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AN INVERSE RESULT IN SIMULTANEOUS APPROXIMATION BY MODIFIED BETA OPERATORS

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Abstract. In this paper we study the modified Beta operators. We extend the result of [4] and obtain an inverse result for the linear combination of these modified Beta operators in simultaneous approximation.

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1. Introduction

Let f be a function defined on $[0,\infty)$. The modified Beta operators introduced by Gupta and Ahmad [4] are defined by

$$B_n(f,x) = \frac{n-1}{n} \sum_{\nu=0}^{\infty} b_{n,\nu}(x) \int_0^{\infty} p_{n,\nu}(t) f(t)\, dt, \quad x \in [0,\infty), \tag{1.1}$$

where

$$b_{n,\nu} = \frac{1}{B(\nu+1,n)} \frac{x^\nu}{(1+x)^{n+\nu+1}}, \quad p_{n,\nu}(t) = \binom{n+\nu+1}{\nu} \frac{t^\nu}{(1+t)^{n+\nu}}.$$

Let $C_\gamma[0,\infty) = \{f \in [0,\infty) : |f(t)| \le Mt^\gamma$ for some $\gamma > 0$ and some constant $M > 0\}$. It is easily observed that for $n > \gamma$ this class of the operators $B_n(f,x)$ is well defined. We define the norm $\|\cdot\|_\gamma$ on $C_\gamma[0,\infty)$ by $\|f\|_\gamma = \sup\limits_{0\le t<\infty} |f(t)|t^{-\gamma}$.

The order of approximation for these operators (1.1) is at best $O(n^{-1})$. To improve the order of approximation, we consider the linear combination of these operators (1.1). For arbitrary but fixed distinct positive integers $d_0, d_1, \dots, d_k$, the linear combination $B_n(f,k,x)$ of $B_{d_jn}(f,x)$, $j = 0,1,\dots,n$, is defined by

$$B_n(f,k,x) = \sum_{j=0}^{k} C(j,k) B_{d_jn}(f,x), \tag{1.2}$$

where

$$C(j,k) = \prod_{\substack{i=0\\ i\ne j}} \frac{d_j}{d_j - d_i}, \quad k \ne 0 \quad \text{and} \quad C(0,0) = 1.$$

In [4] the authors obtained a Voronvskaja type asymptotic formula and an error estimate in simultaneous approximation. Recently, Maheshwari and Gupta

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[5] have extended the result of [4] and obtained direct theorems for the linear combination $B_n(f,k,x)$ in terms of a higher order modulus of continuity. In this context we mention the recent work of V. Gupta (see, e.g., [2], [3]), who studied another type of discretely defined summation integral type operators and estimated local direct results in ordinary and simultaneous approximation. In the present paper the results of [4] and [5] are extended. It should be noted that there were many typing errors in [5], which are corrected in the present paper. Here we obtain an inverse result in simultaneous approximation by the linear combination $B_n(f,k,x)$.

We may rewrite operators (1.1) as

$$B_n(f,x)=\int_0^\infty P_n(x,t)f(t)\,dt,$$

where the kernel $P_n(x,t)$ is given by

$$P_n(x,t)=\frac{n-1}{n}\sum_{\nu=0}^{\infty} b_{n,\nu}(x)p_{n,\nu}(t).$$

2. Auxiliary Results

This section contains the basic results and definitions needed to prove our main theorem.

Throughout the paper it is assumed that $0<a_1<a_2<b_2<b_1<\infty$.

Definition 1. A continuous function f on the interval $[a,b]$ is said to belong to the generalized Zygmund class $Z_\alpha(k,a,b)$, $0<\alpha<2$, $k\in N$, if there exists a constant C such that

$$\omega_{2k}(f,\delta,a,b)\le C\delta^{\alpha k},\quad \delta>0,$$

where $\omega_{2k}(f,\delta,a,b)$ denotes the modulus of continuity of $2k$-th order of f on the interval $[a,b]$. In particular we denote by Z^*_α the class $Z_\alpha(1,a,b)$.

