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MODULUS OF CONTINUITY AND BEST APPROXIMATION WITH RESPECT TO VILENKIN-LIKE SYSTEMS IN SOME FUNCTION SPACES

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Abstract. We rephrase Fridli's result [2] on the modulus of continuity with respect to a Vilenkin group in the Lebesgue space. We show that this result is valid in the logarithm space and for Vilenkin-like systems. In addition, we prove that there is a strong connection between the best approximation of Fourier series and the modulus of continuity, not only in the Lebesgue space [5] but in the logarithm space too. We formulate two variable generalizations of the obtained results, which have not been known till now even in the Walsh case.

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The basic concepts used in this paper are discussed in [5]. The Vilenkin spaces are generalizations of many well known structures. One of them is the Walsh–Paley system which is a good tool for solving differential and integral equations, and in pattern recognizing and image processing. The Walsh analysis can be easily applied in computer sciences.

Now we introduce the necessary notions of this paper.

Let $m := (m_k)_{k\in\mathbb{N}}$ be a sequence such that $2 \leq m_i \in \mathbb{N}$ for all $i \in \mathbb{N}$. Let G_{m_k} be an arbitrary set with cardinality m_k. Equip G_{m_k} with a discrete topology, which means that every subset of G_{m_k} is open. Thus, all sets will be open and closed and G_{m_k} will be compact. Let the measure μ_k be as follows: if $x \in G_{m_k}$, then $\mu_k(\{x\}) = 1/m_k$ $(k \in \mathbb{N})$. Denote

$$G_m := \mathop{\times}_{k=0}^{\infty} G_{m_k}.$$

G_m is equipped with the product topology and the product measure μ. Thus, the elements of G_m are sequences: $x = (x_i)_{i\in\mathbb{N}} \in G_m$, where $x_i \in G_{m_i}$ $(i \in \mathbb{N})$. As a result, G_m is a compact topological space and a probability space. We call this structure a *Vilenkin space* that is *bounded* if $\sup_{k\in\mathbb{N}} m_k < +\infty$.

A neighbourhood base of G_m is defined by

$$I_0(x) := G_m, \quad I_n(x) := \{y = (y_i)_{i\in\mathbb{N}} \in G_m \mid y_i = x_i \text{ if } i < n\} \quad (n \geq 1).$$

The elements of this base are called *intervals.*

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We introduce the generalized powers: $M_0 := 1$, $M_{n+1} := m_n M_n$ $(n \in \mathbb{N})$. Then all $n \in \mathbb{N}$ can be represented uniquely in the form

$$n = \sum_{i=0}^{\infty} n_i M_i \quad (n_i \in \{0, 1, \ldots, m_i - 1\},\ i \in \mathbb{N}).$$

Note that $\mu(I_n) = M_n^{-1}$. We define an orthonormal system on G_m, which is called a *Vilenkin-like system.* The *generalized Rademacher functions* satisfy the following properties:

(1) $r_n^k : G_m \to \mathbb{C}$.
(2) $r_n^k(x)$ $(x \in G_m)$ depends only on $x_0, \ldots, x_k$ and $r_0^k = 1$, $k, n \in \mathbb{N}$.
(3) If M_k is a divisor of numbers n and l and if $n^{(k+1)} = l^{(k+1)}$ for all $k, l, n \in \mathbb{N}$, then

$$M_k \int\limits_{I_n(z)} r_k^n(x)\overline{r_k^l(x)}d\mu(x) = \begin{cases} 1 & \text{if } n_k = l_k, \\ 0 & \text{if } n_k \neq l_k, \end{cases}$$

where $n^{(k)} = \sum\limits_{j=k}^{\infty} n_j M_j$, $z \in G_m$.

(4) If M_{k+1} is a divisor of n, then

$$\sum_{j=0}^{m_k - 1} |r_k^{jM_k + n}(x)|^2 = m_k \quad (x \in G_m).$$

(5) There exists $\delta > 1$ such that $\|r_k^n\|_\infty \leq \sqrt{\frac{m_k}{\delta}}$.

A Vilenkin-like system is written as

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n^{(k)}}(x) \quad (n \in \mathbb{N}).$$

It can be proved (see [7]) that $(\psi_n)_{n \in \mathbb{N}}$ is really orthonormal, i.e.,

$$\int\limits_{G_m} \psi_n(x)\overline{\psi_m(x)}d\mu(x) = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$$

We define *Fourier coefficients*, *partial sums of Fourier series* and *Dirichlet kernels* with respect to $(\psi_n)_{n \in \mathbb{N}}$ as follows:

$$\hat{f}(n) := \int\limits_{G_m} f\overline{\psi_n},$$

$$S_n f := \sum_{k=0}^{n-1} \hat{f}(k)\psi_k,$$

$$D_n(y, x) := \sum_{k=0}^{n-1} \psi_k(y)\overline{\psi_k(x)}, \quad D_0 := 0.$$

Now we would like to recall some structures that are special cases of a Vilenkin-like system.

- **Walsh(–Paley) and Vilenkin systems.**

  If $G_{m_k} = \mathbb{Z}_{m_k}$ and the Rademacher functions are $r_n(x) = \exp\left(2\pi i \frac{x_n}{m_n}\right)$ $(x \in G_m,\, n \in \mathbb{N})$, then we get a Vilenkin-like system. Here we can define the operation as coordinate-wise addition modulo m_k. The simplest case is $m_k = 2$ $(k \in \mathbb{N})$ (the *Walsh–Paley system*) and it has vast literature, see, e.g., [9].
- **The group of m-adic integers.**

  Let $G_{m_k} = \mathbb{Z}_{m_k}$ for all $k \in \mathbb{N}$. Define on G_m the following commutative addition: let $x, y \in G_m$, then $x + y = z \in G_m$ is defined in a recursive way. First, $x_0 + y_0 = t_0 m_0 + z_0$, where $z_0 \in G_{m_0}$ and $t_0 \in \mathbb{N}$. Now let us assume that we have $z_0, \ldots, z_k$ and $t_0, \ldots, t_k$. Then $x_{k+1} + y_{k+1} + t_k = t_{k+1} m_{k+1} + z_{k+1}$, where $z_{k+1} \in G_{m_{k+1}}$ and $t_{k+1} \in \mathbb{N}$. Then G_m is called the *group of m-adic integers*. The *Rademacher functions* are

  $$r_k^n(x) := \left(\exp\left(2\pi i \left(\frac{x_k}{m_k} + \cdots + \frac{x_0}{m_k m_{k-1} \cdots m_0}\right)\right)\right)^{n_k}.$$

  Let $\psi_n := \prod_{k=0}^{\infty} r_k^{n(k)}$. This system is the *character system* of the group G_m. For more information see, e.g., [3].
- **Almost even arithmetical functions.**

  We can consider a Vilenkin-like system in the Fourier analysis of arithmetical functions. An arithmetical function $f : \mathbb{N} \to \mathbb{C}$ is *even* mod k, if $f(\text{g.c.d.}(n,k)) = f(n)$ $(n \in \mathbb{N})$. These functions form (with the aid of usual pointwise operations) a subalgebra of $\{f \mid f : \mathbb{N} \to \mathbb{C}\}$;

  $$M(f) := \limsup_{x \to \infty} \frac{1}{x} \sum_{1 \le i \le x} |f(i)|;$$

  $$\|f\|_p := M(|f|^p)^{\frac{1}{p}};$$

  $$\mathcal{B} := \overline{\{f : \mathbb{N} \to \mathbb{C} \mid f \text{ even},\ \|f\|_p < \infty\}}\,.$$

  The elements of the Banach space $\mathcal{B}$ are called *almost even arithmetical functions*. (The topology is derived from the above p-norm.) A Vilenkin-like system is a useful tool for the approximation theoretic investigations of the functions mentioned above. One can read more about this in [6].

