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A POSITIVE ANSWER TO VELICHKO'S QUESTION

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Abstract. We give positive answer to Velichko's question in which the quotient and s-map is replaced by a sequence-covering and cs-map. In addition, let X have a star-countable k-network, then X is a sequence-covering and cs-image of a locally separable metric space if and only if X is a sequence-covering and cs-image of a metric space.

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1. Introduction

In recent years, sequence-covering maps introduced by Siwiec in [1] have again been drawing attention [2]–[5]. On the other hand, B. Qu and M. Gao introduced the concept of cs-map in order to study the relationships between spaces with certain compact-countable k-networks and certain images of metric spaces [6]. Velichko [7] posed the following interesting question about quotient and s-images of metric spaces: Find a Φ-property such that a space Y is a quotient and s-image of a metric and Φ-space if and only if Y is a Φ-space which is a quotient and s-image of a metric space. Velichko [7] proved that a space Y is a pseudo-open and s-image of a locally separable metric space if and only if Y is a locally separable space which is a pseudo-open and s-image of a metric space. In this paper, it is shown that a local $\aleph_0$-property is a positive answer to Velichko's question if the quotient and s-map is replaced by a sequence-covering and cs-map. In addition, let X have a star-countable k-network, then X is a sequence-covering and cs-image of a locally separable metric space if and only if X is a sequence-covering and cs-image of a metric space.

In this paper, all spaces are regular and T_1, all mappings are continuous and onto. $\omega = \{0\} \bigcup N$. Let us recall some basic definitions.

Definition 1.1. Let X be a space, and let $\mathcal{P}$ be a cover of X.

(1) $\mathcal{P}$ is called compact-countable (resp. compact-finite) if for any compact subset K of X, only countably (resp. finitely) many members of $\mathcal{P}$ intersect K.

(2) $\mathcal{P}$ is star-countable if for any element P of $\mathcal{P}$, only countably many members of $\mathcal{P}$ intersect P.

(3) $\mathcal{P}$ is point-countable if for each $x \in X$, only countably many members of $\mathcal{P}$ contain x.

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(4) Let $x \in P \subset X$. P is a sequential neighborhood of x in X [8] if whenever $\{x_n\}$ is a sequence converging to the point x, we have $\{x_n : n \geq m\} \subset P$ for some $m \in N$.

(5) Let $P \subset X$. P is a sequentially open subset in X [8] if P is a sequential neighborhood of x in X for each $x \in P$. X is a sequential space if each sequentially open subset in X is open.

(6) $\mathcal{P}$ is a network if whenever $x \in U$ with U open in X, we have $x \in P \subset U$ for some $P \in \mathcal{P}$. A space is a cosmic space [9] if it has a countable network.

(7) $\mathcal{P}$ is a *cs*-network [10] if whenever $\{x_n\}$ is a sequence converging to a point $x \in U$ with U open in X, we have $\{x\} \bigcup \{x_n : n \geq m\} \subset P \subset U$ for some $m \in N$ and some $P \in \mathcal{P}$.

(8) $\mathcal{P}$ is a k-network for X if whenever $K \subset U$ with K compact and U open in X, we have $K \subset \bigcup \mathcal{P}' \subset U$ for some finite $\mathcal{P}' \subset \mathcal{P}$; A space is an $\aleph_0$-space [9] if it has a countable k-network.

(9) $\mathcal{P}$ is an *so*-cover (i.e., sequentially open cover) [12] if each element of $\mathcal{P}$ is sequentially open in X.

Definition 1.2. Let $f : X \to Y$ be a map. Then

(1) f is an s-map if each $f^{-1}(y)$ is separable.

(2) f is a *cs*-map [6] if for each compact subset K of Y, $f^{-1}(K)$ is separable.

(3) f is a sequence-covering map [1] if each convergent sequence of Y is the image of some convergent sequence of X.

(4) f is a quotient map if whenever $f^{-1}(U)$ is open in X, we have U is open in Y.

(5) f is a pseudo-open map if whenever $f^{-1}(y) \subset V$ with V open in X, we have $y \in \operatorname{int}(f(V))$.

Definition 1.3 ([13])**.** A space X is sequentially separable if X has a countable subset D such that for each $x \in X$, there is a sequence $\{x_n\}$ in D with $x_n \to x$. D is called a sequentially dense subset of X.

Liu and Tanaka [14] showed that every cosmic space with a point-countable *cs*-network is an $\aleph_0$-space, the key property of which is that every cosmic space is sequentially separable.

2. Results

Theorem 2.1. *The following statements are equivalent for a space X:*

(1) *X is a sequence-covering and cs-image of a locally separable metric space.*

(2) *X has a compact-countable cs-network consisting of cosmic subspaces.*

(3) *X has a compact-countable cs-network, and an so-cover consisting of $\aleph_0$-subspaces.*

(4) *X is a sequence-covering and cs-image of a metric space, and has an so-cover consisting of $\aleph_0$-subspaces.*

Proof. (1) $\Rightarrow$ (2). Let $f : M \to X$ be a sequence-covering and *cs*-mapping, where M is a locally separable metric space. Suppose $\mathcal{B}$ is a σ-locally finite base for M consisting of separable subspaces. Put $\mathcal{P} = \{f(B) : B \in \mathcal{B}\}$. Then $\mathcal{P}$ is a compact-countable cs-network for X consisting of cosmic subspaces.

