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AN AUTOMATICALLY STABLE AND ORDER THREE SPLIT RATIONAL APPROXIMATION OF A SEMIGROUP

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Abstract. An automatically stable and order three split rational approximation scheme is proposed for solving the Cauchy abstract problem. The third order precision is obtained by introducing the complex parameter $\alpha = \frac{1}{2} \pm i\frac{1}{2\sqrt{3}}$ and performing a rational approximation of the semigroup. For the considered scheme, an explicit a priori estimate is derived.

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Introduction

The study of approximate schemes for the solution of evolution problems leads to the conclusion that to each approximate scheme there corresponds a certain operator (solution operator of a discrete problem), which approximates the solution operator (semigroup) of a continuous problem. The inverse statement is also true: constructing the approximation of a continuous semigroup, we obtain an approximate scheme for the solution of an evolution problem. For example, if we apply the semi-discrete method (method of lines, i.e., discretization with respect to a time variable used for the first time by Rothe) for the solution of an evolution problem, then the solution operator of the obtained semi-discrete problem will be a discrete semigroup. Thus we arrive at the problem of approximation of a continuous semigroup by means of discrete semigroups (T. Kato [17], Ch. IX).

If the decomposition method is applied, then the corresponding solution operator generates the Trotter formula [27] or the Chernoff formula (see [3], [4]) or a formula which is a combination of both formulas. Therefore the estimation of a decomposition method error is equivalent to the approximation of a continuous semigroup by Trotter and Chernoff type formulas. The works by T. Ichinose and S. Takanobu [14], T. Ichinose and H. Tamura [15], J. Rogava [22] (see also [23], Ch. II) are dedicated to error estimation of Chernoff and Trotter type formulas.

There are decomposition schemes of two kinds: differential and difference. Trotter type formulas correspond to differential schemes, and Chernoff type formulas to difference ones.

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We call Trotter type formulas formulas giving an approximation of a semigroup by combining the semigroups generated by the summands of its generating operator.

We call Chernoff type formulas formulas that can be obtained from Trotter type formulas if the semigroups are replaced by the corresponding resolvents.

The decomposition scheme associated with the Trotter formula allows us to split the Cauchy problem for an evolution equation with the operator $A = A_1 + A_2$ into two problems corresponding to the operators A_1 and A_2. These problems are successively solved on each time interval of length $\overline{t}/n$, where $\overline{t}$ is a fixed value of the time variable t.

The decomposition scheme associated with the Chernoff formula is known as the method of fractional steps (N. N. Janenko [16]).

The first works devoted to the construction and investigation of decomposition schemes for non-stationary problems were published in the fifties and sixties of the 20-th century (see the bibliography of [20]) by: V. B. Andreev (1967), G. A. Baker (1959/1960), G. A. Baker and T. A. Oliphant (1959/1960), G. Birkhoff and R. S. Varga (1959), G. Birkhoff, R. S. Varga, and D. Young (1962), J. Douglas (1955), J. Douglas and H. Rachford (1956), E. G. D'jakonov (1962), M. Dryja (1967), G. Fairweather, A. R. Gourlay and A. R. Mitchell (1967), I. V. Fryazinov (1969), D. G. Gordeziani (1965) [10], A. R. Gourlay and A. R. Mitchell (1967), N. N. Janenko (1960, 1967), N. N. Janenko and G. V. Demidov (1966), A. N. Konovalov (1962), G. I. Marchuk (1988), G. I. Marchuk and N. N. Janenko (1964), G. I. Marchuk and U. M. Sultangazin (1965), D. Peaceman and H. Rachford (1955), V. P. Il'in (1965), A. A. Samarskii (1962, 1964), R. Temam (1968). The works by these authors formed the basis of subsequent investigations of decomposition schemes.

From the viewpoint of computation, decomposition schemes can be divided into two groups: schemes that are inherently sequential (see, for example, G. I. Marchuk [20]) and schemes that allow for (at least partially) parallel implementation (D. G. Gordeziani, H. V. Meladze [11], [12], D. G. Gordeziani, A. A. Samarskii [13], A. M. Kuzyk, V. L. Makarov [19]). In [23], Ch. II, explicit estimates are obtained for the decomposition schemes considered in [11]. Presently, there are quite a lot of papers where the decomposition method is discussed (see [16], [20], [24] and the references therein).

In the above-listed works the considered schemes have first or second order global precision. The global precision order of a scheme is defined as an error of the approximate solution obtained by the scheme on the whole interval, and the local precision order of a scheme is defined as an error of the approximate solution obtained by the scheme in the proximity of zero (where the initial condition is given). As far as we know, high order precision decomposition formulas were for the first time obtained by M. Schatzman without any commutative assumption in the case of two summands ($A = A_1 + A_2$) (see [25]). Note that the formulas constructed in this paper are not automatically stable. Decomposition formulas are called automatically stable if the sum of magnitudes of its split coefficients is equal to one, and the real parts of coefficients of the exponential power are positive. Q. Sheng proved [26] that in the field of real number there exists no automatically stable decomposition of the semigroup $exp\,(-tA)$ with precision order higher than two. The works [6]–[9] give the construction of a split exponential approximation of global third order. The new idea is the introduction of a complex parameter, which enables us to break the order 2 barrier. The corresponding formulas of the constructed schemes are automatically stable decomposition formulas without any commutative assumption. These results were obtained in [28] and [29] for a homogeneous evolution problem with the operator $A\,(t) = b\,(t)\,(A_1 + A_2)$, where a function $b\,(t) \geq b_0 > 0$ satisfies the Hölder condition. The split exponential approximation is constructed and investigated in [28] and the split rational approximation of a homogeneous evolution problem in [29].

In the present paper, the split rational approximation scheme of global third order is constructed for a nonhomogeneous evolution problem. This scheme can be obtained on the basis of the decomposition formulas constructed in [6] if we replace semigroups by the corresponding rational approximation of third order global precision. For the considered scheme an explicit a priori estimate is obtained. Under an explicit estimate we imply such an a priori estimate of the solution error, where the constants on the right-hand side do not depend on the solution of the initial continuous problem, i.e., are absolute constants.

1. Statement of the Problem and the Main Result

Let us consider the Cauchy abstract problem in the complex Banach space X:

$$\frac{du(t)}{dt} + Au(t) = f(t), \quad t > 0, \quad u(0) = \varphi, \tag{1.1}$$

where $A : X \to X$ is a closed linear operator with the definition domain $D\,[A]$ and everywhere dense in X, φ is a given element from X, $f(t) \in C^1\,([0;\infty)\,;X)$.

Let the operator $(-A)$ be the generator of a strongly continuous semi-group $\{\exp(-tA)\}_{t\geq 0}$, then a solution of problem (1.1) is given by the formula ([18], Theorem 6.5, p. 166)

$$u(t) = U(t, A)\varphi + \int\limits_0^t U(t - s, A)f(s)ds, \tag{1.2}$$

where $U(t, A) = exp\,(-tA)$ is a strongly continuous semigroup.

Let $A = A_1 + A_2$, where A_j $(j = 1, 2)$ are closed linear, densely defined operators in X.

As is well-known, the decomposition method consists in splitting the semigroup $U\,(t, A)$ by means of the semigroups $U\,(t, A_j)$ $(j = 1, 2)$. In [6] (see also [7],[9]), the following decomposition formula with fourth order local precision is constructed:

$$\begin{aligned} T(\tau) = \frac{1}{2}\big[&U\,(\tau, \overline{\alpha}A_1)\,U\,(\tau, A_2)\,U\,(\tau, \alpha A_1) \\ &+ U\,(\tau, \overline{\alpha}A_2)\,U\,(\tau, A_1)\,U\,(\tau, \alpha A_2)\,\big], \end{aligned} \tag{1.3}$$

where $\alpha = \frac{1}{2} + i\frac{1}{2\sqrt{3}}$ $\left(i = \sqrt{-1}\right)$, τ is the step corresponding to the time variable.