To introduce a logarithm space, we need some preliminary notions: let $(X, \mathcal{A}, \mu)$ be a fixed finite-measure space. The *distribution function* of $f : X \to \mathbb{C}$ is:

$$\mu_f(\lambda) = \mu\{x \in X \mid |f(x)| > \lambda\}.$$

The *nonincreasing rearrangement* of f is the following function:

$$f^*(t) = \inf\{\lambda \mid \mu_f(\lambda) \le t\}, \qquad \text{here } \inf \varnothing = +\infty,$$

The $\mathrm{Llog}^{+}\mathrm{L}$-*norm* of f is

$$\|f\|^*_{\mathrm{Llog}^{+}\mathrm{L}} := \int\limits_0^{\mu(X)} f^*(t)\log_e\left(\frac{1}{t}\right)dt.$$

The *Logarithm space* is defined in the following way:

$$\mathrm{Llog}^{+}\mathrm{L}^*(X) := \left\{ f : X \to \mathbb{C} \,\middle|\, \|f\|^*_{\mathrm{Llog}^{+}\mathrm{L}} < +\infty \right\}.$$

The norm defined above cannot be calculated easily even for simple functions, but we can give a value which is finite if and only if the $\mathrm{Llog}^{+}\mathrm{L}$-norm of f is finite:

$$\|f\|_{\mathrm{Llog}^{+}\mathrm{L}} := \int\limits_{G_m} |f(x)|\log^{+}|f(x)|d\mu(x),$$

where

$$\log^{+}(x) = \begin{cases} 1 & \text{if } x \le e, \\ \log_e(x) & \text{if } x > e. \end{cases}$$

Thus

$$\mathrm{Llog}^{+}\mathrm{L}^*(X) = \mathrm{Llog}^{+}\mathrm{L}(X) := \{ f : X \to \mathbb{C} \,|\, \|f\|_{\mathrm{Llog}^{+}\mathrm{L}} < +\infty \}.$$

One of the notable properties of a logarithm space (which is a Banach space with the norm defined above) is that it is larger than the Lebesgue space L^p if $p > 1$. Its dual space is an exponential space ($L_{\exp}$) and there are the following set theoretic relations between them:

$$L^\infty \subset L_{\exp} \subset L^p \subset \mathrm{Llog}^{+}\mathrm{L} \subset L^1 \quad (1 < p < \infty).$$

One can find detailed information about $\mathrm{Llog}^{+}\mathrm{L}$ and $L_{\exp}$ in [1]. A logarithm space is often called a Zygmund space (A. Zygmund, E. C. Titchmarsh, 1928).

One of the main results of this paper is that we prove the validity of Fridli's theorem [2] for a logarithm space. The history of this is the following:

- Rubinshtein [4] proved Fridli's result in the case of functions in $C(G_m)$, $L^1(G_m)$ and in $L^2(G_m)$, where G_m is a Vilenkin group.
- Fridli verified that Rubinshtein's conjecture is true, i.e., Rubinshtein's theorem holds for any $L^p(G_m)$, where $1 \le p < \infty$.

Now we show that this result is true for Vilenkin spaces and in logarithm spaces too.

Definition.

$$\Lambda_n := \left\{ (\lambda_0, \lambda_1, \dots) : G_m \to G_m \,\middle|\, \begin{array}{ll} \lambda_i = \mathrm{id}_{G_{m_i}}, & \text{if } (0 \le i \le n-1) \\ \lambda_i \text{ bijective} & \text{if } i \ge n \end{array} \right\}.$$

Consequently, Λ_n contains functions of which the first n coordinate functions are identical functions, the rest are coordinate-wise bijections.

We use the following convention: if some definitions in the sequel formally coincide for two space $\mathrm{Llog}^{+}\mathrm{L}$ and L^p, we use $\mathcal{L}$ instead of $\mathrm{Llog}^{+}\mathrm{L}$ or L^p.

Definition. The *modulus of continuity* of $f \in \mathcal{L}(G_m)$ is the sequence

$$\omega_k(f)_{\mathcal{L}} := \sup_{\lambda \in \Lambda_k} \|f - f \circ \lambda\|_{\mathcal{L}} \quad (k \in \mathbb{N}).$$

Theorem 1. *If G_m is a Vilenkin space and $(\omega_k)_{k \in \mathbb{N}}$ is a sequence of real numbers monotonously decreasing to zero, then there exists $f \in \mathcal{L}(G_m)$ such that $\omega_k(f)_{\mathcal{L}} = \omega_k$ $(k \in \mathbb{N})$.*

Conversely, if $f \in \mathcal{L}(G_m)$, then $(\omega_k(f)_{\mathcal{L}})_{k \in \mathbb{N}}$ is a sequence monotonously decreasing to zero.

Definition. The **-modulus of continuity* of $f \in \mathrm{Llog}^+\mathrm{L}(G_m)$ with respect to the norm $\|\cdot\|^*_{\mathrm{Llog}^+\mathrm{L}}$ is the sequence

$$\omega^*_k(f)_{\mathrm{Llog}^+\mathrm{L}} := \sup_{\lambda \in \Lambda_k} \|f - f \circ \lambda\|^*_{\mathrm{Llog}^+\mathrm{L}} \quad (k \in \mathbb{N}).$$

Theorem 2. *If G_m is a Vilenkin space and $(\omega_k)_{k \in \mathbb{N}}$ is a sequence of real numbers monotonously decreasing to zero, then there exists $f \in \mathrm{Llog}^+\mathrm{L}(G_m)$ such that $\omega^*_k(f)_{\mathrm{Llog}^+\mathrm{L}} = \omega_k$ $(k \in \mathbb{N})$.*

Conversely, if $f \in \mathrm{Llog}^+\mathrm{L}(G_m)$, then $(\omega^*_k(f)_{\mathrm{Llog}^+\mathrm{L}})_{k \in \mathbb{N}}$ is a sequence monotonously decreasing to zero.

Definition. The *set of Vilenkin-like polynomials with degree at most n* is defined as

$$\mathcal{P}_n := \{c_0\psi_0 + c_1\psi_1 + \cdots + c_{n-1}\psi_{n-1} \mid c_0, c_1, \ldots, c_{n-1} \in \mathbb{C}\}.$$

Denote

$$\mathcal{P} := \bigcup_{n=0}^{\infty} \mathcal{P}_n.$$

Definition. Let $f \in \mathrm{Llog}^+\mathrm{L}(G_m)$, where G_m is a Vilenkin space. Then the *best approximation of f with polynomials of degree M_n* is

$$E_n(f)_{\mathrm{Llog}^+\mathrm{L}} := \inf_{P \in \mathcal{P}_{M_n}} \|f - P\|_{\mathrm{Llog}^+\mathrm{L}} \quad (n \in \mathbb{N}).$$

By definition, we have

$$\|f\|_1 \le \|f\|_{\mathrm{Llog}^+\mathrm{L}} \quad f \in \mathrm{Llog}^+\mathrm{L}(G_m).$$

The triangle inequality is not valid for the norm $\|\cdot\|_{\mathrm{Llog}^+\mathrm{L}}$, but the following is true:

$$\|f + g\|_{\mathrm{Llog}^+\mathrm{L}} \le 3\|f\|_{\mathrm{Llog}^+\mathrm{L}} + 3\|g\|_{\mathrm{Llog}^+\mathrm{L}} \quad (f, g \in \mathrm{Llog}^+\mathrm{L}(G_m)).$$

In the next theorem we give an upper bound for the modulus of continuity of a function f with respect to partial sums of the Fourier series of f and the best approximation.

