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OSCILLATION THEOREMS FOR CERTAIN EVEN ORDER DELAY DIFFERENTIAL EQUATIONS INVOLVING GENERAL MEANS

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Abstract. By using the general means, we establish some oscillation theorems for the even order delay differential equation

$$(r(t)|x^{(n-1)}(t)|^{\alpha-1}x^{(n-1)}(t))' + F(t, x[g(t)]) = 0,$$

where $\alpha > 0$ is a constant, $r \in C^1([t_0,\infty), \mathbb{R}_+)$, $F \in C([t_0,\infty)\times\mathbb{R},\mathbb{R})$, and $g \in C([t_0,\infty),\mathbb{R})$. The results obtained extend and improve some results known in the literature.

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1. Introduction

In this paper, we study the oscillatory behavior of solutions of the even order delay differential equation

$$(r(t)|x^{(n-1)}(t)|^{\alpha-1}x^{(n-1)}(t))' + F(t, x[g(t)]) = 0, \quad n \text{ is even.} \tag{1.1}$$

Throughout this paper, we assume that the following conditions hold:

$(A1)$ $\alpha > 0$ is a constant;

$(A2)$ $g \in C([t_0,\infty),\mathbb{R})$, $\lim\limits_{t\to\infty} g(t) = \infty$;

$(A3)$ for $F \in C([t_0,\infty)\times\mathbb{R},\mathbb{R})$, there exist functions $q \in C([t_0,\infty),\mathbb{R}_0)$, where $q(t)$ is not identically zero for all large t, $\sigma \in C^1([t_0,\infty),\mathbb{R}_+)$, and a constant $\beta > 0$ such that

$$F(t,x)\operatorname{sign} x \ge q(t)|x|^\beta \quad \text{for} \quad x \ne 0 \text{ and } t \ge t_0,$$

and

$$\sigma(t) \le \min\{t, g(t)\}, \quad \sigma'(t) > 0 \quad \text{for } t \ge t_0, \quad \text{and} \quad \lim_{t\to\infty}\sigma(t) = \infty,$$

where $\mathbb{R}_0 = [0,\infty)$, $\mathbb{R}_+ = (0,\infty)$;

$(A4)$ $r \in C^1([t_0,\infty),\mathbb{R}_+)$, $\lim\limits_{t\to\infty}\int\limits_{t_0}^{t} r^{-1/\alpha}(s)ds = \infty$, $\liminf\limits_{t\to\infty} r(t) = c > 0$. For any $\varepsilon > 0$, there exists a $t_\varepsilon > t_0$ such that $|r'(t)| \le \varepsilon q(t)$ for all $t \ge t_\varepsilon$.

By a solution of equation (1.1), we mean a function $x \in C^{n-1}([T_x,\infty),\mathbb{R})$ for some $T_x \ge t_0$ which has the property that $r(t)|x^{(n-1)}(t)|^{\alpha-1}x^{(n-1)}(t) \in C^1([T_x,\infty),\mathbb{R})$ and satisfies equation (1.1) on $[T_x,\infty)$. A nontrivial solution

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of equation (1.1) is said to be oscillatory if it has arbitrarily large zeros; otherwise it is said to be nonoscillatory. Equation (1.1) is oscillatory if all of its solutions are oscillatory.

The problem of obtaining sufficient conditions to ensure that all solutions of certain classes of n-th order delay differential equations are oscillatory has been studied by many researchers. Some of these results have been obtained for an equation of the form

$$(|x^{(n-1)}(t)|^{\alpha-1}x^{(n-1)}(t))' + F(t, x[g(t)]) = 0, \quad n \text{ is even}, \tag{1.2}$$

where the function F satisfies $(A3)$ with $\alpha = \beta$. For typical results for equation (1.2), we refer the reader to the papers [1], [13]. Recently, in [2], Agarwal and Grace have presented the oscillation criteria for the equation

$$(x^{(n-1)}(t))^{\alpha})' + q(t)x^{\beta}[g(t)] = 0, \quad n \text{ is even}, \tag{1.3}$$

where α and β are the ratio of positive odd integers, $q \in C([t_0, \infty), \mathbb{R}_0)$, $g \in C^1([t_0, \infty), \mathbb{R})$ and $\lim_{t\to\infty} g(t) = \infty$, $g'(t) \geq 0$ for $t \geq t_0$. The obtained theorems extend and improve some well-known oscillation results reported in the literature. For a general interest in the oscillation of high order differential equation, see, for example, [1]–[4], [7], [8], [10], [12], [13] and the references therein.