(2) $\Rightarrow$ (3). Let $\mathcal{P}$ be a compact-countable cs-network of X consisting of cosmic subspaces. For each $P \in \mathcal{P}$, let $D(P)$ be a countable and sequentially dense subset of P. For each $x \in X$, put

$$\mathcal{P}(x,1) = \{P \in \mathcal{P} : x \in P\}, \quad D(x,1) = \bigcup \{D(P) : P \in \mathcal{P}(x,1)\},$$

and for each $n \geq 2$ inductively define that

$$\mathcal{P}(x,n) = \Big\{P \in \mathcal{P} : P \bigcap D(x,n-1) \neq \varnothing\Big\},$$
$$D(x,n) = \bigcup \{D(P) : P \in \mathcal{P}(x,n)\}.$$

Let $\mathcal{P}(x) = \bigcup \{\mathcal{P}(x,n) : n \in N\}$, and $U(x) = \bigcup \mathcal{P}(x)$. To complete the proof of (3), it suffices to show that $U(x)$ is sequentially open in X and $\mathcal{P}(x)$ is a cs-network for $U(x)$. If $\{y_n\}$ is a sequence in X converging to a point $y \in U(x) \bigcap W$ with W open in X, then $y \in P$ for some $m \in N$ and some $P \in \mathcal{P}(x,m)$, and there is a sequence $\{z_n\}$ in $D(P)$ with $z_n \to y$, thus $\{y\} \bigcup \{y_n, z_n : n \geq m\} \subset Q \subset W$ for some $m \in N$ and some $Q \in \mathcal{P}$, so $Q \in \mathcal{P}(x, m+1) \subset \mathcal{P}(x)$ and $\{y\} \bigcup \{y_n : n \geq m\} \subset Q \subset U(X) \bigcap W$. This implies that $U(x)$ is sequentially open and $\mathcal{P}(x)$ is a cs-network for $U(x)$.

(3) $\Rightarrow$ (1). First, we shall show that X has a compact-countable cs-network $\mathcal{P}$ consisting of $\aleph_0$-subspaces. Let $\mathcal{P}'$ be a compact-countable cs-network of X which is closed under finite intersections, and let $\mathcal{U}$ be an so-cover of X consisting of $\aleph_0$-subspaces. Put

$$\mathcal{P} = \{P \in \mathcal{P}' : P \subset U \text{ for some } U \in \mathcal{U}\}.$$

Then $\mathcal{P}$ is still a cs-network for X. Indeed, let $x \in W$ with W open in X. If $\{x_n\}$ is a sequence converging to the point $x \in X$, put

$$\mathcal{P}'_x = \{P \in \mathcal{P}' : x \in P \subset W \text{ and } P \text{ contains all but finite many} x_n\}$$
$$= \{P_n : n \in N\}.$$

For each $n \in N$, take $Q_n = \bigcap\limits_{i \leq n} P_i$, then $Q_n \in \mathcal{P}'_x$. Let $U_x \in \mathcal{U}$ be a sequential neighborhood of x in X. If there is $q_n \in Q_n \backslash U_x$ for each $n \in N$, and G is open in X with $x \in G$, then $P_k \subset G$ for some $k \in N$ because $\mathcal{P}'$ is a cs-network for X, thus $q_n \in Q_n \subset P_k \subset G$ when $n \geq k$, and $q_n \to x$, a contradiction. Hence $Q_m \subset U_x$ for some $m \in N$, and $Q_m \in \mathcal{P}$. Therefore $\mathcal{P}$ is a compact-countable cs-network for X consisting of $\aleph_0$-subspaces. Let $\mathcal{P} = \{P_\alpha : \alpha \in \bigwedge\}$. For each $\alpha \in \bigwedge$, by Theorem in [9], there are a separable metric space M_α and a sequence-covering $f_\alpha : M_\alpha \to P_\alpha$. Put $M = \bigoplus_{\alpha \in \Lambda} M_\alpha$, $Z = \bigoplus_{\alpha \in \bigwedge} P_\alpha$ and $f = \bigoplus_{\alpha \in \bigwedge} f_\alpha : M \to Z$. Then M is a locally separable metric space and f is a sequence-covering map. Assume $h : Z \to X$ to be a natural map, and let $g = h \circ f : M \to X$. Then g is a sequence-covering cs-mapping.