Theorem 3. *If $f \in \mathrm{Llog}^+\mathrm{L}(G_m)$, then*

$$\omega_n(f)_{\mathrm{Llog}^+\mathrm{L}} \le 6\|S_{M_n}f - f\|_{\mathrm{Llog}^+\mathrm{L}}$$

and

$$\omega_n(f)_{\mathrm{Llog}^+\mathrm{L}} \le 6E_n(f)_{\mathrm{Llog}^+\mathrm{L}}.$$

Now, we extend our result to two variables.

Let G_m and $G_{\widetilde{m}}$ be two Vilenkin spaces with measures μ and ν, respectively. Let us denote the product measure on $G_m \times G_{\widetilde{m}}$ by $\mu \otimes \nu$.

Definition. Let for all $n, k \in \mathbb{N}$,

$$\Lambda_n := \left\{ (\lambda_0, \lambda_1, \dots) : G_m \to G_m \;\middle|\; \begin{array}{ll} \lambda_i = \mathrm{id}_{G_{m_i}} & \text{if } (0 \le i \le n-1) \\ \lambda_i \text{ is bijective} & \text{if } i \ge n \end{array} \right\},$$

$$\widetilde{\Lambda}_n := \left\{ (\lambda_0, \lambda_1, \dots) : G_{\widetilde{m}} \to G_{\widetilde{m}} \;\middle|\; \begin{array}{ll} \lambda_i = \mathrm{id}_{G_{\widetilde{m}_i}} & \text{if } (0 \le i \le n-1) \\ \lambda_i \text{ is bijective} & \text{if } i \ge n \end{array} \right\},$$

$$\Lambda_{nk} := \Lambda_n \times \widetilde{\Lambda}_k.$$

One can find the basic definition of different types of the modulus of continuity, e.g., in [8].

Definition. The *partial moduli of continuity* are:

$$\overset{1}{\omega}_k (f)_{\mathcal{L}} = \sup_{\lambda \in \Lambda_k} \|f - f \circ (\lambda, \mathrm{id})\|_{\mathcal{L}} \quad (k \in \mathbb{N}),$$

$$\overset{2}{\omega}_k (f)_{\mathcal{L}} = \sup_{\lambda \in \widetilde{\Lambda}_k} \|f - f \circ (\mathrm{id}, \lambda)\|_{\mathcal{L}} \quad (k \in \mathbb{N}).$$

Theorem 4. *If $(\omega_k)_{k \in \mathbb{N}}$ is a sequence of real numbers monotonously decreasing to zero, then there exists $f \in \mathcal{L}(G_m)$ such that $\overset{i}{\omega}_k (f)_{\mathcal{L}} = \omega_k$ $(k \in \mathbb{N})$ with $i = 1, 2$.*

Conversely, if $f \in \mathcal{L}(G_m)$, then $(\overset{i}{\omega}_k (f)_{\mathcal{L}})_{k \in \mathbb{N}}$ is a sequence monotonously decreasing to zero $(i = 1, 2)$.

Definition. The (*total*) *modulus of continuity* is defined as

$$\omega_k(f)_{\mathcal{L}} = \sup_{(\lambda^1, \lambda^2) \in \Lambda_{kk}} \|f - f \circ (\lambda^1, \lambda^2)\|_{\mathcal{L}}.$$

Theorem 5. *If $(\omega_k)_{k \in \mathbb{N}}$ is a sequence of real numbers monotonously decreasing to zero, then there exists $f \in \mathcal{L}(G_m \times G_{\widetilde{m}})$ such that $\omega_k(f)_{\mathcal{L}} = \omega_k$ $(k \in \mathbb{N})$.*

Remark. Since theorems like the above ones are valid for the *-norm of a logarithm space, we do not formulate them. They are proved similarly to the case of one variable, see the proof of Theorem 2.

To proceed, we must introduce the fundamental notion of the Fourier analysis with two variables with respect to Vilenkin-like systems. For this, let us fix two arbitrary Vilenkin spaces, G_m and $G_{\widetilde{m}}$. The corresponding Vilenkin-like systems are denoted by $(\psi_i)_{i \in \mathbb{N}}$ and $(\widetilde{\psi}_j)_{j \in \mathbb{N}}$, respectively.

Definition. Let $f \in L^1(G_m \times G_{\widetilde{m}})$ and $n, k \in \mathbb{N}$ be arbitrary. Let *two-dimensional Fourier coefficients*, *partial sums of Fourier series* and *polynomials with degree* (n, k) be, respectively, defined by

$$\hat{f}(n, k) := \int\limits_{G_m \times G_{\widetilde{m}}} f(t^1, t^2) \overline{\psi_n(t^1) \widetilde{\psi}_k(t^2)} d\mu \otimes \nu(t^1, t^2),$$

$$S_{n,k}f := \sum_{i=0}^{n-1}\sum_{j=0}^{k-1} \hat{f}(i,j)\psi_i\widetilde{\psi}_j,$$

$$\mathcal{P}_{n,k} := \left\{ \sum_{i=0}^{n-1}\sum_{j=0}^{k-1} c_{ij}\psi_i\widetilde{\psi}_j \;\middle|\; c_{ij} \in \mathbb{C} \right\}.$$

Therefore $S_{n,k} \in \mathcal{P}_{n,k}$. The *best approximation* of f is

$$E_{n,k}(f)_{\mathcal{L}} = \inf_{P\in\mathcal{P}_{n,k}} \|f - P\|_{\mathcal{L}}.$$

To define the *Dirichlet kernel* we fix two elements: $\mathbf{x} = (x^1, x^2)$ and $\mathbf{y} = (y^1, y^2) \in G_m \times G_{\widetilde{m}}$. Then

$$D_{n,k}(\mathbf{x},\mathbf{y}) := \sum_{i=0}^{n-1}\sum_{j=0}^{k-1} \overline{\psi_i(x^1)\widetilde{\psi}_j(x^2)}\psi_i(y^1)\widetilde{\psi}_j(y^2), \quad D_{0,0} := 0.$$

By the above notation and definitions we have for all $n, k \in \mathbb{N}$ and $\mathbf{x} = (x^1, x^2)$, $\mathbf{t} = (t^1, t^2) \in G_m \times G_{\widetilde{m}}$:

$$S_{n,k}f(\mathbf{x}) = \int\limits_{G_m\times G_{\widetilde{m}}} f(\mathbf{t})D_{n,k}(\mathbf{t},\mathbf{x})d\mu\otimes\nu(\mathbf{t}),$$

$$D_{n,k}(\mathbf{t},\mathbf{x}) = D_n(t^1,x^1)\widetilde{D}_k(t^2,x^2),$$

$$D_{M_n,\widetilde{M}_k}(\mathbf{t},\mathbf{x}) = \begin{cases} M_n\widetilde{M}_k & \text{if } \mathbf{t} \in I_n(x^1)\times\widetilde{I}_k(x^2), \\ 0 & \text{otherwise,} \end{cases}$$

$$S_{M_n,\widetilde{M}_k}f(\mathbf{x}) = M_n\widetilde{M}_k \int\limits_{I_n(x^1)\times\widetilde{I}_k(x^2)} f(\mathbf{t})d\mu\otimes\nu(\mathbf{t}).$$

Now we give the theorems showing that the two-dimensional partial sums of Fourier series – as operators – are of type (p,p) $(1 < p < \infty)$.

Theorem 6. *Let $f \in L^1(G_m \times G_{\widetilde{m}})$ and $n, k \in \mathbb{N}$ be arbitrary. Then the operator $S_{n,k}$ is of type (strongly) (p,p) if $1 < p < \infty$, i.e.,*

$$\|S_{n,k}f\|_p \le c_p^2\|f\|_p,$$

where c_p is the constant of the one-dimensional operator and depends only on p.