Motivated by the idea of [1], [9], [11], [13], in this paper, we establish some oscillation theorems for equation (1.1). In fact, by using general means [9], [11], we extend the results of [6], [9], [11], [14] to the general equation (1.1), which improves the main results in [1], [13]. We believe that our approach is simple and also provides a more unified tool for the study of Kamenev-type oscillation theorems. To show the importance of our results, two interesting examples are included.

2. Main Results

First of all, we introduce the general means [9], [11] and present some properties which will be used in the proof of our main results.

Let $D = \{(t, s) : t \geq s \geq t_0\}$ and $D_0 = \{(t, s) : t > s \geq t_0\}$. We say that a function $H \in C(D, \mathbb{R})$ belongs to a function class $\Im$, written as $H \in \Im$, if

$(H1)$ $H(t, t) = 0$ for $t \geq t_0$, $H(t, s) > 0$ on D_0;

$(H2)$ H has a continuous and nonpositive partial derivative in D_0 with respect to the second variable;

$(H3)$ There exist functions $\rho \in C^1([t_0, \infty), \mathbb{R}_+)$ and $h \in C(D, \mathbb{R})$ such that

$$\frac{\partial}{\partial s}[H(t, s)\rho(s)] = -H(t, s)h(t, s), \quad \text{(t,s)} \in D_0.$$

Let $\rho \in C^1([t_0,\infty),\mathbb{R}_+)$ and $H \in \Im$, we take the integral operator A, which is defined in [11], in terms of $H(t,s)$ and $\rho(s)$ as

$$A_T(\phi;t) := \int_T^t H(t,s)\phi(s)\rho(s)ds \quad \text{for } t \geq T \geq t_0, \tag{2.1}$$

where $\phi \in C([t_0,\infty),\mathbb{R})$. It is easily seen that the integral operator A satisfies the following properties:

$$A_T(\alpha_1 h_1 + \alpha_2 h_2;t) = \alpha_1 A_T(h_1,t) + \alpha_2 A_T(h_2;t); \tag{2.2}$$

$$A_T(h_3;t) \geq 0 \quad \text{whenever} \quad h_3 \geq 0; \tag{2.3}$$

$$A_T(h_4';t) = -H(t,T)h_4(T)\rho(T) + A_T(\rho^{-1} h_4 h;t). \tag{2.4}$$

Here $h_1, h_2, h_3 \in C([t_0,\infty),\mathbb{R})$, $h_4 \in C^1([t_0,\infty),\mathbb{R})$, and $\alpha_1,\ \alpha_2 \in \mathbb{R}$.

The following two lemmas will be needed in proving our results. The first is the well-known Kiguradze's Lemma [7]. The second can be easily obtained by Kiguradze and Koplatadze's lemmas (see [8], Ch. 1).

Lemma 2.1 ([7]). *Let $u \in C^n([t_0,\infty),\mathbb{R}_+)$. If $u^{(n)}(t)$ is of constant sign and not identically zero on any interval of the form $[\,t^*,\infty\,)$, then there exist a $t_u \geq t_0$ and an integer l, $0 \leq l \leq n$, with $n+l$ even for $u^{(n)}(t) \geq 0$, or $n+l$ odd for $u^{(n)}(t) \leq 0$ such that*

$$l > 0 \quad \textit{implies that} \quad u^{(k)}(t) > 0 \quad \textit{for} \quad t \geq t_u, \quad k = 0,1,\cdots,l-1,$$

and

$$l \leq n-1 \quad \textit{implies that} \quad (-1)^{l+k}u^{(k)}(t) > 0 \quad \textit{for} \quad t \geq t_4, \quad k = l, l+1, \ldots, n-1.$$

Lemma 2.2 ([8]). *If the function $u(t)$ is as in Lemma 2.1 and $u^{(n-1)}(t) \times u^{(n)}(t) \leq 0$ for any $t \geq t_u$, then*

$$u(\frac{1}{2}t) \geq \frac{2^{1-n}}{(n-1)!}t^{n-1}|u^{(n-1)}(t)| \quad \textit{for all large} \quad t.$$

We are now able to state and prove the main results.

