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# WEYL'S THEOREM FOR ALGEBRAICALLY $(p,k)$-QUASIHYPONORMAL OPERATORS

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**Abstract.** Let $A$ be a bounded linear operator acting on a Hilbert space $H$. The $B$-Weyl spectrum of $A$ is the set $\sigma_{Bw}(A)$ of all $\lambda \in \mathbb{C}$ such that $A - \lambda I$ is not a $B$-Fredholm operator of index 0. Let $E(A)$ be the set of all isolated eigenvalues of $A$. Recently, in [3] the author showed that if $A$ is hyponormal, then $A$ satisfies the generalized Weyl's theorem $\sigma_{Bw}(A) = \sigma(A) \setminus E(A)$, and the $B$-Weyl spectrum $\sigma_{Bw}(A)$ of $A$ satisfies the spectral mapping theorem. Lee [13] showed that Weyl's theorem holds for algebraically hyponormal operators. In this paper the above results are generalized to an algebraically $(p,k)$-quasihyponormal operator which includes an algebraically hyponormal operator.

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## 1. Introduction

Let $B(H)$ and $K(H)$ denote, respectively, the algebra of bounded linear operators and the ideal of compact operators acting on an infinite dimensional separable Hilbert space $H$. If $A \in B(H)$ we shall write $N(A)$ and $R(T)$ for the null space and the range of $A$, respectively. Also, let $\alpha(A) := \dim N(A)$, $\beta(A) := \dim N(A^*)$, and let $\sigma(A)$, $\sigma_a(A)$ and $\pi_0(A)$ denote the spectrum, approximate point spectrum and point spectrum of $A$, respectively. An operator $A \in B(H)$ is called Fredholm if it has a closed range, a finite dimensional null space, and its range has finite co-dimension. The index of a Fredholm operator is given by

$$I(A) := \alpha(A) - \beta(A).$$

$A$ is called Weyl if it is of index zero, and Browder if it is Fredholm of finite ascent and descent, equivalently ([15], Theorem 7.9.3) if $A$ is Fredholm and $A - \lambda$ is invertible for sufficiently small $|\lambda| > 0$, $\lambda \in \mathbb{C}$. The essential spectrum $\sigma_e(A)$, the Weyl spectrum $\sigma_w(A)$ and the Browder spectrum $\sigma_b(A)$ of $A$ are defined by [14, 15]

$$\sigma_e(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is not Fredholm}\},$$
$$\sigma_w(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is not Weyl}\},$$
$$\sigma_b(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is not Browder}\},$$

respectively. Evidently

$$\sigma_e(A) \subseteq \sigma_w(A) \subseteq \sigma_b(A) = \sigma_e(A) \cup \operatorname{acc} \sigma(A),$$

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where we write $\operatorname{acc} K$ for the accumulation points of $K \subseteq \mathbb{C}$. If we write $\operatorname{iso} K = K \setminus \operatorname{acc} K$, then we let

$$\pi_{00}(A) := \{\lambda \in \operatorname{iso} \sigma A : 0 < \alpha(A - \lambda) < \infty\},$$
$$p_{00}(A) := \sigma(A) \setminus \sigma_b(A).$$

We say that Weyl's theorem holds for $A$ if

$$\sigma(A) \setminus \sigma_w(A) = \pi_{00}(A).$$

More generally, Berkani in [2] says that the generalized Weyl's theorem holds for $A$ provided

$$\sigma(A) \setminus \sigma_{Bw}(A) = E(A),$$

where $E(A)$ and $\sigma_{Bw}(A)$ denote the isolated point of the spectrum which are eigenvalues (with no restriction on multiplicity) and the set of complex numbers $\lambda$ for which $A - \lambda I$ fails to be Weyl, respectively. Let $X$ be a Banach space. An operator $A \in B(X)$ is called $B$-Fredholm by Berkani [2] if there exists $n \in \mathbb{N}$ for which the induced operator