Theorem 7. *If $f \in L^p(G_m \times G_{\widetilde{m}})$ $(1 < p < \infty)$, then*

$$\frac{1}{2}\omega_n(f)_{L^p} \le \|S_{M_n,\widetilde{M}_n}f - f\|_p \le \omega_n(f)_{L^p}$$

and

$$\frac{1}{2}\omega_n(f)_{L^p} \le E_{n,n}(f)_{L^p} \le \omega_n(f)_{L^p}.$$

Corollary. *If $f \in L^p(G_m \times G_{\widetilde{m}})$, then*

$$\lim_{n\to\infty} \|S_{Mn,\widetilde{M}_n}f - f\|_p = 0.$$

We formulate the two-dimensional version of Theorem 4.

Theorem 8. *If* $f \in \mathrm{Llog}^+\mathrm{L}(G_m \times G_{\widetilde{m}})$ $(1 < p < \infty)$, *then*

$$\omega_n(f)_{\mathrm{Llog}^+\mathrm{L}} \le 6\|S_{M_n,\widetilde{M}_n} f - f\|_{\mathrm{Llog}^+\mathrm{L}}$$

and

$$\omega_n(f)_{\mathrm{Llog}^+\mathrm{L}} \le 6E_{n,n}(f)_{\mathrm{Llog}^+\mathrm{L}}.$$

Proofs

Proof of Theorem 1. For convenience, we write $(\tau_\lambda f)(x) := (f \circ \lambda)(x)$ and $g_\lambda := f - \tau_\lambda f$. Thus

$$\omega_k(f)_{\mathcal{L}} = \sup_{\lambda \in \Lambda_k} \|g_\lambda\|_{\mathcal{L}}.$$

Let us fix an arbitrary element $z = (z_0, z_1, \dots) \in G_m$ and denote it and its coordinates by "0". This makes the reading easier. Let us divide G_m into disjoint sets:

$$G_m = \{0\} \cup \left(\bigcup_{i=0}^{\infty} I_i(0) \setminus I_{i+1}(0) \right).$$

In the sequel we do not write z and 0 whenever possible, so $I_i = I_i(z) = I_i(0)$. Let us investigate a monotonously increasing step function ($\mathbb{1}$ denotes the characteristic function):

$$f(x) = \sum_{i=0}^{\infty} F_i \mathbb{1}_{I_i \setminus I_{i+1}}(x) \quad (F_i \ge F_j \text{ if } i \ge j).$$

Hence

$$g_\lambda(x) = \sum_{i=0}^{\infty} (F_i - \tau_\lambda f(x)) \mathbb{1}_{I_i \setminus I_{i+1}}(x).$$

To study g_λ, first consider the values of $\tau_\lambda f$.

Let $z = 0 \ne x \in G_m$ be fixed and let, after fixing k, $\lambda \in \Lambda_k$ be arbitrary. Thus $x \in I_i \setminus I_{i+1}$ for some i,

$$x = (0, \dots, 0, x_i \ne 0, x_{i+1}, \dots),$$
$$\lambda = (\mathrm{id}, \dots, \mathrm{id}, \lambda_k, \lambda_{k+1}, \dots).$$

Hence

$$i < k \Longrightarrow \lambda(x) = (0, \dots, 0, x_i \ne 0, \dots, x_{k-1}, \lambda_k(x_k), \dots) \in I_i \setminus I_{i+1}$$
$$\Longrightarrow \tau_\lambda f(x) = F_i,$$
$$i = k \Longrightarrow \lambda(x) = (0, \dots, 0, \lambda_i(x_i), \lambda_{i+1}(x_{i+1}), \dots) \in I_i,$$
$$i > k \Longrightarrow \lambda(x) = (0, \dots, 0, \lambda_k(0), \dots, \lambda_{i-1}(0), \lambda_i(x_i), \dots) \in I_k.$$

Since, for $i = k$ either $\lambda(x) \in I_k \setminus I_{k+1} = I_i \setminus I_{i+1}$ and thus $\tau_\lambda f = F_i$ or $\lambda(x) \in I_{k+1} \setminus I_{k+2} \cup \cdots$ and thus $\tau_\lambda f = F_l$ for some $l \ge k+1$. This case will be considered below. Therefore we can start the summation only from $k+1$. To find the supremum of g_λ, we need to minimize $\tau_\lambda f$. Since $i > k$, $\lambda(x) \in I_k$. Hence

$$\tau_\lambda f(x) \in \{F_l \mid l \ge k\}$$

which is minimal if $l = k$. Therefore it is enough to consider λ's such that $\lambda_k(0) \neq 0$, i.e., $\lambda \in \Lambda_k \setminus \Lambda_{k+1}$. Let us fix such λ, for example, $\widetilde{\lambda}$ and let $g := g_{\widetilde{\lambda}}$. Thus we have to take the integral multiplied by 2 because $\widetilde{\lambda}$ maps $I_k(0)$ to $I_k(w)$ (for some $w \in G_m$ where $w_0 = \cdots = w_{k-1} = 0$, $w_k \neq 0$) and – because of the maximality assumption and bijectivity – vice versa. So if we integrate only over $I_k = (I_k \setminus I_{k+1}) \cup (I_{k+1} \setminus I_{k+2}) \cup \cdots$, we get $I_k(w)$. Therefore

$$\omega_k(f)_{\mathcal{L}} = \sup_{\lambda\in\Lambda_k} \|g_\lambda\|_{\mathcal{L}} = \|g_{\widetilde{\lambda}}\|_{\mathcal{L}} = \|g\|_{\mathcal{L}},$$

$$\begin{aligned}\|g\|_{\mathrm{Llog^+L}} &= \int\limits_{G_m} |g(x)| \log^+ |g(x)| d\mu(x)\\ &= \sum_{i=0}^{\infty} \int\limits_{I_i\setminus I_{i+1}} |g(x)| \log^+ |g(x)| d\mu(x)\\ &= 2\sum_{i=k+1}^{\infty} (F_i - F_k)\log^+(F_i - F_k) \int\limits_{I_i\setminus I_{i+1}} 1 d\mu(x)\\ &= 2\sum_{i=k+1}^{\infty} (F_i - F_k)\log^+(F_i - F_k)\left(\frac{1}{M_i} - \frac{1}{M_{i+1}}\right).\end{aligned}$$

For the space L^p we have

$$\|g\|_p^p = \int\limits_{G_m} |g(x)|^p d\mu(x) = 2\sum_{i=k+1}^{\infty} |F_i - F_k|^p \left(\frac{1}{M_i} - \frac{1}{M_{i+1}}\right).$$

This is the same result as in Fridli's paper and so it is enough to consider the space Llog$^+$L.

Lemma 1. *Let $n \in \mathbb{N}$ be given and let $(x_l)_{l\in\mathbb{N}}$ be a real sequence such that $x_{n+i} := x_{n+1}$ $(i > 1)$. There exist real numbers $x_{n+1} \geq x_n \geq \cdots \geq x_0$ such that for all $k \leq n$,*

$$2\sum_{i=k+1}^{\infty} (x_i - x_k)\log^+(x_i - x_k)\left(\frac{1}{M_i} - \frac{1}{M_{i+1}}\right) = \omega_k(f)_{\mathrm{Llog^+L}} \in [0, +\infty[.$$

Proof. Let us use the method of descent. For $n = k$

$$2(x_{n+1} - x_n)\log^+(x_{n+1} - x_n)\sum_{i=n+1}^{\infty}\left(\frac{1}{M_i} - \frac{1}{M_{i+1}}\right) = \omega_n(f)_{\mathrm{Llog^+L}}.$$

The series written above is convergent because

$$0 \leq \sum_{i=n+1}^{\infty}\left(\frac{1}{M_i} - \frac{1}{M_{i+1}}\right) \leq \sum_{i=0}^{\infty}\left(\frac{1}{M_i} - \frac{1}{M_{i+1}}\right) = \mu(G_m) = 1.$$

By the substitution $x_{n+1} - x_n \rightsquigarrow z$ we get $z\log^+ z = c$, where c is the quotient of $\omega_n(f)_{\mathrm{Llog^+L}}/2$ and the sum of the series. There are two cases:

1) $c \leq e$. Then let $z = c$.