(3) $\Leftrightarrow$ (4). It suffices to show that X has a compact-countable cs-network if and only if X is a sequence-covering and cs-image of a metric space. Let X

be a space with a compact-countable cs-network $\mathcal{P}$. We can suppose that $\mathcal{P}$ is closed under finite intersections. Denote $\mathcal{P}$ by $\{P_\alpha : \alpha \in A\}$. Let A_i denote the set A with a discrete topology for each $i \in N$. Put

$$M = \left\{ \beta = (\alpha_i) \in \prod_{i \in N} A_i : \{P_{\alpha_i} : i \in N\} \text{ is a network at some point } x(\beta) \text{ in } X \right\},$$

then M is a metric space, and $f : M \to X$ defined by $f(\beta) = x(\beta)$ is a cs-map. We shall show that f is sequence-covering. For a sequence $\{x_n\}$ of X converging to a point x_0 in X, we can assume that all x_n's are distinct. Let $K = \{x_m : m \in \omega\}$, and let $K \subset U$ with U open in X. A subset $\mathcal{F}$ of $\mathcal{P}$ is said to have the property $F(K, U)$ if $\mathcal{F}$ satisfies the following conditions:

(1) $\mathcal{F}$ is finite;
(2) $\varnothing \neq P \bigcap K \subset P \subset U$ for each $P \in \mathcal{F}$;
(3) for each $x \in K$ there is a unique $P_x \in \mathcal{F}$ with $x \in P_x$;
(4) if $x_0 \in P \in \mathcal{F}$, then $K \backslash P$ is finite.

Put

$$\{\mathcal{F} \subset \mathcal{P} : \mathcal{F} \text{ has the property } F(K, X)\} = \{\mathcal{F}_i : i \in N\}.$$

For each $i \in N$ and each $m \in \omega$, there is $\alpha_{im} \in A_i$ with $x_m \in P_{\alpha_{im}} \in \mathcal{F}_i$. It can be checked that $\{P_{\alpha_{im}} : i \in N\}$ is a network at the point x_m. Let $\beta_m = (\alpha_{im})$ for each $m \in \omega$, then $\beta_m \in M$ and $f(\beta_m) = x_m$. For each $i \in N$, there is $n(i) \in N$ such that $\alpha_{in} = \alpha_{i0}$ if $n \geq n(i)$. Thus the sequence $\{\alpha_{in}\}$ converges to α_{i0} in A_i, and the sequence $\{\beta_n\}$ converges to β_0 in M. This shows that f is a sequence-covering map.

Conversely, suppose that $f : M \to X$ is a sequence-covering and cs-map, where M is a metric space. Let $\mathcal{B}$ be a σ-locally-finite base for M, then $\{f(B) : B \in \mathcal{B}\}$ is a compact-countable cs-network for X. □

Corollary 2.2. *The following statements are equivalent for a space X:*

(1) *X is a sequence-covering and quotient cs-image of a locally separable metric space.*

(2) *X is a local $\aleph_0$-space and a sequence-covering, and a quotient cs-image of a metric space.*

(3) *X is a sequential and local $\aleph_0$-space with a compact-countable cs-network.*

Theorem 2.3. *Let X have a star-countable k-network. Then X is a sequence-covering and cs-image of a locally separable metric space if and only if X is a sequence-covering and cs-image of a metric space.*

Proof. We prove only the "if" part. We have that a space X is a sequence-covering and cs-image of a metric space if and only if X has a compact-countable cs-network from the proof of Theorem 2.1. Let $\mathcal{P}$ be a compact-countable cs-network for X, here we can assume that $\mathcal{P}$ is closed under finite intersections. Since X has a star-countable k-network $\mathcal{R}$, in view of Lemma 1.1 in [15], X is the disjoint union of $\{X_\alpha : \alpha \in \bigwedge\}$ satisfying the following conditions:

(a) each X_α is an $\aleph_0$-space which is the countable union of elements of $\mathcal{R}$.
(b) $\{X_\alpha : \alpha \in \bigwedge\}$ is compact-finite.

Let $\mathcal{P}' = \{P \in \mathcal{P} : P$ is an $\aleph_0$-space of $X\}$. Then $\mathcal{P}'$ is a cs-network for X. Indeed, let $L = \{x_n : n \in N\}$ converge to x with $x \in U$ and U is open. Let $\mathcal{P}_x = \{P \in \mathcal{P} : x \in P,\ P \subset U$ and L is eventually in $P\}$. Then $\mathcal{P}_x$ is countable. We take a subfamily $\mathcal{P}'_x = \{P_n : n \in N\}$ of $\mathcal{P}_x$ such that $P_n \supset P_{n+1}$ for any $n \in N$ and $\mathcal{P}'_x$ is a network of x in X. If $\mathcal{P}'_x \bigcap \mathcal{P}' = \varnothing$, this means that each P_n meets uncountably many X_α. Pick $z_n \in P_n \bigcap X_{\alpha(n)}$ for some $\alpha(n) \in \bigwedge$, where $X_\alpha(n) \neq X_\alpha(m)$ if $n \neq m$. Then $\{z_n : n \in N\}$ is a convergent sequence with $z_n \to x$. The compact set $\{x\} \bigcup \{z_n : n \in N\}$ meets infinitely many X_α. This is a contradiction to (b). Thus $\mathcal{P}'$ is a compact-countable cs-network for X consisting of $\aleph_0$-spaces. By Theorem 2.1, X is a sequence-covering and cs-image of a locally separable metric space. □

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