikl. Mat. Dokl.* No. 18, (1984), 34–38, 83.

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