Definition 2. Let C_0 denote the class of continuous functions on the interval $[0,\infty)$ having a compact support, and C_0^k be a subset of C_0 of k times continuously differentiable functions. Suppose $m\in N_0\equiv N\cup\{0\}$, $[a',b']\subset(a,b)$, $[a,b]\subset(0,\infty)$ and for a fixed $k\in N_0$, let $G^{(m)}=\{g : g\in C_0^{2k+m+2}$, $\operatorname{supp}(g)\subset[a',b']\}$. For m times continuously differentiable functions f with $\operatorname{supp}(f)\subset[a',b']$, the Peetre's K-functional is defined as

$$K_m(\xi,f) =\inf_{g\in G^{(m)}}\left[\|f^{(m)}-g^{(m)}\|_{C[a',b']}+\xi\{\|g^{(m)}\|_{C[a',b']}+\|g^{(2k+m+2)}\|_{C[a',b']}\}\right],$$

where $0<\xi<1$. For $0<\alpha<2$, we define by $C_0^m(a,k,a',b')$ the class of m times continuously differentiable functions f with $\operatorname{supp}(f)\subset[a',b']$ satisfying the condition

$$\sup_{0<\xi\le1}\xi^{-\alpha/2}K_m(\xi,f)<M\quad\text{for some constant}\quad M>0.$$

In the following two lemmas $[\beta]$ denotes the integer part of β and a constant in $O(\cdot)$ depends on x.

Lemma 2.1 ([4]). *For $m \in N_0$, the polynomial $U_{n,m}(x) = \frac{1}{n}\sum\limits_{\nu=0}^{\infty} b_{n,\nu}(x)\big(\frac{\nu}{n+1} - x\big)^m$ satisfies the following recurrence relation:*

$$(n+1)U_{n,m+1}(x) = x(1+x)\left[U'_{n,m}(x) + mU_{n,m+1}(x)\right]$$

which implies that
(i) *$U_{n,m}(x)$ is a polynomial of x of degree $\leq m$;*
(ii) *$U_{n,m}(x) = O(n^{-[(m+1)/2]})$.*

Lemma 2.2 ([4]). *For $m \in N_0$, $n \in N$, $x \in [0,\infty)$ the m-th order moment is defined by*

$$T_{n,m}(x) = \frac{n-1}{n}\sum_{\nu=0}^{\infty} b_{n,\nu}(x)\int\limits_0^{\infty} p_{n,\nu}(t)(t-x)^m dt,$$

then $T_{n,0} = 1$, $T_{n,1} = \frac{3x+1}{n-2}$ and we have the recurrence relation

$$\begin{aligned}(n-m-2)T_{n,m+1}(x) &= x(1+x)\left[T_{n,m}(x) + 2mT_{n,m-1}(x)\right]\\ &\quad + \left[(1+2x)(m+1)+x\right]T_{n,m}(x), \quad n > m+2,\end{aligned}$$

which for all $x \in [0,\infty)$ implies $T_{n,m}(x) = O(n^{[(m+1)/2]})$.

Corollary 2.3 ([4]). *Let δ be a positive number, then for every $n > \gamma > 0$ and $x \in [0,\infty)$, there exists a constant $K_{m,x}$ depending on m and x:*

$$\int\limits_{|t-x|>\delta} P_n(x,t)t^{\gamma}dt \leq K_{m,x}n^{-m} \quad \text{for some} \quad m \in N.$$

Lemma 2.4. *There exist polynomials $\phi_{i,j,r}(x)$ independent of n and ν such that*

$$[x(1+x)]^r \frac{d^r}{dx^r}\left(b_{n,\nu}(x)\right) = \sum_{\substack{2i+j\leq r,\\ i,j\geq 0}} (n+1)^i[\nu-(n+1)x]^j\phi_{i,j,r}(x)b_{n,\nu}(x).$$

Theorem 2.5 ([5]). *Let $f \in C_{\gamma}[0,\infty)$. If $f^{(2k+m+2)}$ exists at a point $x \in [0,\infty)$, then*

$$\lim_{n\to\infty} n^{k+1}\left\{B_n^{(m)}(f,k,x) - f^{(m)}(x)\right\} = \sum_{i=r}^{2k+m+2} Q(i,k,m,x)f^{(i)}(x),$$

where $Q(i,k,m,x)$ are certain polynomials in x.