$$A_n : A^n(X) \to A^n(X)$$

is Fredholm in the usual sense, and $B$-Weyl if in addition $A_n$ has index zero. Note that if the generalized Weyl's theorem holds for $A$, then so does Weyl's theorem [2]. We say that Browder's theorem holds for $A$ if

$$\sigma(A) \setminus \sigma_w(A) = p_{00}(A).$$

Recently, in [3] the author showed that if $A$ is a hyponormal operator, then $A$ satisfies the generalized Weyl's theorem $\sigma_{Bw}(A) = \sigma(A) \setminus E(A)$, and the $B$-Weyl spectrum $\sigma_{Bw}(A)$ of $A$ satisfies the spectral mapping theorem. Lee [13] showed that Weyl's theorem holds for algebraically hyponormal operators. In this paper the above results are generalized to the case where $A$ is an algebraically $(p,k)$-quasihyponormal operator which includes an algebraically hyponormal operator.

## 2. Main Results

Before proving the following lemma, we need some notation and definitions.

For any operator $A$ in $B(H)$ set, as usual, $|A| = (A^*A)^{\frac{1}{2}}$ and $[A^*, A] = A^*A - AA^* = \mid A \mid^2 - \mid A^* \mid^2$ (the self commutator of A), and consider the following standard definitions: $A$ is normal if $A^*A = AA^*$, hyponormal if $A^*A - AA^* \geq 0$, $p$-hyponormal if $(|A|^{2p} - |A^*|^{2p}) \geq 0$.

$A$ is said to be $p$-quasihyponormal if $A^*((A^*A)^p - (AA^*)^p)A \geq 0$ $(0 < p \leq 1)$, $(p,k)$-quasihyponormal if $A^{*k}((A^*A)^p - (AA^*)^p)A^k \geq 0$ $(0 < p \leq 1, k \in \mathbb{N})$, if $p = 1$, $k$=1 and $p = k = 1$, then $A$ is $k$-quasihyponormal, $p$-quasihyponormal and quasihyponormal, respectively. $A$ is normaloid if $\|A\| = r(A)$ (the spectral radius of $A$). Let $(pH)$, $(HN)$, $Q(p)$, $(Q(p,k))$ and $(NL)$ denote the classes consisting of $p$-hyponormal, hyponormal, $p$-quasihyponormal,

$(p,k)$-quasihyponormal, and normaloid operators. These classes are related by the proper inclusion

$$(HN) \subset (pH) \subset (Q(p)) \subset (Q(p,k)) \subset (NL)$$

(see [20]). $A$ is said to be algebraically $(p,k)$-quasihyponormal if there exists a nonconstant complex polynomial $p$ such that $p(A)$ is $(p,k)$-quasihyponormal. We say that $A \in B(H)$ has the single valued extension property (SVEP) if for every open set $U \subseteq \mathbb{C}$ the only analytic function $f : U \to H$ which satisfies the equation $(A-\lambda)f(\lambda) = 0$ is the constant function $f \equiv 0$.

**Lemma 2.1.** *Let $A \in B(H)$ be a $(p,k)$-quasihyponormal operator. Then $f(A)$ has SVEP for each analytic function $f$ on a neighborhood of $\sigma(A)$.*

*Proof.* It is known that SVEP is stable under the functional calculus, i.e. if $A \in B(H)$ has SVEP, then so does $f(A)$ for each $f$ analytic in a neighborhood of $\sigma(A)$. Hence to prove the lemma it is sufficient to prove that $A$ has SVEP. If $A$ is $(p,k)$-quasihyponormal, then it follows from ([26], Theorem 4) that