2) $c > e$. Now $f(z) := z \log^+ z = z \log_e z$ $(z > e)$; $\operatorname{dom} f = \operatorname{ran} f =]e, +\infty[$; f is strictly increasing and thus injective, therefore z exists.

The statement is true for $n = k$.

Now let us assume that we have already got the numbers $x_{n+1} \geq x_n \geq \cdots \geq x_{j+1}$ which satisfy the statement. Denote

$$a_j(x) := 2 \sum_{i=j+1}^{\infty} (x_i - x) \log^+(x_i - x) \left(\frac{1}{M_i} - \frac{1}{M_{i+1}} \right).$$

Let us observe that in the definition of a_j the coefficients of $1/M_i - 1/M_{i+1}$ are independent of i after finitely many indexes, since x_i's coincide for $i = n+1, n+2, \ldots$. Because of the induction hypotheses we know that $a_j(x_{j+1}) = \omega_{j+1}(f)_{\mathrm{Llog^+L}}$. Since a_j is continuous (as it is the product and the finite sum of continuous functions), monotonously decreasing (as its derivative is nowhere positive) and $\lim_{x \to -\infty} a_j(x) = +\infty$, there exists $x_j \leq x_{j+1}$ such that $a_j(x_j) = \omega_j(f)_{\mathrm{Llog^+L}}$. ☐

Lemma 2. *There exists a sequence of functions* (f_n) *in* $\mathrm{Llog^+L}(G_m)$*, containing step functions, such that*

1) $(F_i(f_n))_{i \in \mathbb{N}}$ *is increasing and* $F_0(f_n) = 0$*;*

2) $\omega_k(f_n)_{\mathrm{Llog^+L}} = \begin{cases} \omega_k, & k \leq n \\ 0, & k > n \end{cases} \quad (k \in \mathbb{N}).$

Proof. It is clear that if we define f_n as

$$f_n = \sum_{i=0}^{\infty} (x_i - x_0) \mathbb{1}_{I_i \setminus I_{i+1}},$$

where $(x_l)_{l \in \mathbb{N}}$ is derived from the preceding lemma, then the statement of this lemma is satisfied. ☐

We continue the proof of Theorem 1. We have

$$2 \sum_{i=k+1}^{\infty} (x_i - x_k) \log^+(x_i - x_k) \left(\frac{1}{M_i} - \frac{1}{M_{i+1}} \right) = \omega_k(f)_{\mathrm{Llog^+L}}$$

If $k = 0$, then we get the norms of f_n from the preceding lemma:

$$\|f_n\|_{\mathrm{Llog^+L}} = \omega_0(f)_{\mathrm{Llog^+L}} < \infty \quad (n \in \mathbb{N}),$$

whence because of the continuity of the norm we get

$$\left\| \lim_{n \to \infty} f_n \right\|_{\mathrm{Llog^+L}} = \omega_0(f)_{\mathrm{Llog^+L}} \Longrightarrow \lim_{n \to \infty} f_n \in \mathrm{Llog^+L}(G_m).$$

By part 2) of Lemma 2 we get

$$\lim_{n \to \infty} \omega_k(f_n)_{\mathrm{Llog^+L}} = \omega_k(f)_{\mathrm{Llog^+L}} \quad (k \in \mathbb{N}).$$

The space $\mathrm{Llog^+L}(G_m)$ is complete and thus the function we needed is

$$f := \lim_{n \to \infty} f_n.$$

The converse statement of the theorem is obvious because $I_{k+1} \subset I_k$. Therefore $(\omega_k(f)_{\mathrm{Llog^+L}})_{k\in\mathbb{N}}$ is monotonously decreasing. If $k \to \infty$,then the limit of the sequence is 0 by the definition of I_k. ☐

Proof of Theorem 2. The only difference between the proof of Theorem 1 and this proof is the determination of the norm. We use the notation introduced above.

Let us calculate $\mu_g(s)$:

$$0 \le s < F_{k+1} - F_k \Longrightarrow \mu_g(s) = 2\mu(I_{k+1})(= 2\mu(I_{k+1} \setminus I_{k+2} \cup \cdots),$$

$$F_{k+1} - F_k \le s < F_{k+2} - F_k \Longrightarrow \mu_g(s) = 2\mu(I_{k+2})(= 2\mu(I_{k+1} \setminus I_{k+2} \cup \cdots),$$

and so on. Generally,

$$F_{k+i} - F_k \le s < F_{k+i+1} - F_k \Longrightarrow \mu_g(s) = 2\mu(I_{k+i+1}) = \frac{2}{M_{k+i+1}}.$$

Now we determine $g^*(t)$:

$$\frac{2}{M_{k+i+1}} \le t < \frac{2}{M_{k+i}} \Longrightarrow g^*(t) = F_{k+i} - F_k \quad (i \in \mathbb{N}).$$

If $t \ge \frac{2}{M_{k+1}}$, then by the last formula (because $i = 0$) $g^*(t) = 0$. Therefore

$$\omega_k^*(f)_{\mathrm{Llog^+L}} = \|g\|^*_{\mathrm{Llog^+L}} = \int_0^1 g^*(t)\log\left(\frac{1}{t}\right) dt$$

$$= \int_0^{\frac{2}{M_{k+1}}} g^*(t)\log\left(\frac{1}{t}\right) dt = \sum_{i=1}^{\infty} \int_{\frac{2}{M_{k+i+1}}}^{\frac{2}{M_{k+i}}} (F_{k+i} - F_k)\log\left(\frac{1}{t}\right) dt.$$

Using the equality $\int \log\left(\frac{1}{t}\right) dt = \log\left(\frac{1}{x}\right) x + x + c$, replacing the index i by $k+i$, we get

$$\|g\|^*_{\mathrm{Llog^+L}} = 2\sum_{i=k+1}^{\infty} (F_i - F_k)\left[\frac{\log(M_i/2)+1}{M_i} - \frac{\log(M_{i+1}/2)+1}{M_{i+1}}\right].$$

This is convergent because

$$\sum_{i=k+1}^{\infty} \frac{\log(M_i/2)+1}{M_i} \le \sum_{n=1}^{\infty} \frac{\log(2^{n-1})+1}{2^n} = \log(2) + 1,$$

$$a_j(x) := 2\sum_{i=j+1}^{\infty} (x_i - x)\left(\frac{\log(M_i/2)+1}{M_i} - \frac{\log(M_{i+1}/2)+1}{M_{i+1}}\right).$$

Now we are absolutely in the same situation as above. ☐

Proof of Theorem 3. Let $P \in \mathcal{P}_{M_n}$ and $\lambda \in \Lambda_n$. Then [5] we get $P \circ \lambda = P$,

$$\|f - f\circ\lambda\|_{\mathrm{Llog^+L}} = \|f - P + P - P\circ\lambda + P\circ\lambda - f\circ\lambda\|_{\mathrm{Llog^+L}}$$
$$= \|f - P + P\circ\lambda - f\circ\lambda\|_{\mathrm{Llog^+L}}$$

$$\leq 3\|f-P\|_{\mathrm{Llog^+L}} + 3\|P\circ\lambda - f\circ\lambda\|_{\mathrm{Llog^+L}} = 6\|f-P\|_{\mathrm{Llog^+L}}.$$

Thus we get $\omega_n(f)_{\mathrm{Llog^+L}} \leq 6E_n(f)_{\mathrm{Llog^+L}}$. Since $S_{M_n}f \in \mathcal{P}_{M_n}$, we also get $\omega_n(f)_{\mathrm{Llog^+L}} \leq 6\|S_{M_n}f - f\|_{\mathrm{Llog^+L}}$.