In what follows $C_1, C_2, \ldots$ stand for the positive constant.

Lemma 2.6. *Let $0 < a < a' < a'' < b'' < b' < b < \infty$. If $f^{(m)} \in C_0$,* $\mathrm{supp}(f) \in [a'', b'']$ *and*

$$\|B_n^{(m)}(f,k,\cdot) - f^{(m)}(x)\|_{C[a,b]} = O(n^{-\alpha(k+1)/2}),$$

then

$$K_m(\eta,f)=C_1\left\{n^{-\alpha(k+1)/2}+n^{k+1}\eta K_m(n^{-(k+1)},f)\right\}. \tag{2.2}$$

As a consequence, $K_m(\eta,f)\le C_2\eta^{\alpha/2}$, *i.e.,* $f\in C_0^m(\alpha,k+1,a',b')$.

Proof. To prove (2.2), it is sufficient to show that

$$K_m(\eta,f)=C_1\left\{n^{-\alpha(k+1)/2}+n^{k+1}\eta K_m(n^{-(k+1)},f)\right\} \quad \text{for sufficiently large } n.$$

Now as $\operatorname{supp}(f)\subset[a'',b'']$ in view of Theorem 2.5 there exists a function $g^i\in G^{(m)}$ such that for $i=m$ and $i=2k+m+2$

$$\|B_n^{(i)}(f,k,\cdot)-g^{(i)}\|_{C[a,b]}\le C_2 n^{-(k+1)},$$
$$K_m(\eta,f)\le 3C_3n^{-1}+\|B_n^{(m)}(f,k,\cdot)-f^{(m)}\|_{C[a',b']}$$
$$+\eta\{\|B_n^{(m)}(f,k,\cdot)\|_{C[a',b']}+\|B_n^{(2k+m+2)}(f,k,\cdot)\|_{C[a',b']}\}.$$

Thus it suffices to show that there exists a constant C_4 such that for each $h\in G^{(m)}$

$$\|B_n^{(2k+m+2)}(f,k,\cdot)\|_{C[a',b']}$$
$$\le C_4n^{k+1}\left\{\|f^{(m)}-h^{(m)}\|_{C[a',b']}+n^{-(k+1)}\|h^{(2k+m+2)}\|_{C[a',b']}\right\}. \tag{2.3}$$

Again $B_n^{(2k+m+2)}(f,k,\cdot)$ satisfies the linearity property

$$\|B_n^{(2k+m+2)}(f,k,\cdot)\|_{C[a',b']}$$
$$\le\|B_n^{(2k+m+2)}(f-h,k,\cdot)\|_{C[a',b']}+\|B_n^{(2k+m+2)}(h,k,\cdot)\|_{C[a',b']}. \tag{2.4}$$

Using Lemma 2.4, we have

$$\int_0^\infty\left|\frac{\partial^{2k+m+2}}{\partial x^{2k+m+2}}P_n(x,t)\right|dt\le\sum_{\substack{2i+j<2k+m+2\\ i,j\ge0}}\frac{n-1}{n}\sum_{\nu=1}^{\infty}(n+1)^i|\nu-(n+1)x|^j$$
$$\times\frac{|\phi_{i,j,2k+m+2}(x)|}{\{x(1+x)\}^{2k+m+2}}b_{n,\nu}(x)\int_0^\infty p_{n,\nu}(t)\,dt.$$

Hence, by the Cauchy–Schwarz inequality, Lemma 2.1 and the fact $\int_0^\infty p_{n,\nu}(t)\,dt=\frac{1}{n-1}$, we obtain

$$\left\|B_n^{(2k+m+2)}(f-h,k,\cdot)\right\|_{C[a',b']}\le C_5n\|f^{(m)}-g^{(m)}\|_{C[a',b']},$$

where the constant C_5 is independent of f and g.