In [6]–[9] it is shown that

$$U(\tau, A) - T(\tau) = Op\left(\tau^4\right),$$

where $Op(\tau^4)$ is an operator whose norm is of fourth order with respect to τ; more precisely, in the case of the unbounded operator $\|Op(\tau^4)\varphi\| = O(\tau^4)$, where $\|\cdot\|$ is the norm in X, for any φ from the definition domain of $Op(\tau^4)$. It can be also shown using $CBDH$ (Campbell–Baker–Dynkin–Hausdorff) formulas or the formal Taylor expansions of the semigroups. The detailed proof is given in [6] and [7]. In [8], we constructed the following rational approximation of a semigroup with fourth order local precision:

$$W(\tau, A) = aI + b(I + \lambda\tau A)^{-1} + c(I + \lambda\tau A)^{-2}, \tag{1.4}$$

where $\lambda = \frac{1}{2} + \frac{1}{2\sqrt{3}}$, $a = 1 - \frac{2}{\lambda} + \frac{1}{2\lambda^2}$, $b = \frac{3}{\lambda} - \frac{1}{\lambda^2}$, $c = \frac{1}{2\lambda^2} - \frac{1}{\lambda}$.

In the scalar case the rational approximation defined by formula (1.4) is the Padé approximation (the numerator degree is two and the denominator degree is also two) for an exponential function (see [2]).

Using simple transformation, we can show that the operator $W(\tau, A)$ defined by formula (1.4) coincides with the transition operator of the Calahan scheme (see [30]). The stability of the Calahan scheme for an abstract parabolic equation is investigated in [1].

By (1.3) and (1.4) we can construct the decomposition formula

$$\begin{aligned} V(\tau) = \frac{1}{2}\big[&W(\tau, \overline{\alpha}A_1)\, W(\tau, A_2)\, W(\tau, \alpha A_1) \\ &+ W(\tau, \overline{\alpha}A_2)\, W(\tau, A_1)\, W(\tau, \alpha A_2)\big]. \end{aligned} \tag{1.5}$$

It will be shown below that this formula has fourth order local precision:

$$U(\tau, A) - V(\tau) = Op\left(\tau^4\right).$$

Using formula (1.5), a decomposition scheme with third order global precision will be constructed here for the solution of problem (1.1).

Let us introduce the grid set

$$\overline{\omega}_\tau = \{t_k = k\tau, \quad k = 0, 1, 2, \ldots, \tau > 0\}.$$

According to (1.2), we have

$$u(t_k) = U(\tau, A)u(t_{k-1}) + \int_{t_{k-1}}^{t_k} U(t_k - s, A)f(s)ds. \tag{1.6}$$

To calculate the integral on the right-hand side of this equality we use the quadrature formula of fourth order local precision. Then formula (1.6) can be rewritten as

$$\begin{aligned} u\left(t_k\right) &= U(\tau, A) u\left(t_{k-1}\right) \\ &\qquad + \frac{\tau}{4}\left(3U\left(\tau, \frac{1}{3}A\right) f\left(t_{k-1/3}\right) + U\left(\tau, A\right) f\left(t_{k-1}\right)\right) + R_{k,4}\left(\tau\right), \\ u\left(t_0\right) &= \varphi \quad \left(k = 1, 2, \dots\right), \end{aligned} \tag{1.7}$$

where $R_{k,4}\left(\tau\right)$ is the remainder term of the quadrature formula

$$\begin{aligned} R_{k,4}\left(\tau\right) = \int\limits_{t_{k-1}}^{t_k} & U(t_k - s, A) f(s) ds \\ & - \frac{\tau}{4}\left(3U\left(\tau, \frac{1}{3}A\right) f\left(t_{k-1/3}\right) + U\left(\tau, A\right) f\left(t_{k-1}\right)\right). \end{aligned} \tag{1.8}$$

For a sufficiently smooth function f we have (see Lemma 2.3)

$$\left\| R_{k,4}\left(\tau\right)\right\| = O\left(\tau^4\right).$$

Note that the above quadrature formula has fourth order local precision and contains a minimal number of nodes, which facilitates numerical calculations.

Let us introduce the notation

$$\begin{aligned} K(\tau, A) &= \left(I - \frac{1}{2}\tau A\right)\left(I + \frac{1}{2}\tau A\right)^{-1}, \\ S(\tau) &= K\left(\tau, \frac{1}{2}A_1\right) K\left(\tau, A_2\right) K\left(\tau, \frac{1}{2}A_1\right). \end{aligned}$$

Using this notation and (1.7), we construct the scheme

$$\begin{aligned} u_k &= V(\tau) u_{k-1} + \frac{\tau}{4}\left(3S\left(\frac{1}{3}\tau\right) f\left(t_{k-1/3}\right) + S\left(\tau\right) f\left(t_{k-1}\right)\right), \\ u_0 &= \varphi \quad \left(k = 1, 2, \dots\right). \end{aligned} \tag{1.9}$$

Note that the operator $K\left(\tau, A\right)$ is the transition operator of the Crank–Nicolson scheme.

Calculate $u_{k,0}$ by the scheme

$$\begin{aligned} v_{k-2/3} &= W\left(\tau, \alpha A_1\right) u_{k-1}, & w_{k-2/3} &= W\left(\tau, \alpha A_2\right) u_{k-1}, \\ v_{k-1/3} &= W\left(\tau, A_2\right) v_{k-2/3}, & w_{k-1/3} &= W\left(\tau, A_1\right) w_{k-2/3}, \\ v_k &= W\left(\tau, \overline{\alpha} A_1\right) v_{k-1/3}, & w_k &= W\left(\tau, \overline{\alpha} A_2\right) w_{k-1/3}, \\ u_{k,0} &= \frac{1}{2}[v_k + w_k], & u_0 &= \varphi. \end{aligned} \tag{1.10}$$

Assuming $\gamma_1=\frac{1}{3}$, $\gamma_2=1$, we calculate $u_{k,s}$ $(s=1,2)$ by the scheme

$$\begin{aligned} u_{k-2/3,s} &= K\left(\tau,\frac{1}{2}\gamma_s A_1\right) f\left(t_k-\gamma_s\tau\right), \\ u_{k-1/3,s} &= K\left(\tau,\gamma_s A_2\right) u_{k-2/3,s}, \\ u_{k,s} &= K\left(\tau,\frac{1}{2}\gamma_s A_1\right) u_{k-1/3,s}. \end{aligned}$$

Scheme (1.9) is calculated by the algorithm

$$u_k=u_{k,0}+\frac{\tau}{4}\left(3u_{k,1}+u_{k,2}\right).$$

From the definition of operator power it obviously follows that

$$A^2=\left(A_1+A_2\right)A=A_1A+A_2A.$$

Moreover, in the case of an unbounded operator the following inclusion is true:

$$A_jA=A_j\left(A_1+A_2\right)\supset A_jA_1+A_jA_2 \quad (j=1,2),$$

which implies that

$$A^2\supset\left(A_1^2+A_2^2\right)+\left(A_1A_2+A_2A_1\right).$$

Analogously, we can obtain the operator inclusions

$$\begin{aligned} A^3 &\supset \left(A_1^3+A_2^3\right)+\left(A_1^2A_2+\cdots+A_2^2A_1\right)+\left(A_1A_2A_1+A_2A_1A_2\right), \\ A^4 &\supset \left(A_1^4+A_2^4\right)+\left(A_1^3A_2+\cdots+A_2^3A_1\right) \\ &\quad +\left(A_1^2A_2A_1+\cdots+A_2^2A_1A_2\right)+\left(A_1A_2A_1A_2+A_2A_1A_2A_1\right). \end{aligned}$$

Let us denote the definition domains of the operators on the right-hand side of these inclusions by D_2, D_3 and D_4, respectively. From these operator inclusions it follows that $D_k\subset D\left(A^k\right)$, $k=2,3,4$.