Proof of Theorem 4. Let

$$f(x,y) = \sum_{(i,l)\in\mathbb{N}\times\mathbb{N}} F_{il}\mathbb{1}_{il}(x,y),$$

where

$$\mathbb{1}_{il}(x,y) := \begin{cases} 1 & \text{if } (x,y)\in(I_i\setminus I_{i+1})\times(\widetilde{I}_l\setminus\widetilde{I}_{l+1}), \\ 0 & \text{otherwise}\end{cases} \qquad (x,y)\in G_m\times G_{\widetilde{m}}.$$

Let $x\in I_i\setminus I_{i+1}$, $y\in\widetilde{I}_l\setminus\widetilde{I}_{l+1}$. By the same reasoning as that used at the beginning of the proof of Theorem 1 we obtain

$$\inf_{\lambda\in\Lambda_k} f\circ(\lambda(x),y) = \begin{cases} F_{il} & \text{if } i<k, \\ F_{il} & \text{if } i=k, \\ F_{kl} & \text{if } i>k.\end{cases}$$

Let $g_\lambda = f - f\circ(\lambda,\mathrm{id})$,

$$g_\lambda(x,y) = \sum_{(i,l)\in\mathbb{N}\times\mathbb{N}} (f(x,y) - f\circ(\lambda(x),y))\mathbb{1}_{(I_i\setminus I_{i+1})\times(\widetilde{I}_l\setminus\widetilde{I}_{l+1})}(x,y).$$

This sum is maximal if $f\circ(\lambda(x),y)$ is minimal. We denote this g_λ by g. From the maximality condition we can calculate the p-norm of g:

$$\|g\|_p^p = \int\limits_{G_m\times G_{\widetilde{m}}} |g|^p\,d\mu\otimes\nu$$

$$= \sum_{i=0}^{\infty}\int\limits_{I_i\setminus I_{i+1}}\sum_{l=0}^{\infty}\int\limits_{\widetilde{I}_l\setminus\widetilde{I}_{l+1}} |g(x,y)|^p\,d\nu(y)d\mu(x)$$

$$= 2\sum_{i=k+1}^{\infty}\int\limits_{I_i\setminus I_{i+1}}\sum_{l=0}^{\infty}\int\limits_{\widetilde{I}_l\setminus\widetilde{I}_{l+1}} |F_{il}-F_{kl}|^p\,d\nu d\mu$$

$$= 2\sum_{i=k+1}^{\infty}\sum_{l=0}^{\infty}|F_{il}-F_{kl}|^p\mu\otimes\nu[(I_i\setminus I_{i+1})\times(\widetilde{I}_l\setminus\widetilde{I}_{l+1})].$$

We know that

$$\mu\otimes\nu[(I_i\setminus I_{i+1})\times(\widetilde{I}_l\setminus\widetilde{I}_{l+1})] = \left(\frac{1}{M_i}-\frac{1}{M_{i+1}}\right)\left(\frac{1}{\widetilde{M}_l}-\frac{1}{\widetilde{M}_{l+1}}\right).$$

We can define f such that it be constant in the second variable:

$$F_{il} = F_{i0},\quad l=1,2,\dots\ .$$

Hence for the p-norm of g we have

$$\begin{aligned}\|g\|_p^p &= 2\sum_{i=k+1}^{\infty}\sum_{l=0}^{\infty}|F_{il}-F_{kl}|^p\left(\frac{1}{M_i}-\frac{1}{M_{i+1}}\right)\left(\frac{1}{\widetilde{M_l}}-\frac{1}{\widetilde{M_{l+1}}}\right)\\ &= 2\sum_{i=k+1}^{\infty}|F_{i0}-F_{k0}|^p\left(\frac{1}{M_i}-\frac{1}{M_{i+1}}\right)\sum_{l=0}^{\infty}\left(\frac{1}{\widetilde{M_l}}-\frac{1}{\widetilde{M_{l+1}}}\right).\end{aligned}$$

The inner sum is $\nu(G_{\widetilde{m}})=1$,

$$\overset{1}{\omega}_k\,(f)_{L^p}=\|g\|_p\,.$$

The rest of the proof is the same as in the one variable case both for the Lebesgue space and logarithm space.

If we consider a function which is constant in its first variable, we can apply this proof for $\overset{2}{\omega}_k\,(f)_{\mathcal{L}}$ too. □

Proof of Theorem 5. Let $x\in I_i\setminus I_{i+1}$, $y\in\widetilde{I}_l\setminus\widetilde{I}_{l+1}$. From this – as above – we get

$$\inf_{\lambda_1,\lambda_2\in\Lambda_k} f\circ(\lambda_1(x),\lambda_2(y))=\begin{cases}F_{il} & \text{if } i<k,\ \ l<k,\\ F_{ik} & \text{if } i<k,\ \ l\geq k,\\ F_{kl} & \text{if } i\geq k,\ \ l<k,\\ F_{kk} & \text{if } i\geq k,\ \ l\geq k,\end{cases}$$

$$g(x,y):=\left(\sup_{\lambda_1,\lambda_2\in\Lambda_n}(f-f\circ(\lambda_1,\lambda_2))\right)(x,y).$$

It is enough to take the integral over the following three sets:

$$\begin{aligned}&\cup\{I_i\setminus I_{i+1}\times\widetilde{I}_l\setminus\widetilde{I}_{l+1}\,|\,i<k,\ l\geq k\}=:A,\\ &\cup\{I_i\setminus I_{i+1}\times\widetilde{I}_l\setminus\widetilde{I}_{l+1}\,|\,i\geq k,\ l<k\}=:B,\\ &\cup\{I_i\setminus I_{i+1}\times\widetilde{I}_l\setminus\widetilde{I}_{l+1}\,|\,i\geq k,\ l\geq k\}=:C.\end{aligned}$$

It is obvious that $C=I_k\times\widetilde{I}_k$.

By the minimum condition we get

$$\lambda:=(\lambda_1,\lambda_2)\in(\Lambda_k\setminus\Lambda_{k+1})\times(\widetilde{\Lambda}_k\setminus\widetilde{\Lambda}_{k+1}),$$

therefore

$$\begin{aligned}\lambda(A)&=\bigcup\left\{(I_i\setminus I_{i+1})\times\widetilde{I}_k(\widetilde{w})\,\middle|\,i<k\right\},\\ \lambda(B)&=\bigcup\left\{I_k(w)\times(\widetilde{I}_l\setminus\widetilde{I}_{l+1})\,\middle|\,l<k\right\},\\ \lambda(C)&=I_k(w)\times\widetilde{I}_k(\widetilde{w}).\end{aligned}$$

where $(w,\widetilde{w})\in G_m\times G_{\widetilde{m}}$ such that $w_0=\cdots=w_{k-1}=0, w_k\neq 0$ and $\widetilde{w}_0=\cdots=\widetilde{w}_{k-1}=0,\widetilde{w}_k\neq 0$. Actually, we should differentiate between $0\in G_m$ and $\widetilde{0}\in\widetilde{G}_m$, but we will not do it.

We can see that $A, B, C, \lambda(A), \lambda(B), \lambda(C)$ are pairwise disjoint. So if we integrate over $A \cup B \cup C$, then we have to take the integral multiplied by 2.