Theorem 2.1. *Suppose that there exist functions $\rho \in C^1([t_0,\infty),\mathbb{R}_+)$, $H, h \in C(D,\mathbb{R})$ with $H \in \Im$, and for any $k_1,\ k_2 > 0$ such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} A_{t_0}\left(q - \theta\kappa^{-\alpha}\rho^{-(\alpha+1)}|h|^{\alpha+1};t\right) = \infty, \tag{2.5}$$

where

$$\theta = (\alpha+1)^{-(\alpha+1)},$$

and

$$\kappa(t) = \begin{cases} \dfrac{\beta}{\alpha}\dfrac{2^{1-n}}{(n-2)!}\, k_1^{(\beta/\alpha)-1} r^{-1/\alpha}(t)\sigma^{n-2}(t)\sigma'(t), & \text{when } \beta > \alpha; \\ \dfrac{\beta}{\alpha}\dfrac{2^{1-n}}{(n-2)!}\, r^{-1/\alpha}(t)\sigma^{n-2}(t)\sigma'(t), & \text{when } \beta = \alpha; \\ \dfrac{\beta}{\alpha}\dfrac{2^{1-n}}{(n-2)!}\, k_2^{(\beta/\alpha)-1} r^{-1/\alpha}(t)\sigma^{(n-1)(\beta/\alpha)-1}(t)\sigma'(t), & \text{when } \beta < \alpha. \end{cases}$$

Then equation (1.1) *is oscillatory.*

Proof. Suppose to the contrary that equation (1.1) has a nonoscillatory solution $x(t)$. Without loss of generality, we may assume that $x(t) > 0$ for $t \geq t_1 \geq t_0 \geq 0$. Since

$$(r(t)|x^{(n-1)}(t)|^{\alpha-1}x^{(n-1)}(t))' = -F(t, x[g(t)]) \leq 0,$$

the function $r(t)|x^{(n-1)}(t)|^{\alpha-1}x^{(n-1)}(t)$ is decreasing and $x^{(n-1)}(t)$ is eventually of one sign. If $x^{(n-1)}(t) < 0$ eventually, then there exists a constant $\delta > 0$ such that

$$-r(t)(-x^{(n-1)}(t))^{\alpha} \leq -\delta^{\alpha} < 0.$$

Integrating the above inequality from t_1 to t, we get

$$x^{(n-2)}(t) \leq x^{(n-2)}(t_1) - \delta \int_{t_1}^{t} \left(\frac{1}{r(s)}\right)^{1/\alpha} ds.$$

By $(A4)$ we find that $x^{(n-2)}(t) < 0$ eventually. But then Lemma 2.1 (note that n is even) implies that $x(t) < 0$ eventually, which is a contradiction. So $x^{(n-1)}(t) > 0$ eventually, then again from Lemma 2.1 we have $x'(t) > 0$ eventually. Thus there exists a $t_2 \geq t_1$ such that

$$x'(t) > 0 \quad \text{and} \quad x^{(n-1)}(t) > 0 \quad \text{for } t \geq t_2. \tag{2.6}$$

Observing that the function $r(t)(x^{n-1)}(t))^{\alpha}$ is decreasing for $t \geq t_2$, by $(A4)$, there exists a $t_3 \geq t_2$ such that

$$(x^{(n-1)}(t))^{\alpha} \leq \frac{r(t_3)}{r(t)}(x^{(n-1)}(t_3))^{\alpha} \leq \frac{r(t_3)}{c}(x^{(n-1)}(t_3))^{\alpha} \quad \text{for } t \geq t_3. \tag{2.7}$$