We introduce the notation

$$\begin{aligned} \|\varphi\|_A &= \|A_1\varphi\|+\|A_2\varphi\|, \quad \varphi\in D[A], \\ \|\varphi\|_{A^2} &= \sum_{i,j=1}^{2}\|A_iA_j\varphi\|, \quad \varphi\in D_2\subset D\left[A^2\right], \\ \|\varphi\|_{A^3} &= \sum_{i,j,k=1}^{2}\|A_iA_jA_k\varphi\|, \quad \varphi\in D_3\subset D\left[A^3\right], \\ \|\varphi\|_{A^4} &= \sum_{i,j,k,l=1}^{2}\|A_iA_jA_kA_l\varphi\|, \quad \varphi\in D_4\subset D\left[A^4\right]. \end{aligned}$$

The following statement is true.

Theorem 1.1. *Let the following conditions be satisfied:*

(a) *There exists* $\tau_0 > 0$ *such that for any* $0 < \tau \le \tau_0$ *there exist operators* $(I + \gamma\lambda\tau A_j)^{-1}$, $j = 1, 2$, $\gamma = 1, \alpha, \overline{\alpha}$ *and they are bounded. Moreover, the following inequalities hold:*

$$\|W(\tau, \gamma A_j)\| \le e^{\omega\tau}, \quad \omega = const > 0.$$

(b) *The operator* $(-A)$ *generates the strongly continuous semigroup* $U(t, A) = \exp(-tA)$, *for which the following inequality holds:*

$$\|U(t, A)\| \le M e^{\omega t}, \quad M,\ \omega = const > 0.$$

(c) $U(s, A)\varphi \in D_4$ *for any* $s \ge 0$.

(d) $f(t) \in C^3([0,\infty); X)$, $f(t) \in D_3$, $f'(t) \in D_2$, $f''(t) \in D[A]$ *and* $U(s, A) f(t) \in D_4$ *for any fixed* t *and* s $(t, s \ge 0)$.

Then the following estimate is valid:

$$\begin{aligned}\|u(t_k) - u_k\| \le c e^{\omega_0 t_k} t_k \tau^3 \Big(& \sup_{s\in[0,t_k]} \|U(s, A)\varphi\|_{A^4} + t_k \sup_{s,t\in[0,t_k]} \|U(s, A) f(t)\|_{A^4} \\ & + \sup_{t\in[0,t_k]} \|f(t)\|_{A^3} + \sup_{t\in[0,t_k]} \|f'(t)\|_{A^2} \\ & + \sup_{t\in[0,t_k]} \|f''(t)\|_{A} + \sup_{t\in[0,t_k]} \|f'''(t)\| \Big),\end{aligned} \tag{1.11}$$

where c *and* ω_0 *are positive constants.*

2. Some Lemmas

Let us prove some auxiliary lemmas on which the proof of Theorem 1.1 is based.

Lemma 2.1. *If the condition* (b) *of Theorem* 1.1 *is satisfied, then for the operator* $W(t, A)$ *the expansion*

$$W(t, A) = \sum_{i=0}^{k-1} (-1)^i \frac{t^i}{i!} A^i + R_{W,k}(t, A), \quad k = 1, 2, 3, 4, \tag{2.1}$$

is true, where for the remainder term the following estimate holds:

$$\|R_{W,k}(t, A)\varphi\| \le c_0 e^{\omega_0 t} t^k \left\|A^k \varphi\right\|, \quad \varphi \in D\left[A^k\right], \quad c_0,\ \omega_0 = const > 0. \tag{2.2}$$

Proof. We obviously have

$$\begin{aligned}(I + \gamma A)^{-1} &= I - I + (I + \gamma A)^{-1} = I - (I + \gamma A)^{-1}(I + \gamma A - I) \\ &= I - \gamma A (I + \gamma A)^{-1}.\end{aligned}$$

From this equality, for any natural k, we can obtain the expansion

$$(I + \gamma A)^{-1} = \sum_{i=0}^{k-1} (-1)^i \gamma^i A^i + \gamma^k A^k (I + \gamma A)^{-1}. \tag{2.3}$$

Expanding the rational approximation $W(\tau, A)$ up to first order, by (2.3) we obtain

$$\begin{aligned} W(\tau, A) &= aI + b(I+\lambda\tau A)^{-1} + c(I+\lambda\tau A)^{-2} \\ &= (a+b+c)I + R_{W,1}(\tau, A), \end{aligned} \tag{2.4}$$

where

$$R_{W,1}(\tau, A) = -(b+c)\lambda\tau A(I+\lambda\tau A)^{-1} - c\lambda\tau A(I+\lambda\tau A)^{-2}. \tag{2.5}$$

Since by virtue of the condition (b) of Theorem 1.1 the operator $(-A)$ generates a strongly continuous semigroup by the classical Hille–Phillips–Yosida theorem (see [21], p. 274), the operator $(I+\lambda\tau A)$ is reversible for $\tau < 1/(\omega\lambda)$ and the estimate

$$\left\|(I+\lambda\tau A)^{-1}\right\| = \frac{1}{\lambda\tau}\left\|\left(\frac{1}{\lambda\tau}I + A\right)^{-1}\right\| \le \frac{M}{\lambda\tau}\left(\frac{1}{\lambda\tau} - \omega\right)^{-1} = \frac{M}{1-\omega\lambda\tau} \tag{2.6}$$

is true, where ω and M are constants from the condition (b) of Theorem 1.1. If $0 < \tau \le \tau_0 < 1/(\omega\lambda)$, then (2.6) obviously implies the inequality

$$\left\|(I+\lambda\tau A)^{-1}\right\| \le Me^{\omega_1\tau}, \tag{2.7}$$

where $\omega_1 = \omega\lambda/(1-\omega\lambda\tau_0)$.

By (2.7), from (2.5) we have

$$\|R_{W,1}(\tau, A)\varphi\| \le c_0 e^{\omega_1\tau}\tau\|A\varphi\|, \quad \varphi \in D[A]. \tag{2.8}$$

Substituting the values of the parameters a, b and c into (2.4), we obtain

$$W(\tau, A) = I + R_{W,1}(\tau, A). \tag{2.9}$$

Using (2.3) we expand the rational approximation $W(\tau, A)$ up to second order to obtain

$$W(\tau, A) = (a+b+c)I - (b+2c)\lambda\tau A + R_{W,2}(\tau, A), \tag{2.10}$$

where

$$R_{W,2}(\tau, A) = (b+2c)\lambda^2\tau^2A^2(I+\lambda\tau A)^{-1} + \lambda^2\tau^2(I+\lambda\tau A)^{-2}A^2.$$

According to (2.7), we have

$$\|R_{W,2}(\tau, A)\varphi\| \le c_0 e^{\omega_1\tau}\tau^2\left\|A^2\varphi\right\|, \quad \varphi \in D\left[A^2\right]. \tag{2.11}$$

If we substitute the values of the parameters a, b and c into (2.10), we obtain

$$W(\tau, A) = I - \tau A + R_{W,2}(\tau, A). \tag{2.12}$$

Again using (2.3) we expand the rational approximation $W(\tau, A)$ up to third order to obtain

$$W(\tau, A) = (a+b+c)I - (b+2c)\lambda\tau A + (b+3c)\lambda^2\tau^2A^2 + R_{W,3}(\tau, A), \tag{2.13}$$

where

$$R_{W,3}(\tau, A) = -(b+3c)\lambda^3\tau^3(I+\lambda\tau A)^{-1}A^3 - c\lambda^3\tau^3(I+\lambda\tau A)^{-2}A^3,$$

According to (2.7), we have

$$\|R_{W,3}(\tau,A)\varphi\| \le c_0 e^{\omega_1\tau}\tau^3 \left\|A^3\varphi\right\|, \quad \varphi \in D\left[A^3\right]. \tag{2.14}$$