Now, by Taylor's expansion, we have

$$h(t)=\sum_{i=0}^{2k+m+1}\frac{h^{(i)}(t)}{i!}(t-x)^i+\frac{h^{(2k+m+2)}(\xi)}{(2k+m+2)!}(t-x)^{2k+m+2}, \tag{2.6}$$

where ξ lies between t and x. Using (2.6), we have

$$\left\| \frac{\partial^{2k+m+2}}{\partial x^{2k+m+2}} B_n(g,k,\cdot) \right\|_{C[a',b']} \leq \sum_{j=0}^{k} \frac{|C(j,k)|}{(2k+m+2)!} \|g^{(2k+m+2)}\|_{C[a',b']}$$
$$\times \left\| \int_0^\infty \frac{\partial^{2k+m+2}}{\partial x^{2k+m+2}} B_{d_j n}(x,t)(t-x)^{2k+m+2} dt \right\|_{C[a',b']}. \quad (2.7)$$

We shall now calculate the term given in the second norm on the right-hand side. It is sufficient to consider the expression without the linear combination. Using Lemma 2.4 and the Cauchy–Schwarz inequality we have

$$\begin{aligned} I &= \int_0^\infty \left| \frac{\partial^{2k+m+2}}{\partial x^{2k+m+2}} P_n(x,t) \right| dt \\ &\leq \frac{n-1}{n} \sum_{\substack{2i+s<2k+m+2 \\ i,s\geq 0}} \sum_{r=0}^{\infty} (n+1)^i |\nu-(n+1)x|^s \frac{|\phi_{i,s,2k+m+2}(x)|}{\{x(1+x)\}^{2k+m+2}} b_{n,\nu}(x) \\ &\qquad \times \int_0^\infty p_{n,\nu}(t)(t-x)^{2k+m+2}\, dt. \end{aligned}$$

Next, using Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned} I &\leq \frac{n-1}{n} \sum_{\substack{2i+s<2k+m+2 \\ i,s\geq 0}} (n+1)^i \frac{|\phi_{i,s,2k+m+2}(x)|}{\{x(1+x)\}^{2k+m+2}} b_{n,\nu}(x) \\ &\quad \times \left(\frac{1}{n} \sum_{r=0}^{\infty} b_{n,\nu}(x)(\nu-(n+1)x)^{2s} \right)^{1/2} \\ &\quad \times \left(\sum_{r=0}^{\infty} b_{n.\nu}(x) \int_0^\infty p_{n,\nu}(t)(t-x)^{2k+2m+4} dt \right)^{1/2} \left(\int_0^\infty p_{n,\nu}(t)\, dt \right)^{1/2} \\ &= \sum_{\substack{2i+s<2k+m+2 \\ i,s\geq 0}} (n+1)^i \frac{|\phi_{i,s,2k+m+2}(x)|}{\{x(1+x)\}^{2k+m+2}} O(n^{s/2}) O(n^{-(k+m)/2+1}), \\ I &= \sum_{\substack{2i+s<2k+m+2 \\ i,s\geq 0}} (n+1)^i \frac{|\phi_{i,s,2k+m+2}(x)|}{\{x(1+x)\}^{2k+m+2}} O(n^{(2i+s)/2}) O(n^{-(2k+m+2)/2}). \end{aligned}$$

Hence by using (2.7) and the above estimate we have

$$\left\| B_n^{(2k+m+2)}(h,k,\cdot) \right\|_{C[a',b']} \leq C_6 \|h^{(2k+m+2)}\|_{C[a',b']}. \quad (2.8)$$