Equation (1.1) implies that

$$\begin{aligned} \big(r(t)(x^{(n-1)}(t)\big)^{\alpha})' &= r'(t)(x^{(n-1)}(t))^{\alpha} + \alpha r(t)(x^{(n-1)}(t))^{\alpha-1}x^{(n)}(t) \\ &= -F(t, x(g(t)) \leq -q(t)x^{\beta}[\sigma(t)]. \end{aligned} \tag{2.8}$$

Now, in view of $(A4)$, let $\varepsilon = \dfrac{c}{2r(t_3)}\dfrac{x^{\beta}[\sigma(t_3)]}{(x^{(n-1)}(t_3))^{\alpha}}$, then there exists a $t_4 \geq t_3$ such that, taking into account (2.7) and (2.8), for $t \geq t_4$,

$$\alpha r(t)(x^{(n-1)}(t))^{\alpha-1}x^{(n)}(t) \leq |r'(t)|(x^{(n-1)}(t))^{\alpha} - q(t)x^{\beta}[\sigma(t)]$$

$$\leq \varepsilon\, q(t)\, \frac{r(t_3)}{c} (x^{(n-1)}(t_3))^{\alpha} - q(t) x^{\beta}[\sigma(t_3)]$$
$$\leq q(t) \left(\frac{1}{2} x^{\beta}[\sigma(t_3)] - x^{\beta}[\sigma(t_3)] \right)$$
$$= -\frac{1}{2}\, q(t) x^{\beta}[\sigma(t_3)] \leq 0.$$

Thus $x^{(n)}(t) \leq 0$ for $t \geq t_4$. It is easy to check that we can apply Lemma 2.2 for $x' = u$ and conclude that there exists a $t_5 \geq t_4$ such that

$$x' \left[\frac{\sigma(t)}{2} \right] \geq \frac{2^{2-n}}{(n-2)!} \sigma^{n-2}(t) x^{(n-1)}(t) \quad \text{for } t \geq t_5, \tag{2.9}$$

since $x^{(n-1)}[\sigma(t)] \geq x^{(n-1)}(t)$ for $t \geq t_5$. Put

$$W(t) = \frac{r(t)(x^{(n-1)}(t))^{\alpha}}{x^{\beta}[\sigma(t)/2]}.$$

Taking into account (1.1) and (2.9), for $t \geq t_5$, we have

$$W'(t) \leq -q(t) - \frac{\beta\, 2^{1-n}}{(n-2)!}\, r^{-1/\alpha}(t) \sigma^{n-2}(t) \sigma'(t) x^{(\beta/\alpha)-1} \left[\frac{\sigma(t)}{2} \right] W^{1+1/\alpha}(t). \tag{2.10}$$

Next we consider (2.10) in the following three cases.

Case 1. $\beta > \alpha$. In view of $x'(t) > 0$, for $t \geq t_5$ there exist constants $k_1 > 0$ and $T_1 \geq t_5$ such that

$$x \left[\frac{\sigma(t)}{2} \right] \geq k_1 \quad \text{for } t \geq T_1.$$

Thus (2.10) takes the following form

$$\begin{aligned} W'(t) &\leq -q(t) - \frac{\beta\, 2^{1-n}}{(n-2)!}\, k_1^{(\beta/\alpha)-1} r^{-1/\alpha}(t) \sigma^{n-2}(t) \sigma'(t) W^{1+1/\alpha}(t) \\ &= -q(t) - \alpha \kappa(t) W^{1+1/\alpha}(t). \end{aligned} \tag{2.11}$$

Applying the operator A_T, $t > T \geq T_0$, to (2.11), and using (2.4), we have

$$A_T(q;t) \leq H(t,T)\rho(T)W(T) + A_T(\rho^{-1}|h|W;t) - \alpha A_T(\kappa\, W^{1+1/\alpha};t). \tag{2.12}$$

The Young inequality [5, Theorem 61] gives

$$\rho^{-1}|h|W \leq \alpha\kappa\, W^{1+1/\alpha} + \theta\, \kappa^{-\alpha} \rho^{-(\alpha+1)} |h|^{\alpha+1}.$$