$$\|A^k x\|^2 \le \|A^{k-1}x\| \, \|A^{k+1}x\|,$$

for every unit vectors $x \in H$. If $x \in N(A^{k+1})$, then

$$\|A^k x\|^2 \le \|A^{k-1}x\| \, \|A^{k+1}x\| = 0.$$

Thus $x \in N(A^k)$. Since the non-zero eigenvalues of a $(p,k)$-quasihyponormal operator $A$ are the normal eigenvalues of $A$ ([26], Lemma 3), for $0 \ne \lambda \in \sigma_p(A)$ and $(A-\lambda)^{k+1}x = 0$ we have

$$(A-\lambda)(A-\lambda)^k x = 0 = (A-\lambda)^*(A-\lambda)^k x$$

and

$$\|(A-\lambda)^k x\|^2 = \big\langle (A-\lambda)^*(A-\lambda)^k x, (A-\lambda)^{k-1}x \big\rangle = 0.$$

Hence, if $A$ is $(p,k)$-quasihyponormal, then $asc(A-\lambda) \le k$.

Since operators with finite ascent have SVEP [17], $A$ also has SVEP. Therefore $f(A)$ has SVEP. □

**Theorem 2.1.** *Let $A \in B(H)$ be a $(p,k)$-quasihyponormal operator. Then $f(A)$ satisfies Browder's theorem for each analytic function $f$ in a neighborhood of $\sigma(A)$ and we have*

$$f(\sigma(A) \setminus \pi_0(A)) = f(\sigma_b(A)) = \sigma_b(f(A)) = \sigma(f(A)) \setminus \pi_0(f(A))$$
$$= f(\sigma(A)) \setminus \pi_0(f(A)) = f(\sigma(A)) \setminus \pi_0(f(A)), \text{ and } f(\sigma_{Bw}(A)) = \sigma_{Bw}(f(A)).$$

*Proof.* It is known that operators with SVEP satisfy Browder's theorem [10]. Then $f(A)$ satisfies Browder's theorem. Since $f(A)$ satisfies Browder's theorem,

$$f(\sigma(A) \setminus \pi_0(A)) = f(\sigma_b(A)) = \sigma_b(f(A)) = \sigma(f(A)) \setminus \pi_0(f(A))$$
$$= f(\sigma(A)) \setminus \pi_0(f(A)), \text{ and } f(\sigma_{bw}(A)) = \sigma_{bw}(f(A)).$$

This completes the proof. □

Let $r(A)$ and $W(A)$ denote the spectral radius and the numerical range of $A$, respectively. It is well known that $r(A) \leq \|A\|$ and that $W(A)$ is convex with convex hull $conv\sigma(A) \subseteq \overline{W(A)}$. $A$ is said convexoid if $conv\sigma(A) = \overline{W(A)}$.

**Lemma 2.2.** *Let $A$ be a $(p,k)$-quasihyponormal operator and $\lambda \in \mathbb{C}$. If $\sigma(A) = \{\lambda\}$, then $A = \lambda$.*

*Proof.* We consider two cases:

*Case* 1 ($\lambda = 0$). Since $A$ is $(p,k)$-quasihyponormal, $A$ is normaloid [20]. Therefore $A = 0$.

*Case* 2 ($\lambda \neq 0$). Here $A$ is invertible, and since $A$ is $(p,k)$-quasihyponormal, $A^{-1}$ is also $(p,k)$-quasihyponormal ([21], Lemma 3). Therefore $A^{-1}$ is normaloid. On the other hand, $\sigma(A^{-1}) = \{\frac{1}{\lambda}\}$. Hence $\|A\|\|A^{-1}\| = |\lambda||\frac{1}{\lambda}| = 1$. It follows from ([22], Lemma 3) that $A$ is convexoid. Hence $W(A) = \{\lambda\}$ and $A = \lambda$. □