Combining estimates (2.4), (2.5) and (2.8), the result (2.3) follows. This completes the proof of (2.2). The other consequences are standard and can be found in [1]. □

Lemma 2.7. *Let $0 < a < a' < a'' < b'' < b' < b$ and $f^{(m)} \in C_0$ with $\operatorname{supp}(f) \subset [a'', b'']$, then if $f \in C_0^{(m)}(\alpha, k+1, a', b')$, we have $f^{(m)} \in \operatorname{Liz}(\alpha, k+1, a', b')$.*

Proof. Let $|\delta| < g$ and $h \in G^m$, then we have with $f \in C_0^{(m)}(\alpha, k+1, a', b')$.

$$\begin{aligned} |\Delta_\delta^{2k+2} f^{(m)}(x)| &\le |\Delta_\delta^{2k+2}(f^{(m)}(x) - h^{(m)}(x))| + |\Delta_\delta^{2k+2} h^{(m)}(x)| \\ &\le 2^{2k+2} \|f^{(m)} - h^{(m)}\|_{C[a',b']} \\ &\quad + \delta^{2k+2} \|g^{(2k+m+2)}\|_{C[a',b']} + \|h^{(2k+m+2)}\|_{C[a',b']} \\ &\le C_7 2^{2k+2} K_\infty(\delta^{2k+2}, f) \le C_8 2^{2k+2} \delta^{\alpha(k+1)}. \end{aligned}$$

It follows that $f^{(m)} \in Z_\alpha(k+1, a', b')$. □

Theorem 2.8 ([5])**.** *Let $f^{(m)} \in C_\gamma[0,\infty)$ and $0 < a < a' < b' < b < \infty$, then for n sufficiently large,*

$$\left\|B_n^{(m)}(f,k,\cdot) - f^{(m)}\right\|_{C[a',b']} = \max\left\{C_9 \omega_{2k+2}(f^{(m)}, n^{-1/2}, a, b), C_{10} n^{-(k+1)} \|f\|_\gamma\right\},$$

where $C_9 = C_9(k,m)$ and $C_{10} = C_{10}(k,m,f)$.

3. The Main Result

In this section, we shall prove the following inverse result.

Theorem 3.1. *If $0 < \alpha < 2$, $0 < a_1 < a_2 < b_2 < b_1 < \infty$ and $f \in C_\gamma[0,\infty)$, then for the following statements the implication* (i)⇒(ii) *is true:*

(i) *$f^{(m)}$ exist on the interval $[a_1, b_1]$ and*

$$\left\|B_n^{(m)}(f,k,\cdot) - f^{(m)}\right\|_{C[a_1,b_1]} = O(n^{-\alpha(k+1)/2});$$

(ii) *$f^{(m)} \in Z_\alpha(k+1, a_2, b_2)$.*

Proof. We shall prove this theorem by the principle of mathematical induction. Assuming (i), put $\tau = \alpha(k+1)$ and first consider the case $0 < \tau \le 1$. Let us choose a', a'', b', b'' in such a way that $a_1 < a' < a'' < a_2 < b_2 < b'' < b' < b_1$. Also suppose $g \in C_0^\infty$ with $\operatorname{supp}(g) \subset [a'', b'']$ and $g(x) = 1$ on $[a_2, b_2]$ for $x \in [a', b']$ with $D = \frac{d}{dx}$. We have

$$\begin{aligned} B_n^{(m)}(fh,k,x) - (fh)^{(m)}(x) &= D^m(B_n((fh)(t) - (fh)(x), k, x)) \\ &= D^m(B_n(f(t)(h(t) - h(x)), k, x)) + D^m(B_n(h(x)(f(t) - f(x)), k, x)) \\ &= J_1 + J_2, \quad \text{say}. \end{aligned}$$