Substituting the above inequality into (2.12), we get

$$A_T(q;t) \leq H(t,T)\rho(T)W(T) + \theta A_T(\kappa^{-\alpha}\rho^{-(\alpha+1)}|h|^{\alpha+1};t). \tag{2.13}$$

Furthermore, we may rewrite (2.13) as follows:

$$A_{T_1}(q - \theta\, \kappa^{-\alpha}\rho^{-(\alpha+1)}|h|^{\alpha+1};t) \leq H(t,T_1)\rho(T_1)W(T_1) \leq H(t,t_0)\rho(T_1)|W(T_1)|.$$

Now it is easy to see that for all $t \geq T_1$

$$\begin{aligned} &A_{t_0}(q - \theta\, \kappa^{-\alpha}\rho^{-(\alpha+1)}|h|^{\alpha+1};t) \\ &\quad = A_{t_0}(q - \theta\, \kappa^{-\alpha}\rho^{-(\alpha+1)}|h|^{\alpha+1};T_1) + A_{T_1}(q - \theta\, \kappa^{-\alpha}\rho^{-(\alpha+1)}|h|^{\alpha+1};t) \end{aligned}$$

$$\leq H(t,t_0)\left[\int_{t_0}^{T_1} q(s)\rho(s)ds + \rho(T_1)|W(T_1)|\right].$$

This gives

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}A_{t_0}(q-\theta\,\kappa^{-\alpha}\rho^{-(\alpha+1)}|h|^{\alpha+1};t) \leq \int_{t_0}^{T_1} q(s)\rho(s)ds + \rho(T_1)W(T_1).$$

which contradicts (2.5).

Case 2. $\beta = \alpha$. In this case, inequality (2.10) takes the form

$$\begin{aligned} W'(t) &\leq -q(t) - \frac{\beta\, 2^{1-n}}{(n-2)!}\, r^{-1/\alpha}(t)\sigma^{n-2}(t)\sigma'(t)W^{1+1/\alpha}(t) \\ &= -q(t) - \alpha\kappa(t)W^{1+1/\alpha}(t). \end{aligned} \tag{2.14}$$

Once again, we can complete the proof by proceeding as in the proof of case 1.

Case 3. $\beta < \alpha$. Note that $x^{(n)}(t) \leq 0$ for $t \geq t_4$, then there exist a $t_6 \geq t_4$ and a constant $b > 0$ such that

$$x^{(n-1)}(t) \leq b \quad \text{for } t \geq t_6.$$

Integrating the above inequality $(n-1)$ times, there exist a $T_2 \geq t_6$ and a positive constant $k_2 > 0$ such that

$$x\left[\frac{\sigma(t)}{2}\right] \leq k_2\sigma^{n-1}(t) \quad \text{for } t \geq T_2.$$

Thus inequality (2.10) takes the form

$$\begin{aligned} W'(t) &\leq -q(t) - \frac{\beta\, 2^{1-n}}{(n-2)!}\, k_2^{(\beta/\alpha)-1} r^{-1/\alpha}(t)\sigma^{n-2}(t)\sigma'(t)\sigma^{(n-1)[(\beta/\alpha)-1]}(t)W^{1+1/\alpha}(t) \\ &= -q(t) - \alpha\kappa(t)W^{1+1/\alpha}(t). \end{aligned} \tag{2.15}$$

The rest of the proof is similar to that of Case 1 and hence is omitted. □

Remark 2.1. For equation (1.2), Theorem 2.1 improves Theorem 2.1 in [1], and also improves Theorem 2.1 dropping the restriction "$\rho(t) \geq 0$" in [13].

As an immediate consequence of Theorem 2.1 we get the following corollary.

Corollary 2.1. *Let condition* (2.5) *in Theorem* 2.1 *be replaced by*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}A_{t_0}(q\,;t) = \infty, \tag{2.16}$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}A_{t_0}(\kappa^{-\alpha}\rho^{-(\alpha+1)}|h|^{\alpha+1}\,;t) < \infty. \tag{2.17}$$

Then the conclusion of Theorem 2.1 *holds.*

It is clear that (2.16) is the necessary condition for (2.5) to hold. In case (2.16) fails, then the following theorem may be applicable.