The substitution of the values of the parameters a, b and c into (2.13) gives

$$W(\tau,A) = I - \tau A + \frac{1}{2}\tau^2 A^2 + R_{W,3}(\tau,A). \tag{2.15}$$

Finally, we use (2.3) to decompose the rational approximation $W(\tau,A)$ up to fourth order:

$$\begin{aligned} W(\tau,A) = {} & (a+b+c)\, I - (b+2c)\,\lambda\tau A + (b+3c)\,\lambda^2\tau^2 A^2 \\ & - (b+4c)\lambda^3\tau^3 A^3 + R_{W,4}(\tau,A), \end{aligned} \tag{2.16}$$

where

$$R_{W,4}(\tau,A) = (b+4c)\,\lambda^4\tau^4\,(I+\lambda\tau A)^{-1} A^4 + c\lambda^4\tau^4\,(I+\lambda\tau A)^{-2} A^4.$$

According to (2.7), we have

$$\|R_{W,4}(\tau,A)\varphi\| \le c_0 e^{\omega_1\tau}\tau^4 \left\|A^4\varphi\right\|, \quad \varphi \in D\left[A^4\right]. \tag{2.17}$$

If we substitute the values of the parameters a, b and c into (2.16), we obtain

$$W(\tau,A) = I - \tau A + \frac{1}{2}\tau^2 A^2 - \frac{1}{6}\tau^3 A^3 + R_{W,4}(\tau,A). \tag{2.18}$$

Combining (2.9), (2.12), (2.15) and (2.18), we obtain formula (2.1), and combining inequalities (2.8), (2.11), (2.14) and (2.17), we obtain estimate (2.2). □

Lemma 2.2. *If the conditions* (a), (b) *and* (c) *of Theorem* 1.1 *are satisfied, then the estimate*

$$\left\|\left[U^k(\tau,A) - V^k(\tau)\right]\varphi\right\| \le c e^{\omega_1 t_k} t_k \tau^3 \sup_{s\in[0,t_k]} \|U(s,A)\varphi\|_{A^4}, \quad k=1,2,\ldots, \tag{2.19}$$

holds, where c and ω_0 are positive constants.

Proof. The following formula is valid (see Kato [17], Ch. IX, p. 603):

$$A\int_{t_0}^{t_1} U(s,A)\,ds = U(t_0,A) - U(t_1,A), \quad 0 \le t_0 \le t_1. \tag{2.20}$$

Thus we obtain the expansion

$$U(t,A) = \sum_{i=0}^{k-1} (-1)^i \frac{t^i}{i!} A^i + R_k(t,A), \tag{2.21}$$

where

$$R_k(t,A) = (-A)^k \int_0^t \int_0^{s_1} \cdots \int_0^{s_{k-1}} U(s,A)\,ds\,ds_{k-1}\cdots ds_1. \tag{2.22}$$

Let us expand all rational approximations in the operator $V(\tau)$ according to formula (2.1) from right to left so that each remainder term be of fourth order. We have

$$V(\tau) = I - \tau A + \frac{1}{2}\tau^2 A^2 - \frac{1}{6}\tau^3 A^3 + R_{V,4}(\tau), \tag{2.23}$$

where

$$R_{V,4}(\tau) = \frac{1}{2}\left[R_{1,2}(\tau) + R_{2,1}(\tau)\right]$$

and

$$\begin{aligned} R_{l,j}(\tau) = {} & R_{W,4}(\tau, \overline{\alpha}A_l) - \tau R_{W,3}(\tau, \overline{\alpha}A_l)A_j + \frac{1}{2}\tau^2 R_{W,2}(\tau, \overline{\alpha}A_l)A_j^2 \\ & - \frac{1}{6}\tau^3 R_{W,1}(\tau, \overline{\alpha}A_l)A_j^3 + W(\tau, \overline{\alpha}A_l)R_{W,4}(\tau, A_j) \\ & - \alpha\tau R_{W,3}(\tau, \overline{\alpha}A_l)A_l + \alpha\tau^2 R_{W,2}(\tau, \overline{\alpha}A_l)A_j A_l \\ & - \frac{1}{2}\alpha\tau^3 R_{W,1}(\tau, \overline{\alpha}A_l)A_j^2 A_l - \alpha\tau W(\tau, \overline{\alpha}A_l)R_{W,3}(\tau, A_j)A_l \\ & + \frac{1}{2}\alpha^2\tau^2 R_{W,2}(\tau, \overline{\alpha}A_l)A_l^2 - \frac{1}{2}\alpha^2\tau^3 R_{W,1}(\tau, \overline{\alpha}A_l)A_j A_l^2 \\ & + \frac{1}{2}\alpha^2\tau^2 W(\tau, \overline{\alpha}A_l)R_{W,2}(\tau, A_j)A_l^2 - \frac{1}{6}\alpha^3\tau^3 R_{W,1}(\tau, \overline{\alpha}A_l)A_l^3 \\ & - \frac{1}{6}\alpha^3\tau^3 W(\tau, \overline{\alpha}A_l)R_{W,1}(\tau, A_j)A_l^3 + W(\tau, \overline{\alpha}A_l)W(\tau, A_j)R_{W,4}(\tau, \alpha A_l), \\ & \qquad l, j = 1, 2. \end{aligned}$$

Hence, according to the condition (a) of Theorem 1.1, we obtain the estimate

$$\|R_{V,4}(\tau)\varphi\| \le ce^{\omega_1\tau}\tau^4 \|\varphi\|_{A^4}, \quad \varphi \in D_4. \tag{2.24}$$

From (2.21) ($k = 4$) and (2.23) it follows that

$$U(\tau, A) - V(\tau) = R_4(\tau, A) - R_{V,4}(\tau).$$

From here, according to (2.22) and (2.24), we obtain the estimate

$$\|[U(\tau, A) - V(\tau)]\varphi\| \le ce^{\omega_1\tau}\tau^4 \|\varphi\|_{A^4}, \quad \varphi \in D_4. \tag{2.25}$$

The following representation is obvious:

$$[U^k(\tau, A) - V^k(\tau)]\varphi = \sum_{i=1}^{k} V^{k-i}(\tau)[U(\tau, A) - V(\tau)]U^{i-1}(\tau, A)\varphi.$$

Hence, according to the conditions (a), (b), (c) of Theorem 1.1 and inequality (2.25), we have

$$\begin{aligned} \|u(t_k) - u_k(t_k)\| &\le \sum_{i=1}^{k} \|V(\tau)\|^{k-i} \|[U(\tau, A) - V(\tau)]U((i-1)\tau, A)\varphi\| \\ &\le c\sum_{i=1}^{k} e^{\omega_2(k-i)\tau} e^{\omega_1\tau}\tau^4 \|U((i-1)\tau, A)\varphi\|_{A^4} \end{aligned}$$

$$\leq ce^{\omega_0 t_k}\tau^4 \sum_{i=1}^{k} \left\| U\left((i-1)\tau, A\right)\varphi\right\|_{A^4}$$
$$\leq ce^{\omega_0 t_k} t_k \tau^3 \sup_{s\in[0,t_k]} \left\| U\left(s, A\right)\varphi\right\|_{A^4}. \qquad \square$$

Lemma 2.3. *Let the following conditions be satisfied:*
(a) *The operator A satisfies the conditions of Theorem* 1.1*;*
(b) $f(t) \in C^3([0,\infty);X)$*, and* $f(t) \in D\left[A^3\right]$ *for every fixed* t*,* $f^{(k)}(t) \in D\left[A^{3-k}\right]$*,* $k = 1, 2$*.*
Then the estimate

$$\left\| \int_0^\tau U(\tau-s,A)f\left(s\right)ds - \frac{\tau}{4}\left[U\left(\tau,A\right)f\left(0\right) + 3U\left(\frac{1}{3}\tau,A\right)f\left(\frac{2}{3}\tau\right)\right]\right\|$$
$$\leq ce^{\omega_0\tau}\tau^4\left[\left\|A^3 f\left(\frac{2}{3}\tau\right)\right\| + \sup_{\xi\in[0,\tau]}\left\|A^2 f'\left(\xi\right)\right\|\right.$$
$$\left. + \sup_{\xi\in[0,\tau]}\left\|Af''\left(\xi\right)\right\| + \sup_{\xi\in[0,\tau]}\left\|f'''\left(\xi\right)\right\|\right] \quad (2.26)$$

holds, where c and ω_0 are positive constants.