To estimate J_1, by the Leibniz Theorem, we have

$$\begin{aligned} J_1 &= \sum_{j=0}^{k} C(j,k) \frac{\partial^m}{\partial x^m} \int_0^\infty P_{d_j n}(x,t) f(t)(h(t) - h(x))\,dt \\ &= \sum_{j=0}^{k} C(j,k) \sum_{i=1}^{m} \binom{m}{i} \int_0^\infty P^i_{d_j n}(x,t) \frac{\partial^{m-i}}{\partial x^{m-i}} [f(t)(h(t) - h(x))]\,dt \end{aligned}$$

$$
\begin{aligned}
&= -\sum_{i=1}^{m-1} \binom{m}{i} h^{(m-i)}(x) B_n^{(i)}(f,k,x) \\
&\quad + \sum_{j=0}^{k} C(j,k) \int_0^\infty P_{d_j n}^{(m)}(x,t) f(t)(h(t)-h(x))\,dt \\
&= J_3 + J_4, \quad \text{say.}
\end{aligned}
$$

Now, using Theorem 2.8, we obtain

$$
J_3 = -\sum_{i=1}^{m-1} \binom{m}{i} h^{(m-i)}(x) f^{(i)}(x) + O(n^{-\tau/2}), \quad \text{uniformly in } x \in [a',b'].
$$

Using Theorem 2.5, the Cauchy–Schwarz inequality, Taylor's expansion of f and h and Lemma 2.2, we get

$$
\begin{aligned}
J_4 &= \sum_{i=1}^{m-1} \frac{h^{(i)}(x) f^{(m-1)}(x)}{i!(m-i)!}\, m! + O(n^{-\tau/2}) \\
&= \sum_{i=1}^{m-1} \binom{m}{i} h^{(i)}(x) f^{(m-i)}(x) + O(n^{-\tau/2})
\end{aligned}
$$

uniformly in $x \in [a',b']$ by Corollary 2.3.

Finally, applying the Leibniz theorem, we obtain

$$
\begin{aligned}
J_2 &= \sum_{j=0}^{k} C(j,k) \sum_{i=0}^{m} \binom{m}{i} \int_0^\infty P_{d_j n}^{i}(x,t) \frac{\partial^{m-i}}{\partial x^{m-i}} [h(t)(f(t)-f(x))]\,dt \\
&= \sum_{i=0}^{k} \binom{m}{i} h^{(m-i)}(x) B_n^{(i)}(f,k,x) - (fh)^{(m)}(x) \\
&= \sum_{i=0}^{k} \binom{m}{i} h^{(m-i)}(x) f^{(i)}(x) - (fh)^{(m)}(x) + O(n^{-\tau/2}) \\
&= O(n^{-\tau/2}) \quad \text{uniformly in } x \in [a',b'].
\end{aligned}
$$

Combining these estimates, we have

$$
\left\| B_n^{(m)}(fh,k,\cdot) - (fh)^{(m)} \right\|_{C[a',b']} = O(n^{-\tau/2}).
$$

Hence by Lemma 2.6 and Lemma 2.7, we have $(fh)^{(m)} \in Z_\alpha(k+1,a',b')$. Since $h(x)=1$ on $[a_2,b_2]$, it follows that $f^{(m)} \in Z_\alpha(k+1,a_2,b_2)$, This proves (i)⇒(ii) for the case $0<\tau\le 1$. Now to prove the implication for $0<\tau<2k+2$ it is sufficient to assume it for $\tau \in (p'-1,p')$ and prove it for $\tau \in (p',p'+1)$ ($p'=1,2,\dots,2k+1$). Since the result holds for $\tau \in (p'-1,p')$, $f^{(p'+m+1)}$ exists and belongs to the class $Z_\alpha(1-\delta,a^*,b^*)$ for any $\delta>0$ and for any interval $(a^*,b^*) \subset (a_1,b_1)$. Let a_i^*, b_i^*, $i=1,2$, be such that $(a_2,b_2) \subset (a_2^*,b_2^*) \subset (a_1^*,b_1^*)$.