Theorem 2.2. *Let ρ, H and h be as in Theorem* 2.1. *Suppose that there exist functions $\phi_1, \phi_2 \in C([t_0,\infty),\mathbb{R})$ and for any $k_1, k_2 > 0$, $T \geq t_0$ such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} A_T(q\,;t) \geq \phi_1(T), \tag{2.18}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} A_T(\kappa^{-\alpha}\rho^{-(\alpha+1)}|h|^{\alpha+1}\,;t) \leq \phi_2(T), \tag{2.19}$$

where ϕ_1 and ϕ_2 satisfy

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} A_T(\kappa\,\rho^{-(1+1/\alpha)}(\phi_1-\theta\phi_2)_+^{1+1/\alpha};t) = \infty, \tag{2.20}$$

where θ and κ are as in Theorem 2.1, *and $\phi_+(s) = \max\{\phi(s),0\}$. Then equation* (1.1) *is oscillatory.*

Proof. Proceeding as in the proof of Theorem 2.1, we get that (2.10) holds for $t \geq t_5$. Next, like in the proof of Theorem 2.1, we consider the following three cases.

Case 1. $\beta > \alpha$. In view of Case 1 of Theorem 2.1, we get that (2.12) and (2.13) hold for $t > T \geq T_1$. Then, by (2.13), we have

$$\frac{1}{H(t,T)} A_T(q\,;t) - \frac{\theta}{H(t,T)} A_T(\kappa^{-\alpha}\rho^{-(\alpha+1)}|h|^{\alpha+1};t) \leq \rho(T)W(T), \quad t > T \geq T_1.$$

Taking limsup in the above inequality as $t\to\infty$ and applying (2.18) and (2.19), we obtain

$$\phi_1(T) - \theta\phi_2(T) \leq \rho(T)W(T),$$

from which it follows that

$$\frac{1}{H(t,T)} A_T\left(\kappa\,\rho^{-(1+1/\alpha)}(\phi_1-\theta\phi_2)_+^{1+1/\alpha};t\right) \leq \frac{1}{H(t,T)} A_T\left(\kappa\,W^{1+1/\alpha};t\right). \tag{2.21}$$

On the other hand, by (2.12), we have

$$\frac{\alpha}{H(t,T)} A_T\left(\kappa\,W^{1+1/\alpha};t\right) - \frac{1}{H(t,T)} A_T\left(\rho^{-1}|h|\,W;t\right) \leq \rho(T)W(T) - \frac{1}{H(t,T)} A_T\left(q\,;t\right).$$

Thus, by (2.18), we have

$$\liminf_{t\to\infty}\left\{\frac{\alpha}{H(t,T_1)} A_{T_1}\left(\kappa\,W^{1+1/\alpha};t\right) - \frac{1}{H(t,T_1)} A_{T_1}\left(\rho^{-1}|h|\,W;t\right)\right\} \leq \rho(T_1)W(T_1) - \phi_1(T_1) \leq C_0. \tag{2.22}$$

where C_0 is a constant. According to (2.22), there exists a sequence $\{t_j\}_{j=1}^{\infty} \in [t_0,\infty)$ with $\lim_{j\to\infty} t_j = \infty$ such that for j large enough, we have

$$\frac{\alpha}{H(t_j,T_1)} A_{T_1}\left(\kappa\,W^{1+1/\alpha};t_j\right) - \frac{1}{H(t_j,T_1)} A_{T_1}\left(\rho^{-1}|h|\,W;t_j\right) \leq C_0 + 1. \tag{2.23}$$

Now we claim that

$$\liminf_{t\to\infty} \frac{1}{H(t,T_1)} A_{T_1}\left(\kappa W^{1+1/\alpha}; t\right) < \infty. \tag{2.24}$$

If (2.24) does not hold, then

$$\lim_{j\to\infty} \frac{1}{H(t_j,T_1)} A_{T_1}\left(\kappa W^{1+1/\alpha}; t_j\right) = \infty. \tag{2.25}$$