Proof. Performing a simple transformation, we obtain the representation

$$\int_0^\tau U(\tau-s,A)f\left(s\right)ds - \frac{\tau}{4}\left[U\left(\tau,A\right)f\left(0\right) + 3U\left(\frac{1}{3}\tau,A\right)f\left(\frac{2}{3}\tau\right)\right]$$
$$= r\left(\tau\right) - U\left(\tau,A\right)z\left(\tau\right) - R\left(\tau,A\right)f\left(\frac{2}{3}\tau\right), \quad (2.27)$$

where

$$z\left(\tau\right) = \frac{1}{4}\int_0^\tau f\left(0\right)ds + \frac{3}{4}\int_0^\tau f\left(\frac{2}{3}\tau\right)ds - \int_0^\tau f\left(s\right)ds,$$
$$R\left(\tau,A\right) = \frac{3}{4}\int_0^\tau U\left(\frac{1}{3}\tau,A\right)ds + \frac{1}{4}\int_0^\tau U\left(\tau,A\right)ds - \int_0^\tau U(\tau-s,A)ds$$

and

$$r\left(\tau\right) = \int_0^\tau \left[U(\tau-s,A) - U\left(\tau,A\right)\right]\left[f\left(s\right) - f\left(\frac{2}{3}\tau\right)\right]ds.$$

According to formula (2.20) we can obtain for $r\left(\tau\right)$ the representation

$$r\left(\tau\right) = \int_0^\tau \left(\int_0^s AU(\tau-\xi,A)d\xi \int_{\frac{2}{3}\tau}^s f'\left(\xi\right)d\xi\right)ds$$

$$
\begin{aligned}
&= \int_0^\tau \left(\int_0^s A\left(U(\tau-\xi,A) - U\left(\tau,A\right)\right) d\xi \int_{\frac{2}{3}\tau}^s f'\left(\xi\right) d\xi \right) ds \\
&\quad + \int_0^\tau \left(\int_0^s AU\left(\tau,A\right) d\xi \int_{\frac{2}{3}\tau}^s f'\left(\xi\right) d\xi \right) ds \\
&= A \int_0^\tau \left[\int_0^s A \int_0^\xi U(\tau-\eta,A) d\eta d\xi \int_{\frac{2}{3}\tau}^s f'\left(\xi\right) d\xi \right] ds \\
&\quad + \int_0^\tau \left(\int_0^s AU\left(\tau,A\right) d\xi \int_{\frac{2}{3}\tau}^s \left(f'\left(\xi\right) - f'\left(0\right)\right) d\xi \right) ds \\
&\quad + \int_0^\tau \left(\int_0^s AU\left(\tau,A\right) d\xi \int_{\frac{2}{3}\tau}^s f'\left(0\right) d\xi \right) ds \\
&= A \int_0^\tau \left[\int_0^s A \int_0^\xi U(\tau-\eta,A) d\eta d\xi \int_{\frac{2}{3}\tau}^s f'\left(\xi\right) d\xi \right] ds \\
&\quad + A \int_0^\tau \left[\int_0^s U\left(\tau,A\right) d\xi \int_{\frac{2}{3}\tau}^s \int_0^\xi f''\left(\eta\right) d\eta d\xi \right] ds.
\end{aligned}
$$

If we take into account that A and $U\left(\tau,A\right)$ are commutative operators, then by the condition (b) of Theorem 1.1 we obtain the estimate

$$
\begin{aligned}
\left\| r\left(\tau\right) \right\| &= \left\| \int_0^\tau \left[\int_0^s \int_0^\xi U(\tau-\eta,A) d\eta d\xi \int_{\frac{2}{3}\tau}^s A^2 f'\left(\xi\right) d\xi \right] ds \right\| \\
&\quad + \left\| \int_0^\tau \left[\int_0^s U\left(\tau,A\right) d\xi \int_{\frac{2}{3}\tau}^s \int_0^\xi Af''\left(\eta\right) d\eta d\xi \right] ds \right\| \\
&\le \int_0^\tau \left[\int_0^s \int_0^\xi d\eta d\xi \int_{\frac{2}{3}\tau}^s d\xi \right] ds \; e^{\omega\tau} \sup_{\xi\in[0,\tau]} \left\| A^2 f'\left(\xi\right) \right\| \\
&\quad + \int_0^\tau \left[\int_0^s d\xi \int_{\frac{2}{3}\tau}^s \int_0^\xi d\eta d\xi \right] ds \; e^{\omega\tau} \sup_{\xi\in[0,\tau]} \left\| Af''\left(\xi\right) \right\|
\end{aligned}
$$

$$\leq ce^{\omega\tau}\tau^4\Big[\sup_{\xi\in[0,\tau]}\left\|A^2 f'(\xi)\right\| + \sup_{\xi\in[0,\tau]}\left\|Af''(\xi)\right\|\Big]. \tag{2.28}$$

For the function $(-z(\tau))$ we have the representation

$$-z(\tau) = \frac{1}{4}\int\limits_0^\tau\int\limits_0^s\int\limits_0^\xi\int\limits_0^\eta f'''(\zeta)d\zeta d\eta d\xi ds + \frac{3}{4}\int\limits_0^\tau\int\limits_{\frac{2}{3}\tau}^s\int\limits_0^\xi\int\limits_0^\eta f'''(\zeta)d\zeta d\eta d\xi ds.$$

Hence we obtain the estimate

$$\|U(\tau,A)\,z(\tau)\| \leq ce^{\omega\tau}\tau^4\sup_{s\in[0,\tau]}\|f'''(s)\|. \tag{2.29}$$

Finally, let us transform the integral $R(\tau,A)$ according to formula (2.20):

$$\begin{aligned}-R(\tau,A) = &-\frac{3}{4}A^3\int\limits_0^\tau\int\limits_{\frac{2}{3}\tau}^s\int\limits_0^\xi\int\limits_0^\eta U(\tau-\zeta,A)d\zeta d\eta d\xi ds\\ &-\frac{1}{4}A^3\int\limits_0^\tau\int\limits_0^s\int\limits_0^\xi\int\limits_0^\eta U(\tau-\zeta,A)d\zeta d\eta d\xi ds.\end{aligned}$$

Hence we obtain the estimate

$$\left\|R(\tau,A)\,f\left(\frac{2}{3}\tau\right)\right\| \leq ce^{\omega\tau}\tau^4\left\|A^3 f\left(\frac{2}{3}\tau\right)\right\|. \tag{2.30}$$

Using inequalities (2.28), (2.29) and (2.30), from equality (2.27) we obtain the desired estimate. □

According to Lemma 2.3, for $R_{k,4}(\tau)$ (see formula (1.8)) the following estimate holds:

$$\begin{aligned}\|R_{k,4}(\tau)\| \leq ce^{\omega_0\tau}\tau^4\Bigg[&\left\|A^3 f\left(\frac{2}{3}\tau\right)\right\| + \sup_{\xi\in[t_{k-1},t_k]}\left\|A^2 f'(\xi)\right\|\\ &+\sup_{\xi\in[t_{k-1},t_k]}\left\|Af''(\xi)\right\| + \sup_{\xi\in[t_{k-1},t_k]}\left\|f'''(\xi)\right\|\Bigg].\end{aligned} \tag{2.31}$$

3. Proof of the Theorem

Let us return to the proof of Theorem 1.1.