Let $h \in C_0^\infty$ be such that $h(x) = 1$ on $[a_2, b_2]$ and $\operatorname{supp}(h) \subset [a_2^*, b_2^*]$. Then $\zeta_2(t)$ denotes the characteristic function of the interval $[a_1^*, b_1^*]$, we have

$$\begin{aligned}\left\|B_n^{(m)}(fh,k,\cdot) - (fh)^{(m)}\right\|_{C[a_2^*,b_2^*]} &\le \left\|D^m[B_n(h(x)(f(t)-f(x),k,\cdot)]\right\|_{C[a_2^*,b_2^*]} \\ &\quad + \left\|D^m[B_n(f(x)(h(t)-h(x),k,\cdot)]\right\|_{C[a_2^*,b_2^*]} \\ &= I_1 + I_2 \quad \text{(say)}.\end{aligned}$$

To estimate I_1, by Theorem 2.5, we have

$$\begin{aligned}I_1 &\le \left\|D^m[B_n(h(x)f(t),k,\cdot)] - (fh)^{(m)}\right\|_{C[a_2^*,b_2^*]} \\ &= \left\|\sum_{i=0}^{\infty}\binom{m}{i} h^{(m-i)} B_n^{(i)}(f,k,\cdot) - (fh)^{(m)}\right\|_{C[a_2^*,b_2^*]} \\ &= \left\|\sum_{i=0}^{\infty}\binom{m}{i} h^{(m-i)} f^{(i)} - (fh)^{(m)}\right\|_{C[a_2^*,b_2^*]} + O(n^{-\tau/2}) = O(n^{-\tau/2}).\end{aligned}$$

Also by Leibniz Theorem and Theorem 2.5, we have

$$\begin{aligned}I_2 &= \left\| -\sum_{i=0}^{\infty}\binom{m}{i} h^{(m-i)} B_n^{(i)}(f,k,\cdot) + B_n^{(m)}(f(t)(h(t)-h(\cdot))\zeta_2(t),k,\cdot)\right\}_{C[a_2^*,b_2^*]} \\ &= \|I_3 + I_4\|_{C[a_2^*,b_2^*]} + O(n^{-(k+1)}), \quad \text{say}.\end{aligned}$$

Then by Theorem 2.5, we get

$$I_3 = -\sum_{i=0}^{m-1}\binom{m}{i} h^{(m-i)}(x) f^{(i)}(x) + O(n^{-\tau/2}), \quad \text{uniformly in } x \in [a_2^*, b_2^*].$$

Applying Taylor's expansion of f, we have

$$\begin{aligned}I_4 &= \sum_{j=0}^{k} C(j,k)\int_0^\infty P_{d_jn}^{(m)}(x,t)[f(t)(h(t)-h(x))\zeta_2(t)]\,dt \\ &= \sum_{j=0}^{k} C(j,k)\sum_{i=0}^{p'+m+1}\frac{f^{(i)}(x)}{i!}\int_0^\infty P_{d_jn}^{(m)}(x,t)(t-x)^i(h(t)-h(x))\zeta_2(t)\,dt \\ &= \sum_{j=0}^{k} C(j,k)\int_0^\infty P_{d_jn}^{(m)}(x,t)\left(\frac{f^{(p'+m-1)}(\eta) - f^{(p'+m-1)}(x)}{(p'+m-1)!}\right) \\ &\qquad \times (t-x)^{(p'+m-1)}(h(t)-h(x))\zeta_2(t)\,dt \\ &\qquad (\eta \text{ lying between } t \text{ and } x) \\ &= I_5 + I_6, \quad \text{say}.\end{aligned}$$