So, (2.23) and (2.25) give

$$\frac{A_{T_1}(\rho^{-1}|h|\,W; t_j)}{A_{T_1}\left(\kappa W^{(1+1/\alpha)}; t_j\right)} - \alpha \geq -\frac{\alpha}{2} \quad \text{for } j \text{ large enough,}$$

that is,

$$A_{T_1}(\rho^{-1}|h|\,W; t_j) \geq \frac{\alpha}{2} A_{T_1}\left(\kappa W^{1+1/\alpha}; t_j\right) \quad \text{for all large } j. \tag{2.26}$$

By the Hölder inequality [5, Theorem 189], we have

$$A_{T_1}(\rho^{-1}|h|\,W; t_j) \leq \left[A_{T_1}\left(\kappa W^{1+1/\alpha}; t_j\right)\right]^{\alpha/(\alpha+1)} \times \left[A_{T_1}\left(\kappa^{-\alpha}\rho^{-(\alpha+1)}|h|^{\alpha+1}; t_j\right)\right]^{1/(\alpha+1)}. \tag{2.27}$$

From (2.26) and (2.27), we obtain

$$\frac{1}{H(t_j,T_1)} A_{T_1}\left(\kappa^{-\alpha}\rho^{-(\alpha+1)}|h|^{\alpha+1}; t_j\right) \geq \left(\frac{\alpha}{2}\right)^{\alpha+1} \frac{1}{H(t_j,T_1)} A_{T_1}\left(\kappa W^{1+1/\alpha}; t_j\right). \tag{2.28}$$

By (2.19), the left-hand side of (2.28) is bounded, which contradicts (2.25). Therefore (2.24) holds. Hence, by (2.21),

$$\liminf_{t\to\infty} \frac{1}{H(t,T_1)} A_{T_1}\left(\kappa\,\rho^{-(1+1/\alpha)}(\phi_1-\theta\phi_2)_+^{1+1/\alpha}; t\right) \leq \liminf_{t\to\infty} \frac{1}{H(t,T_1)} A_{T_1}\left(\kappa W^{1+1/\alpha}; t\right) < \infty,$$

which contradicts (2.20).

Case 2. $\beta = \alpha$. Then (2.14) holds for $t \geq t_5$. Once again, we can complete the proof by proceeding as in the proof of Case 1.

Case. 3. $\beta < \alpha$. Then (2.15) holds for $t \geq T_2$, the rest of the proof is the same as in Case 1. □

Remark 2.2. For equation (1.2), Theorem 2.2 extends and improves Theorems 2.2–2.4 in [13].

Remark 2.3. Our results are presented in the form which is essentially new. Note that we do not assume that the function $q(t)$ satisfies the condition "$\int_t^\infty q(s)ds < \infty$" in [2].

3. Examples and Some Remarks

This final section presents two examples that illustrate the results obtained in Section 2. It is easy to see that the results in [1]–[4], [7], [8], [10], [12], [13] are not applicable in these examples.

Example 3.1. Consider the delay differential equation

$$(r(t)|x^{(n-1)}(t)|^{\alpha-1}x^{(n-1)}(t))' + \gamma_1\, t^{\lambda-1}\left|x\left(\frac{t}{2}\right)\right|^{\beta-1} x\left(\frac{t}{2}\right) = 0, \quad n \text{ is even}, \tag{3.1}$$

where $t \geq 1$, $\beta > \alpha > 0$, $\gamma_1 > 0$, and $r(t) = t^{-(\alpha+1)} + c$, $c > 0$.