Let us calculate the p-norm of g (just like in the proof of Theorem 5):

$$\begin{aligned}\frac{1}{2}\|g\|_p^p = \sum_{i<k,\, l>k} |F_{il} - F_{ik}|^p c_{il} + \sum_{i>k,\, l<k} |F_{il} - F_{kl}|^p c_{il} \\ + \sum_{i\geq k,\, l\geq k} |F_{il} - F_{kk}|^p c_{il}, \qquad (1)\end{aligned}$$

where

$$c_{il} = (\mu \otimes \nu)[(I_i \setminus I_{i+1}) \times (\widetilde{I}_l \setminus \widetilde{I}_{l+1})] = \left(\frac{1}{M_i} - \frac{1}{M_{i+1}}\right)\left(\frac{1}{\widetilde{M}_l} - \frac{1}{\widetilde{M}_{l+1}}\right).$$

We will show that there is a double sequence (F_{il}) such that (1) is equal to $\omega_k(f)_{L^p}$. For this reason in (1) the first and the second sum will be zero for properly chosen values of F_{il}. Let all F_{pq} in the sequence (F_{il}) be equal for $n < p$, $n < q$ for a fixed $n \in \mathbb{N}$. Furthermore,

$$\begin{aligned} F_{il} &:= F_{ik}, \quad i = 0, 1, \dots, k-1; \;\; l = k+1, k+2, \dots, n+1, \\ F_{il} &:= F_{kl}, \quad l = 0, 1, \dots, k-1; \;\; i = k+1, k+2, \dots, n+1. \end{aligned}$$

The third sum in (1) can be decomposed as

$$\sum_{i>k,\, l=k} |F_{il} - F_{kk}|^p c_{il} + \sum_{i=k,\, l>k} |F_{il} - F_{kk}|^p c_{il} + \sum_{i>k,\, l>k} |F_{il} - F_{kk}|^p c_{il}.$$

Again, the first and the second sum will be zero if we assume that

$$\begin{aligned} F_{il} &:= F_{kk}, \quad i = k+1, \dots, n+1; \;\; l = k, \\ F_{il} &:= F_{kk}, \quad l = k+1, \dots, n+1; \;\; i = k. \end{aligned}$$

The third sum can be written in the form

$$\sum_{i=l=k+1}^{\infty} |F_{ii} - F_{kk}|^p c_{il} + \sum_{i\neq l} |F_{il} - F_{kk}|^p c_{il}.$$

The second sum will be zero if we assume that

$$F_{il} := F_{kk}, \quad i, l = k+1, k+2, \dots, n+1; \;\; i \neq l.$$

Finally, the problem is reduced to checking whether

$$\left(2 \sum_{i=l=k+1}^{\infty} |F_{ii} - F_{kk}|^p c_{il}\right)^{\frac{1}{p}} = \omega_k(f)_{L^p}.$$

Since the sum above contains only one and the same double indexes we can consider it as a simple sequence. For this, the proof of the one variable case can be applied. Thus, we have constructed the wanted double sequence.

Let (f_n) be a function sequence such that

$$f_n|_{(I_i \setminus I_{i+1}) \times (\widetilde{I}_l \setminus \widetilde{I}_{l+1})} = F_{il} - F_{00}.$$

Hence

$$\omega_k(f_n)_{L^p} = \begin{cases} \omega_k, & k \le n \\ 0, & k > n \end{cases} \quad (k \in \mathbb{N}).$$

The function we need is

$$f = \lim_{n \to \infty} f_n.$$

The construction can be applied for a logarithm space, but we must use the approach of the proof of Theorem 1 instead of Fridli's one.

The constructed double sequence is monotonously increasing on the diagonal, i.e., in the sense of

$$F_{ii} \ge F_{jj} \quad \text{if} \quad i \ge j. \qquad \square$$

Proof of Theorem 6. Let $\mathbf{x} = (x, y)$.

$$\|S_{n,k}f\|_p^p = \int\limits_{G_m \times G_{\widetilde{m}}} |S_{n,k}f(\mathbf{x})|^p d\mu \otimes \nu(\mathbf{x})$$

$$= \int\limits_{G_m} \int\limits_{G_{\widetilde{m}}} \left| \int\limits_{G_m} D_n(t^1, x) \int\limits_{\widetilde{G}_m} f(t^1, t^2) \widetilde{D}_k(t^2, y) d\nu(t^2) d\mu(t^1) \right|^p d\nu(y) d\mu(x).$$

Introduce the mapping $g_y(t^1) := \int\limits_{G_{\widetilde{m}}} f(t^1, t^2) \widetilde{D}_k(t^2, y) d\nu(t^2)$, this equality can be continued in the following manner:

$$= \int\limits_{G_m} \int\limits_{G_{\widetilde{m}}} \left| \int\limits_{G_m} D_n(t^1, x) g_y(t^1) d\mu(t^1) \right|^p d\nu(y) d\mu(x)$$

$$= \int\limits_{G_{\widetilde{m}}} \int\limits_{G_m} \left| \int\limits_{G_m} g_y(t^1) D_n(t^1, x) d\mu(t^1) \right|^p d\mu(x) d\nu(y)$$

$$= \int\limits_{G_{\widetilde{m}}} \|S_n g_y\|_p^p d\nu(y) \le c_p^p \int\limits_{G_{\widetilde{m}}} \|g_y\|_p^p d\nu(y)$$

$$= c_p^p \int\limits_{G_{\widetilde{m}}} \int\limits_{G_m} \left| \int\limits_{\widetilde{G}_m} f(t^1, t^2) \widetilde{D}_k(t^2, y) d\nu(t^2) \right|^p d\mu(t^1) d\nu(y)$$

$$= c_p^p \int\limits_{G_m} \int\limits_{G_{\widetilde{m}}} \left| \int\limits_{\widetilde{G}_m} f(t^1, t^2) \widetilde{D}_k(t^2, y) d\nu(t^2) \right|^p d\nu(y) d\mu(t^1).$$

Let us define h_{t^1} as

$$h_{t^1}(t^2) := f(t^1, t^2).$$

Hence we can continue the previous calculation as

$$= c_p^p \int\limits_{G_m} \int\limits_{\widetilde{G}_m} \left| \int\limits_{\widetilde{G}_m} h_{t^1}(t^2) \widetilde{D}_k(t^2, y) d\nu(t^2) \right|^p d\nu(y) d\mu(t^1)$$

$$= c_p^p \int\limits_{G_m} \|\widetilde{S}_k h_{t^1}\|_p^p d\mu(t^1) \leq (c_p^p)^2 \int\limits_{G_m} \|h_{t^1}\|_p^p d\mu(t^1)$$

$$= (c_p^p)^2 \int\limits_{G_m} \int\limits_{\widetilde{G}_m} |f(t^1, t^2)|^p d\nu(t^2) d\mu(t^1) = (c_p^2)^p \|f\|_p^p.$$

The proof of the theorem is complete. □

Proof of Theorem 7. Let $P \in \mathcal{P}_{M_n, \widetilde{M}_n}$. Then $P \circ \lambda = P$ for all $\lambda \in \Lambda_{nn}$.

$$\|f - f \circ \lambda\|_p \leq \|f - P\|_p + \|P - P \circ \lambda\|_p + \|P \circ \lambda - f \circ \lambda\|_p,$$

whence $\omega_n(f)_{L^p} \leq 2E_{n,n}(f)_{L^p}$. Since $S_{M_n, \widetilde{M}_n} \in \mathcal{P}_{M_n, M_n}$, we get

$$\frac{1}{2} \omega_n(f)_{L^p} \leq \|S_{M_n, \widetilde{M}_n} f - f\|_p.$$

We will prove that if $P \in \mathcal{P}_{M_N, \widetilde{M}_N}$, $\lambda \in \Lambda_{nn}$ $(n < N)$, then $\|S_{M_n, \widetilde{M}_n} P - P\|_p \leq \omega_n(P)_{L^p}$. For this reason we define the following coordinate-wise bijections:

$$\lambda_i : G_{m_i}^2 \to G_{m_i} \text{ such that } \lambda_i(\cdot, x_i) \text{ and } \lambda_i(y_i, \cdot) \text{ are 1-1 functions}$$

for all $x_i, y_i \in G_{m_i}$ $(i \geq n)$,

$$\widetilde{\lambda}_i : G_{\widetilde{m}_i}^2 \to G_{\widetilde{m}_i} \text{ such that } \widetilde{\lambda}_i(\cdot, x_i) \text{ and } \widetilde{\lambda}_i(y_i, \cdot) \text{ are 1-1 functions}$$

for all $x_i, y_i \in G_{\widetilde{m}_i}$ $(i \geq n)$.