From (1.7) we have

$$u(t_k) = U^k(\tau,A)\varphi + \sum_{i=1}^k U^{k-i}(\tau,A)\left(F_i^{(1)} + R_{k,4}(\tau)\right), \tag{3.1}$$

where

$$F_i^{(1)} = \frac{\tau}{4}\left(3U\left(\frac{1}{3}\tau,A\right)f\left(t_{i-1/3}\right) + U(\tau,A)\,f\left(t_{i-1}\right)\right). \tag{3.2}$$

Analogously, let us represent u_k as

$$u_k = V^k(\tau)\varphi + \sum_{i=1}^{k} V^{k-i}(\tau)F_i^{(2)}, \tag{3.3}$$

where

$$F_i^{(2)} = \frac{\tau}{4}\left(3S\left(\frac{1}{3}\tau\right) f\left(t_{i-1/3}\right) + S\left(\tau\right) f\left(t_{i-1}\right)\right). \tag{3.4}$$

Equalities (3.1) and (3.3) imply

$$\begin{aligned} u(t_k) - u_k = {} & \left[U^k(\tau, A) - V^k(\tau)\right]\varphi \\ & + \sum_{i=0}^{k}\left[U^{k-i}(\tau, A)F_i^{(1)} - V^{k-i}(\tau)F_i^{(2)}\right] + \sum_{i=0}^{k} U^{k-i}(\tau, A)R_{k,4}\left(\tau\right) \\ = {} & \left[U^k(\tau, A) - V^k(\tau)\right]\varphi + \sum_{i=1}^{k}\left[\left(U^{k-i}(\tau, A) - V^{k-i}(\tau)\right)F_i^{(1)}\right. \\ & \left. + V^{k-i}(\tau)\left(F_i^{(1)} - F_i^{(2)}\right)\right] + \sum_{i=0}^{k} U^{k-i}(\tau, A)R_{k,4}\left(\tau\right). \end{aligned} \tag{3.5}$$

From formulas (3.2) and (3.4) we have

$$\begin{aligned} F_i^{(1)} - F_i^{(2)} = \frac{\tau}{4}\Bigg(3\left(U\left(\frac{1}{3}\tau, A\right) - S\left(\frac{1}{3}\tau\right)\right) f\left(t_{i-1/3}\right) + \\ + \left(U\left(\tau, A\right) - S\left(\frac{1}{3}\tau\right)\right) f\left(t_{i-1}\right)\Bigg). \end{aligned} \tag{3.6}$$

Next we easily obtain the inequality

$$\left\|\left[U\left(\tau, A\right) - K\left(\tau, A\right)\right]\varphi\right\| \le ce^{\omega_0\tau}\tau^3\left\|\varphi\right\|_{A^3}, \quad \varphi \in D_3.$$

Hence, analogously to estimate (2.25), we obtain

$$\left\|\left[U\left(\tau, A\right) - S\left(\tau\right)\right]\varphi\right\| \le ce^{\omega_0\tau}\tau^3\left\|\varphi\right\|_{A^3}, \quad \varphi \in D_3.$$

According to this inequality, from equality (3.6) we obtain the estimate

$$\left\|F_k^{(1)} - F_k^{(2)}\right\| \le ce^{\omega_0\tau}\tau^4 \sup_{t\in[t_{k-1}, t_k]}\left\|f(t)\right\|_{A^3}. \tag{3.7}$$

By Lemma 2.2 we have

$$\left\|\sum_{i=1}^{k}\left(U^{k-i}(\tau, A) - V^{k-i}(\tau)\right)F_i^{(1)}\right\| \le ce^{\omega_0 t_k}t_k^2\tau^3 \sup_{s,t\in[0,t_k]}\left\|U(s, A)f\left(t\right)\right\|_{A^4}. \tag{3.8}$$

Using inequalities (3.7), (3.8), (2.19), (2.31) and the condition (b) of Theorem 1.1, from equality (3.5) we obtain

$$\|u(t_k) - u_k\| \leq ce^{\omega_0 t_k} t_k \tau^3 \left(\sup_{s\in[0,t_k]} \|U(s,A)\varphi\|_{A^4} + t_k \sup_{s,t\in[0,t_k]} \|U(s,A)f(t)\|_{A^4} + \sup_{t\in[0,t_k]} \|f(t)\|_{A^3} + \sup_{t\in[0,t_k]} \|f'(t)\|_{A^2} + \sup_{t\in[0,t_k]} \|f''(t)\|_A + \sup_{t\in[0,t_k]} \|f'''(t)\| \right) \qquad \square$$

Remark 3.1. Analogously to the discussion of the rational approximation (1.4), we can show that Lemma 2.1 is valid for the rational approximation

$$W_0(\tau, A) = \left(I - \frac{1}{3}\tau A\right)(I + \lambda\tau A)^{-1}\left(I + \overline{\lambda}\tau A\right)^{-1},$$

where $\lambda = \frac{1}{3} + i\frac{1}{3\sqrt{2}} \quad \left(i = \sqrt{-1}\right)$.

Remark 3.2. The operator $V^k(\tau)$ is the solution operator of the above-considered decomposed problem. It is obvious that, according to the condition (a) of Theorem 1.1 and automatical stability of the decomposition formula (1.5) follows the stability of the above-stated decomposition scheme on each finite time interval.

As is known the norm of the operator polynomial when the argument is a self-adjoint bounded operator is equal to the C-norm of the corresponding scalar polynomial (see [21], p. 248). Hence it follows that if A is a self-adjoint non-negative operator, then the relations

$$\|W(\tau, A)\| = \max_{x\in[0,\infty)} \left| a + \frac{b}{1+\lambda\tau x} + \frac{c}{(1+\lambda\tau x)^2} \right| = 1, \tag{3.9}$$

$$\|W_0(\tau, A)\| = \max_{x\in[0,\infty)} \left| \frac{1 - \frac{1}{3}\tau x}{1 + \frac{1}{6}\tau x + \frac{1}{6}\tau^2 x^2} \right| = 1 \tag{3.10}$$

are true. Analogously, if A is a self-adjoint, positive definite operator, then for $W_0(\tau, A)$ the estimate

$$\|W_0(\tau, A)\| = \max_{x\in[\gamma_0,\infty)} \left| \frac{1 - \frac{1}{3}\tau x}{1 + \frac{2}{3}\tau x + \frac{1}{6}\tau^2 x^2} \right| \leq \frac{1}{1+\gamma_1\tau}, \tag{3.11}$$

is true, where $\gamma_0 > 0$, $\gamma_1 = 2/(3\gamma_0)$.

From (3.9) and (3.10) follows

Remark 3.3. In the case of the Hilbert space, when A_1, A_2 and $A_1 + A_2$ are self-adjoint non-negative operators, in case of $W(\tau, A)$ and $W_0(\tau, A)$ rational approximation, in estimate (1.11) ω_0 will be replaced by 0. Alongside with this, for the transition operator of the split problem, the estimate $\left\|V^k(\tau)\right\| \leq 1$ will be true.

(3.10) gives rise to

Remark 3.4. In the case of the Hilbert space, if A_1, A_2 and $A_1 + A_2$ are self-adjoint, positive definite operators and $W_0(\tau, A)$ is a rational approximation, then ω_0 in estimate (1.11) is replaced by $-\alpha_0$, $\alpha_0 > 0$. Moreover, for the transition operator of the split problem, the estimate $\left\|V^k(\tau)\right\| \leq e^{-\alpha_1 t_k}$, $\alpha_1 > 0$ is true.

Remark 3.5. According to the classical Hille–Phillips–Yosida theorem (see [21]), if the operator $(-A)$ generates a strongly continuous semigroup, then the inequality in the condition (b) of Theorem 1.1 is automatically satisfied. The proof of this inequality is based on the uniform boundedness principle, according to which the constants M and ω exist, but generally cannot be explicitly constructed (according to the method of the proof). That is why we demand the inequality in the condition (b) of Theorem 1.1 be satisfied.