Applying Theorem 2.5, we get

$$I_5 = \sum_{j=0}^{k} C(j,k)\sum_{i=0}^{p'+m+1}\frac{f^{(i)}(x)}{i!}\int_0^\infty P_{d_jn}^{(m)}(x,t)(t-x)^i(h(t)-h(x))\,dt + O(n^{-(k+1)})$$

$$(\text{uniformly in } x \in [a_2^*, b_2^*])$$

$$= I_7 + O(n^{-(k+1)}), \quad \text{say.}$$

Since $h \in C_0^\infty$, we can write

$$I_7 = \sum_{j=0}^{k} C(j,k) \sum_{i=0}^{p'+m+1} \frac{f^{(i)}(x)}{i!} \sum_{r=0}^{p'+m+1} \frac{h^{(r)}(x)}{r!} \int_0^\infty P_{d_j n}^{(m)}(x,t)(t-x)^{i+r}dt$$
$$+ \sum_{j=0}^{k} C(j,k) \sum_{i=0}^{p'+m+1} \frac{f^{(i)}(x)}{i!} \int_0^\infty P_{d_j n}^{(m)}(x,t)\varepsilon(t,x)(t-x)^{i+p'+m+1}dt$$
$$(\text{where } \varepsilon(t,x) \to 0 \text{ as } t \to x)$$
$$= I_8 + I_9, \quad \text{say.}$$

Next, by Theorem 2.5 we get

$$I_8 = \sum_{r=1}^{m} \frac{h^{(r)}}{r!} \frac{f^{(m-r)}(x)}{(m-r)!} m! + O(n^{-(k+1)}) = \sum_{r=0}^{m} \binom{m}{i} h^{(r)}(x) + O(n^{-(k+1)}).$$

Also, $I_9 = O(n^{-\tau/2})$ uniformly in $x \in [a_2^*, b_2^*]$.

Finally, by the mean value theorem,

$$\|I_6\|_{C[a_2^*,b_2^*]} \le \sum_{j=0}^{k} |C(j,k)|$$
$$\times \left\| \int_0^\infty P_{d_j n}^{(m)}(x,t) \frac{f^{(p'+m-1)}(\eta) - f^{(p'+m-1)}(x)}{(p'+m-1)!} |h'(\xi)|\, |t-x|^{p'+m} \zeta_2(t)dt \right\|_{C[a_2^*,b_2^*]},$$

where ξ and η lie between t and x.

Using Lemma 2.4, we have

$$\|I_6\|_{C[a_2^*,b_2^*]} \le \sum_{j=0}^{k} |C(j,k)| \sum_{\substack{2r+s\le m \\ r,s\ge 0}} (d_j n+1)^r \|h'\|_{C[a_2^*,b_2^*]}$$
$$\times \left\| \frac{\phi_{r,s,m}(x)}{\{x(1+x)\}^m} \int_0^\infty P_{d_j n}(x,t)|\nu - (d_j n+1)x|^s \right.$$
$$\left. \times \frac{f^{(p'+m-1)}(\eta) - f^{(p'+m-1)}(x)}{(p'+m-1)!} |t-x|^{p'+m}\zeta_2(t) \right\|_{C[a_2^*,b_2^*]}.$$

Now using the Cauchy–Schwarz inequality, Lemma 2.1 and Lemma 2.2, we get

$$\|I_6\|_{C[a_2^*,b_2^*]} = O(n^{-(p'+1-\delta)/2}) = O(n^{-\tau/2})$$

by choosing δ such that $0 \le \delta \le p'+1-\tau$.

Combining the above estimates, we get

$$\left\|B_n^{(m)}(fh,k,\cdot) - (fh)^{(m)}\right\|_{C[a_2^*,b_2^*]} = O(n^{-\tau/2}).$$

Since $\mathrm{supp}(fh) \subset [a_2^*, b_2^*]$, it follows from Lemma 2.5 and Lemma 2.7 that

$$(fh)^{(m)} \in Z_\alpha(k+1, a_2^*, b_2^*).$$

Since $h(x) = 1$ on $[a_2, b_2]$, we have $f^{(m)} \in Z_\alpha(k+1, a_2^*, b_2^*)$.

This completes the proof of Theorem 3.1. □

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