**Lemma 2.3.** *Let $A$ be a quasinilpotent algebraically $(p,k)$-quasihyponormal operator. Then $A$ is nilpotent.*

*Proof.* Assume that $p(A)$ is $(p,k)$-quasihyponormal for some nonconstant polynomial $p$. Since $\sigma(p(A)) = p(\sigma(A))$, the operator $p(A) - p(0)$ is quasinilpotent. Thus Lemma 2.2 would imply that

$$cA^m(A - \lambda_1)\cdots(A - \lambda_n) \equiv p(A) - p(0) = 0,$$

where $m \geq 1$. Since $A - \lambda_i$ is invertible for every $\lambda \neq 0$, we must have $A^m = 0$. □

**Lemma 2.4.** *Let $A$ be an algebraically $(p,k)$-quasihyponormal operator. Then $A$ is isoloid.*

*Proof.* Let $\lambda \in \operatorname{iso}\sigma(A)$ and let

$$P := \frac{1}{2\pi i}\int_{\partial D}(\mu - T)^{-1}d\mu$$

be the associated Riesz idempotent, where $D$ is a closed disk centered at $\lambda$ which contains no other points of $\sigma(A)$. We can then represent $A$ as the direct sum

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad \text{where } \sigma(\mathrm{A}_1) = \{\lambda\} \text{ and } \sigma(\mathrm{A}_2) = \sigma(\mathrm{A}) \setminus \{\lambda\}.$$

Since $A$ is algebraically $(p,k)$-quasihyponormal, $p(A)$ is $(p,k)$-quasihyponormal for some nonconstant polynomial $p$. Since $\sigma(A_1) = \lambda$, we must have

$$\sigma(p(A_1)) = p(\sigma(A_1)) = \{p(\lambda)\}.$$

Therefore $p(A_1) - p(\lambda)$ is quasinilpotent. Since $p(A_1)$ is $(p,k)$-quasihyponormal, it follows from lemma 2.2 that $p(A_1) - p(\lambda) = 0$. Put $q(z) := p(z) - p(\lambda)$. Then $q(A_1) = 0$, so $A_1$ is algebraically $(p,k)$-quasihyponormal. Since $A_1 - \lambda$ is quasinilpotent and algebraically $(p,k)$-quasihyponormal, it follows from Lemma 2.3 that $A_1 - \lambda$ is nilpotent. Therefore $\lambda \in \pi_0(A_1)$, and hence $\lambda \in \pi_0(A)$. This shows that $A$ is isoloid. □

**Theorem 2.2.** *Let $A$ be an algebraically $(p,k)$-quasihyponormal operator. Then Weyl's theorem holds for $f(A)$ and for every function $f$ analytic in a neighborhood of $\sigma(A)$.*

*Proof.* We first show that Weyl's theorem holds for $A$. Assume that $\lambda \in \sigma(A) \setminus \sigma_w(A)$. Then $A-\lambda$ is Weyl and not invertible. We claim that $\lambda \in \partial\sigma(A)$. Assume to the contrary that $\lambda$ is an interior point of $\sigma(A)$. Then there exists a neighborhood $U$ of $\lambda$ such that $\dim(A-\mu) > 0$ for all $\mu \in U$. It follows from ([11], Theorem 10) that $A$ does not have SVEP. On the other hand, since $p(A)$ is $(p,k)$-quasihyponormal for a nonconstant polynomial $p$, it follows from Lemma 2.1 that $p(A)$ has SVEP. Hence by ([18], Theorem 3.3.9), $A$ has SVEP, a contradiction. Therefore $\lambda \in \partial\sigma(A)$. Conversely, assume that $\lambda \in \pi_{00}(A)$ with the associated Riesz idempotent

$$P := \frac{1}{2\pi i}\int_{\partial D} (\mu - T)^{-1} d\mu,$$

where $D$ is a closed disk centered at $\lambda$ which contains no other points of $\sigma(A)$. We can then represent $A$ as the direct sum

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad \text{where } \sigma(\mathrm{A}_1) = \{\lambda\} \text{ and } \sigma(\mathrm{A}_2) = \sigma(\mathrm{A}) \setminus \{\lambda\}.$$