4. Conclusion

When the operators A_1, A_2 are matrices, it is obvious that the conditions (a) and (b) of Theorem 1.1 are automatically satisfied. The conditions (a) and (b) of Theorem 1.1 are also satisfied if A_1, A_2 and A are self-adjoint, positive definite operators. Moreover, the conditions (a) and (b) of Theorem 1.1 are automatically satisfied if the operators A_1, A_2 and A are normal operators. However, in that case, certain restrictions are imposed on the spectra of these operators: the spectrum of the operator A has to be included in the right half-plane and the spectra of the operators A_1 and A_2 have to be included in the sector with an angle of $120°$ so that the spectra of the operators A_1 and A_2 would remain in the right half-plane after rotating by $\pm 30°$ (this is due to multiplication of the operators A_1 and A_2 by the parameters α and $\overline{\alpha}$).

The third order precision is reached by introducing a complex parameter. For this, unlike lower order accuracy schemes, each equation of the considered decomposed system is replaced by a pair of real equations. To solve a specific problem, the matrix factorization can be used, where the coefficients are matrices of second order, while in lower order accuracy schemes the common factorization can be used.

It must be noted that, unlike high order precision decomposition schemes considered in [25], the sum of magnitudes of coefficients of the summands of the transition operator $V(\tau)$ is equal to 1. Hence the scheme considered here is stable for any bounded operators A_1, A_2.

5. A Numerical Example

We performed calculations for the test problem

$$\frac{\partial u(t,x,y)}{\partial t} - a(x,y)\frac{\partial^2 u(t,x,y)}{\partial x^2} - b(x,y)\frac{\partial^2 u(t,x,y)}{\partial y^2} = f(t,x,y), \quad t > 0,$$

$$(x,y) \in \,]0;1[\, \times \,]0;1[\,,$$

$$u(0,x,y) = \varphi(x,y), \quad u(t,x,0) = u(t,x,1) = 0, \quad u(t,0,y) = u(t,1,y) = 0,$$

where

$$f(t,x,y) = e^{\pi t}\pi\left(m + \pi\left(a(x,y) + b(x,y)\right)\right)\sin(\pi x)\sin(\pi y);$$
$$\varphi(x,y) = 0;$$
$$a(x,y) = 2 + \sin(\pi x)\sin(\pi y);$$
$$b(x,y) = 2 + 0.5\sin(\pi x)\sin(\pi y).$$

The solution of the problem is $u(t,x,y) = e^{m\pi t}\sin(\pi x)\sin(\pi y)$.

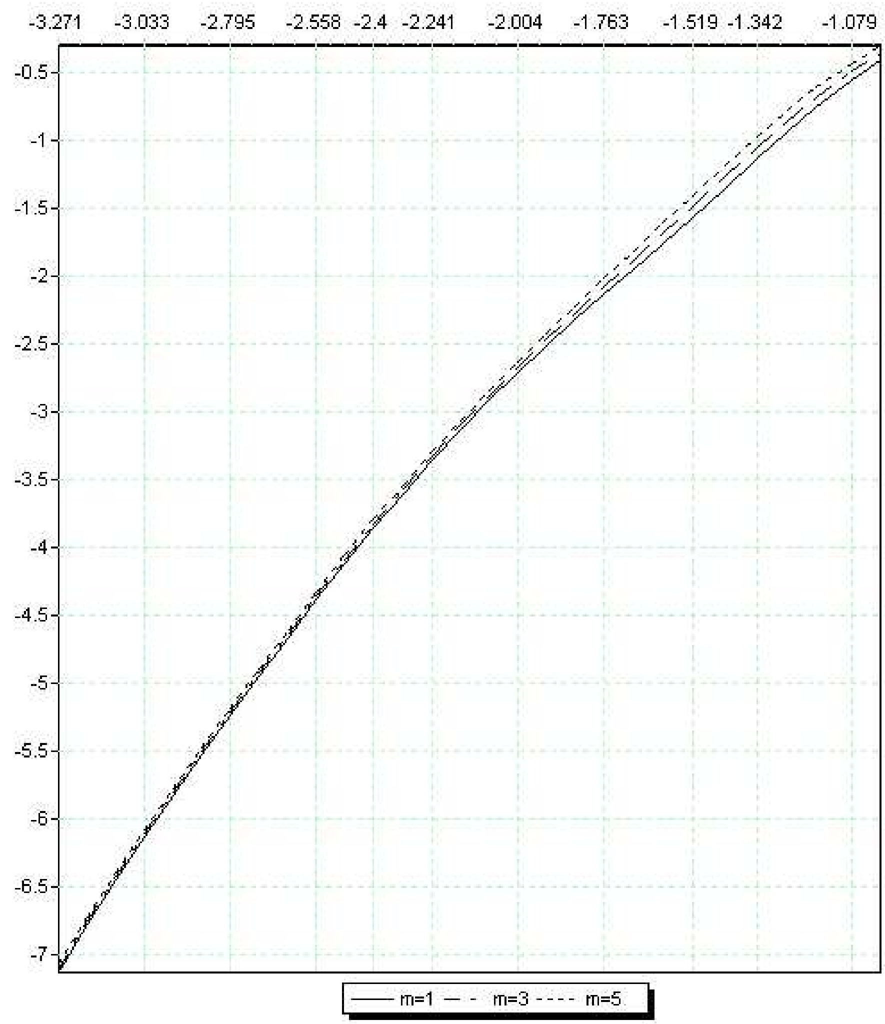

Fig. 1. Dependence of the relative error on the time step

Fig. 1 shows the dependence of a relative error of the approximate solution on the time step logarithm (the time step logarithm is on the horizontal axis and the relative error of the approximate solution is on the vertical axis). Fig. 2 gives the dependence of the absolute error of the approximate solution on the time step logarithm (the time step logarithm is on the horizontal axis and the absolute error of the approximate solution is on the vertical axis). In both figures the calculations are carried out for the following values of the time step: $\tau_k = 1/N_k$, $N_k = \left[10 * 1.2^k\right]$, $k = 0, 1, \ldots, 30$, and the spatial step is assumed to be constant $h_x = h_y = 0.001$. Both figures deal with three cases: $m = 1$, $m = 3$ and $m = 5$. Our aim was to find the convergence rate of the method by means of a numerical experiment. If the method is of third order, then, starting from some value of τ, the graph of the function (solution error logarithm) should approach the straight line, the tangent of which equals three. Both figures clearly show that starting from $\tau = 0.01$ ($Log(\tau) = -2$), the graph approaches the straight line, the tangent of which equals three with sufficient accuracy, which confirms the theoretical result proved in the paper.