For all $t^1, x \in G_m$ and for all $t^2, y \in G_{\widetilde{m}}$ let

$$\lambda(t^1, x; t^2, y) := \Big(x_0, \ldots, x_{n-1}, \lambda_n(t_n^1, x_n), \ldots; y_0, \ldots, y_{n-1}, \widetilde{\lambda}_n(t_n^2, y_n)\Big).$$

Moreover,

$$S_{M_n, \widetilde{M}_n} P(x, y) = M_n \widetilde{M}_n \int\limits_{I_n(x) \times \widetilde{I}_n(y)} P(t^1, t^2) d\mu \otimes \nu(t^1, t^2)$$

$$= M_n \widetilde{M}_n \int\limits_{I_n(x) \times \widetilde{I}_n(y)} P \circ \lambda(t^1, x; t^2, y) d\mu \otimes \nu(t^1, t^2)$$

$$= \frac{M_n \widetilde{M}_n}{M_N \widetilde{M}_N} \sum_{t_n^1 \in G_{m_n}} \cdots \sum_{t_{N-1}^1 \in G_{m_{N-1}}} \sum_{t_n^2 \in G_{\widetilde{m}_n}} \cdots \sum_{t_{N-1}^2 \in G_{\widetilde{m}_{N-1}}} P(x_0, \ldots, x_{n-1},$$

$$\lambda_n(t_n^1, x_n), \ldots, \lambda_{N-1}(t_{N-1}^1, x_{N-1});$$

$$y_0, \ldots, y_{n-1}, \widetilde{\lambda}_n(t_n^2, y_n), \ldots, \widetilde{\lambda}_{N-1}(t_{N-1}^2, y_{N-1})).$$

Hence

$$\|S_{M_n,\widetilde{M}_n}P - P\|_p^p$$

$$= \int\limits_{G_m \times G_{\widetilde{m}}} \left| M_n\widetilde{M}_n \int\limits_{I_n(x)\times \widetilde{I}_n(y)} P(t^1,t^2) - P(x,y) d\mu\otimes\nu(t^1,t^2)\right|^p d\mu\otimes\nu(x,y)$$

$$= \frac{1}{M_N\widetilde{M}_N}\sum_{x_0\in G_{m_0}}\cdots\sum_{x_{N-1}\in G_{m_{N-1}}}\sum_{y_0\in G_{\widetilde{m}_0}}\cdots\sum_{y_{N-1}\in G_{\widetilde{m}_{N-1}}}$$

$$\left|\frac{M_n\widetilde{M}_n}{M_N\widetilde{M}_N}\sum_{t_n^1\in G_{m_n}}\cdots\sum_{t_{N-1}^1\in G_{m_{N-1}}}\sum_{t_n^2\in G_{\widetilde{m}_n}}\cdots\sum_{t_{N-1}^2\in G_{\widetilde{m}_{N-1}}} P\circ\lambda - P\right|^p$$

$$\le \frac{M_n\widetilde{M}_n}{M_N^2\widetilde{M}_N^2}\sum_{x_0\in G_{m_0}}\cdots\sum_{x_{N-1}\in G_{m_{N-1}}}\sum_{y_0\in G_{\widetilde{m}_0}}\cdots\sum_{y_{N-1}\in G_{\widetilde{m}_{N-1}}}$$

$$\sum_{t_n^1\in G_{m_n}}\cdots\sum_{t_{N-1}^1\in G_{m_{N-1}}}\sum_{t_n^2\in G_{\widetilde{m}_n}}\cdots\sum_{t_{N-1}^2\in G_{\widetilde{m}_{N-1}}} |P\circ\lambda - P|^p$$

$$= \frac{M_n\widetilde{M}_n}{M_N\widetilde{M}_N}\sum_{t_n^1\in G_{m_n}}\cdots\sum_{t_{N-1}^1\in G_{m_{N-1}}}\sum_{t_n^2\in G_{\widetilde{m}_n}}\cdots\sum_{t_{N-1}^2\in G_{\widetilde{m}_{N-1}}}$$

$$\frac{1}{M_N\widetilde{M}_N}\sum_{x_0\in G_{m_0}}\cdots\sum_{x_{N-1}\in G_{m_{N-1}}}\sum_{y_0\in G_{\widetilde{m}_0}}\cdots\sum_{y_{N-1}\in G_{\widetilde{m}_{N-1}}} |P\circ\lambda - P|^p$$

$$= \frac{M_n\widetilde{M}_n}{M_N\widetilde{M}_N}\sum_{t_n^1\in G_{m_n}}\cdots\sum_{t_{N-1}^1\in G_{m_{N-1}}}\sum_{t_n^2\in G_{\widetilde{m}_n}}\cdots\sum_{t_{N-1}^2\in G_{\widetilde{m}_{N-1}}} \|P\circ\lambda(t^1,\cdot;t^2,\cdot) - P\|_p^p.$$

Since $\lambda(t^1,\cdot;t^2,\cdot)\in\Lambda_{nn}$,

$$\|S_{M_n,\widetilde{M}_n}P - P\|_p \le \omega_n(P)_{L^p}.$$

Moreover,

$$\begin{aligned}&\|S_{M_n,\widetilde{M}_n}f - f\|_p\\ &\le \|S_{M_n,\widetilde{M}_n}f - S_{M_n,\widetilde{M}_n}P\|_p + \|S_{M_n,\widetilde{M}_n}P - P\|_p + \|P - f\|_p\\ &\le (c_p^2+1)\|f-P\|_p + \omega_n(P)_{L^p}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\omega_n(P)_{L^p} = \sup_{\lambda\in\Lambda_{nn}}\|P - P\circ\lambda\|^p\\ &\sup_{\lambda\in\Lambda_{nn}}\|P-f\|_p + \sup_{\lambda\in\Lambda_{nn}}\|f - f\circ\lambda\|_p + \sup_{\lambda\in\Lambda_{nn}}\|f\circ\lambda - P\circ\lambda\|_p\\ &= 2\|f-P\|_p + \omega_n(f)_{L^p}.\end{aligned}$$

Continuing the above calculations, we have

$$\begin{aligned}\|S_{M_n,\widetilde{M}_n}f - f\|_p &\le (c_p^2+1)\|f-P\|_p + \omega_n(P)_{L^p}\\ &\le (c_p^2+1)\|f-P\|_p + 2\|f-P\|_p + \omega_n(f)_{L^p}\end{aligned}$$

$$= (c_p^2 + 3)\|f - P\|_p + \omega_n(f)_{L^p}.$$

From this and because $S_{M_n,\widetilde{M}_n} \in \mathcal{P}_{M_n,\widetilde{M}_n}$ we get the wanted relations, since Vilenkin-like polynomials are dense in $L^p(G_m \times G_{\widetilde{m}})$. $\square$

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