We consider two cases:

*Case* 1 ($\lambda = 0$). Here $A_1$ is algebraically $(p,k)$-quasihyponormal and quasinilpotent. Hence it follows from Lemma 2.3 that $A_1$ is nilpotent. We claim that $\dim R(P) < \infty$. For, if $N(A_1)$ were infinite dimensional, then $0 \notin \pi_{00}(A)$, a contradiction. Therefore $A_1$ is a finite dimensional operator, and therefore Weyl. But since $A_2$ is invertible, we can conclude that $A$ is Weyl. Thus $0 \in \sigma(A) \setminus \sigma_w(A)$.

*Case* 2 ($\lambda \neq 0$). As in the proof of Lemma 2.3, $A_1 - \lambda$ is nilpotent. Since $\lambda \in \pi_{00}$, $A_1 - \lambda$ is a finite dimensional operator. Therefore $A_1 - \lambda$ is Weyl. Since $A_2 - \lambda$ is invertible, $A - \lambda$ is Weyl and Weyl's theorem holds for $A$.

Next we prove that $f(\sigma_w(A)) = \sigma_w(f(A))$ for every function $f$ analytic in a neighborhood of $\sigma(A)$. Let $f$ be an analytic function in a neighborhood of $\sigma(A)$. Since $\sigma_w(f(A)) \subseteq f(\sigma_w(A))$ with no restriction on $A$, it is sufficient to prove that $f(\sigma_w(A)) \subseteq \sigma_w(f(A))$. Assume that $\lambda \notin \sigma_w(f(A))$. Then $f(A) - \lambda$ is Weyl and

$$f(A) - \lambda = c(A-\alpha_1)(A-\alpha_2)\cdots(A-\alpha_n)g(A), \tag{2.1}$$

where $c, \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ and $g(A)$ is invertible. Since the operators on the right-hand side of (2.1) commute, every $A - \alpha_i$ is Fredholm. Since $A$ is algebraically $(p,k)$-quasihyponormal, $A$ has SVEP by Lemma 2.1. It follows from ([1], Theorem 2.6) that $i(A - \alpha_i) \le 0$ for each $i = 1, 2, \ldots, n$. Hence $\lambda \notin f(\sigma_w(A))$, and so $f(\sigma_w(A)) = \sigma_w(f(A))$.

It is known [19] that if $A$ is isoloid, then

$$f(\sigma(A)) \setminus \pi_{00}(f(A)) = \sigma(f(A)) \setminus \pi_{00}(f(A))$$

for every analytic function in a neighborhood of $\sigma(A)$. Since $A$ is isoloid by Lemma 2.3 and Weyl's theorem holds for $f(A)$,

$$\sigma(f(A)) \setminus \pi_{00}(f(A)) = f(\sigma(A)) \setminus \pi_{00}(f(A)) = f(\sigma_w(A)) = \sigma_w(f(A)).$$

Which achieves the proof. □

**Theorem 2.3.** *Let $A \in B(H)$ be a $(p,k)$-quasihyponormal operator. Then $f(A)$ satisfies the generalized Weyl's theorem for every function $f$ analytic in a neighborhood of $\sigma(A)$. In particular, Weyl's theorem holds for $f(A)$.*

*Proof.* We have already proved that

$$f(\sigma_{Bw}(A)) = \sigma_{Bw}(f(A)) = f(\sigma(A) \setminus E(A)).$$

Hence to prove the theorem it suffices to prove that

$$f(\sigma(A) \setminus E(A)) = \sigma(f(A)) \setminus E(f(A)).$$

But this last equality is satisfied because the operator $A$ is isoloid in the sense that the isolated points of its spectrum are eigenvalues (see Lemma 2.4 and ([3], Lemma 2.9). As it is shown in [2], if the generalized Weyl's theorem holds for $f(A)$, then so does Weyl's theorem. □

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