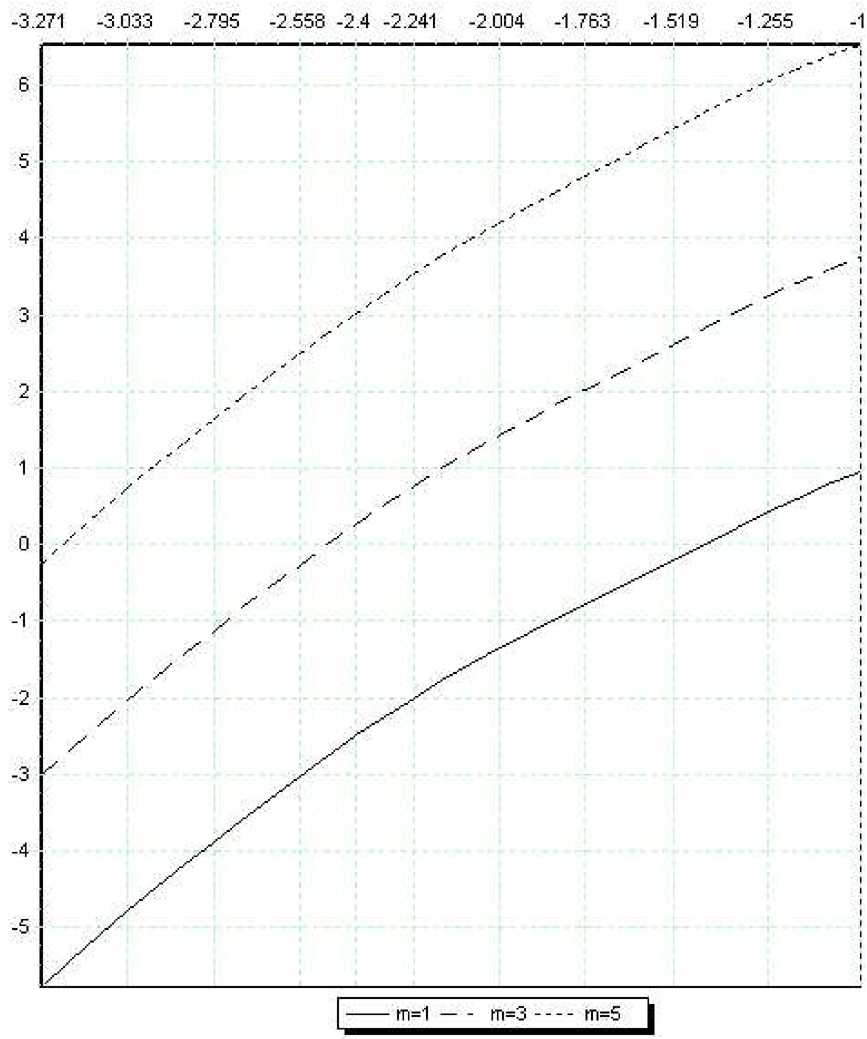

Fig. 2. Dependence of the absolute error on the time step

Note that we used the classical difference formulas for approximation of second derivatives by spatial variables. It is obvious that $u_1, u_2, \dots, u_k$ are complex functions, but their imaginary parts are $O(\tau^3)$.

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References

1. H. A. Alibekov and P. E. Sobolevskiĭ, The stability of difference schemes for parabolic equations. (Russian) *Dokl. Akad. Nauk SSSR* **232**(1977), No. 4, 737–740.
2. G. A. Baker, Jr., and P. Graves-Morris, Pade approximants. (Translated from English into Russian) *Mir, Moscow,* 1986.
3. P. R. Chernoff, Note on product formulas for operator semigroups. *J. Functional Analysis* **2**(1968), 238–242.
4. P. R. Chernoff, Semigroup product formulas and addition of unbounded operators. *Bull. Amer. Math. Soc.* **76**(1970), 395–398.
5. N. Dunford and J. T. Schwartz, Linear operators. Part I: General theory. (Russian) *Izdat. Inostran. Lit., Moscow,* 1962.
6. Z. Gegechkori, J. Rogava, and M. Tsiklauri, Sequential-parallel method of high degree precision of Cauchy abstract problem solution. *Rep. Enlarged Sess. Semin. I. Vekua Inst. Appl. Math.* **14**(1999), No. 3, 45–48.
7. Z. Gegechkori, J. Rogava, and M. Tsiklauri, High-degree precision decomposition method for an evolution problem. *Comput. Methods Appl. Math.* **1**(2001), No. 2, 173–187.
8. Z. Gegechkori, J. Rogava, and M. Tsiklauri, High degree precision decomposition formulas of semigroup approximation. *Rep. Enlarged Sess. Semin. I. Vekua Inst. Appl. Math.* **16**(2001), No. 1-3, 89–92.
9. Z. Gegechkori, J. Rogava, and M. Tsiklauri, High degree precision decomposition method for the evolution problem with an operator under a split form. *M2AN Math. Model. Numer. Anal.* **36**(2002), No. 4, 693–704.
10. D. G. Gordeziani, Application of a locally one-dimensional method to the solution of multi-dimensional parabolic equations of the order $2m$. (Russian) *Soobshch. Akad. Nauk Gruzin. SSR* **39**(1965), 535–541.
11. D. G. Gordeziani and G. V. Meladze, The modelling of multidimensional quasilinear equations of parabolic type by one-dimensional equations. (Russian) *Soobshch. Akad. Nauk Gruzin. SSR* **60**(1970), 537–540.
12. D. G. Gordeziani and G. V. Meladze, The simulation of the third boundary value problem for multidimensional parabolic equations in an arbitrary domain by one-dimensional equations. (Russian) *Ž. Vyčisl. Mat. i Mat. Fiz.* **14**(1974), 246–250.
13. D. G. Gordeziani and A. A. Samarskiĭ, Some problems of the thermoelasticity of plates and shells, and the method of summary approximation. (Russian) *Complex analysis and its applications* (*Russian*), 173–186, 666, *Nauka, Moscow,* 1978.
14. T. Ichinose and S. Takanobu, The norm estimate of the difference between the Kac operator and the Schrdinger semigroup: a unified approach to the nonrelativistic and relativistic cases. *Nagoya Math. J.* **149**(1998), 53–81.
15. T. Ichinose and H. Tamura, The norm convergence of the Trotter-Kato product formula with error bound. *Comm. Math. Phys.* **217**(2001), No. 3, 489–502.

16. N. N. Janenko, The method of fractional steps for solving multidimensional problems of mathematical physics. (Russian) *Academy of Sciences of the USSR. Siberian Branch. Computing Center Izdat. Nauka, Sibirsk. Otdel., Novosibirsk,* 1967.
17. T. Kato, Perturbation theory for linear operators. Reprint of the 1980 edition. *Classics in Mathematics. Springer-Verlag, Berlin,* 1995; Russian transl.: *Mir, Moscow,* 1972.
18. S. G. Krein, Linear differential equations in a Banach space. (Russian) *Nauka, Moscow,* 1967.
19. A. M. Kuzyk and V. L. Makarov, Estimation of the accuracy of the method of summary approximation of the solution of an abstract Cauchy problem. (Russian) *Dokl. Akad. Nauk SSSR* **275**(1984), No. 2, 297–301.
20. G. I. Marchuk, Splitting methods. (Russian) *Nauka, Moscow,* 1988.
21. M. Reed and B. Simon, Methods of modern mathematical physics. II. Fourier analysis, self-adjointness. *Academic Press* [*Harcourt Brace Jovanovich, Publishers*], *New York–London,* 1975.
22. J. L. Rogava, On the error of Trotter-type formulas in the case of selfadjoint operators. (Russian) *Funktsional. Anal. i Prilozhen.* **27**(1993), No. 3, 84–86; English transl.: *Funct. Anal. Appl.* **27**(1993), No. 3, 217–219
23. J. L. Rogava, Semidiscrete schemes for operator-differential equations. (Russian) *Izdatel'stvo Tekhnicheskogo Universiteta, Tbilisi,* 1995.
24. A. A. Samarskiĭ and P. N. Vabishchevich, Additive schemes for problems in mathematical physics. (Russian) *Nauka, Moscow,* 1999.
25. M. Schatzman, Higher order alternate directions methods. *ICOSAHOM*'92 (*Montpellier,* 1992). Comput. Methods Appl. Mech. Engrg. **116**(1994), No. 1-4, 219–225.
26. Q. Sheng, Solving linear partial differential equations by exponential splitting. *IMA J. Numer. Anal.* **9**(1989), No. 2, 199–212.
27. H. F. Trotter, On the product of semi-groups of operators. *Proc. Amer. Math. Soc.* **10**(1959), 545–551.
28. M. A. Tsiklauri, High order of accuracy decomposition method for an evolution problem with variable operator. *Proc. I. Vekua Inst. Appl. Math.* **50(51)**(2000/01), 103–117.
29. M. A. Tsiklauri, High order of accuracy decomposition of the evolution problem with variable operator using resolvent polynoms. *Proc. I. Vekua Inst. Appl. Math.* **50(51)**(2000/01), 118–128.
30. M. Zlámal, Finite element methods for parabolic equations. *Math. Comp.* **28**(1974), 393–404.

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