For Corollary 2.1, let

$$\sigma(t) = \frac{t}{2}, \quad \rho(t) = t^{-\lambda}, \quad H(t,s) = (t-s)^{\lambda}, \quad \text{and} \quad \frac{\alpha(n-1)}{2\alpha+1} > \lambda > \alpha + 1.$$

Then

$$\begin{aligned} h(t,s) &= \frac{\lambda\, s^{\lambda-1} t}{t-s}, \\ \kappa(t) &= \frac{\beta}{\alpha}\frac{2^{\,2(1-n)}}{(n-2)!}k_1^{\beta/\alpha-1}(t^{-(\alpha+1)}+c)^{-1/\alpha}t^{n-2} \\ &\geq \frac{\beta}{\alpha}\frac{2^{\,2(1-n)}}{(n-2)!}k_1^{\beta/\alpha-1}(1+c)^{-1/\alpha}t^{n-2} = C_1 t^{n-2}, \end{aligned}$$

where

$$C_1 := \frac{\beta}{\alpha}\frac{2^{\,2(1-n)}}{(n-2)!}k^{\beta/\alpha-1}(1+c)^{-1/\alpha}.$$

It follows from Theorem 41 in [5] that

$$(t-s)^{\lambda} \geq t^{\lambda} - \lambda s\, t^{\lambda-1} \quad \text{for } t \geq s \geq 1.$$

By the above inequality, we obtain that

$$\begin{aligned} \limsup_{t\to\infty} \frac{1}{H(t,1)} A_1(q;t) &= \lim_{t\to\infty} \frac{\gamma_1}{(t-1)^{\lambda}} \int_1^t (t-s)^{\lambda}\frac{1}{s}ds \\ &\geq \lim_{t\to\infty} \frac{\gamma_1}{(t-1)^{\lambda}} \int_1^t \frac{t^{\lambda} - \lambda\, s t^{\lambda-1}}{s}ds = \gamma_1 \limsup_{t\to\infty}(\ln t - \lambda) = \infty, \end{aligned}$$

and

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{H(t,1)} A_1(\kappa^{-\alpha}\rho^{-(\alpha+1)}|h|^{\alpha+1};t) \\ &\quad\leq C_1^{-\alpha}\lambda^{\alpha+1} \limsup_{t\to\infty} \frac{t^{\alpha+1}}{(t-1)^{\lambda}} \int_1^t (t-s)^{\lambda-\alpha-1} s^{-\alpha(n-1)+\lambda(2\alpha+1)-1} ds \end{aligned}$$

$$\leq C_1^{-\alpha}\lambda^{\alpha+1}\limsup_{t\to\infty}\int_1^t s^{-\alpha(n-1)+\lambda(2\alpha+1)-1}ds<\infty.$$

Thus all conditions of Corollary 2.1 are satisfied and equation (3.1) is oscillatory.

Example 3.2. Consider the delay differential equation

$$(r(t)|x^{(n-1)}(t)|^{\alpha-1}x^{(n-1)}(t))'+\gamma_2\, t^{-5/2}|x\left(\frac{t}{2}\right)|^{\beta-1}x\left(\frac{t}{2}\right)=0,\quad n \text{ is even},\tag{3.2}$$

where $t\geq 1$, $\alpha>\beta>0$, $\gamma_2>0$, $2\beta(n-1)\geq 3(\alpha+1)$, and $r(t)=t^{-3/2}+c$, $c>0$.

For Theorem 2.2, let

$$\sigma(t)=\frac{t}{2},\quad \rho(t)=1,\quad H(t,s)=(t-s)^\lambda,\quad \text{and}\quad \lambda>\alpha+1.$$

Then

$$\begin{aligned}h(t,s)&=\frac{\lambda}{t-s},\\ \kappa(t)&=\frac{\beta}{\alpha}\frac{2^{(1-n)(\beta/\alpha+1)}}{(n-2)!}k_2^{\beta/\alpha-1}(t^{-(3/2)}+c)^{-1/\alpha}t^{(n-1)\beta/\alpha-1}\\ &\geq\frac{\beta}{\alpha}\frac{2^{(1-n)(\beta/\alpha+1)}}{(n-2)!}k_2^{\beta/\alpha-1}(1+c)^{-1/\alpha}t^{(n-1)\beta/\alpha-1}\quad =: C_2t^{(n-1)\beta/\alpha-1}.\end{aligned}$$

By a direct computation, one has, for all $T\geq 1$,

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}A_T(q;t)=\lim_{t\to\infty}\frac{\gamma_2}{(t-T)^\lambda}\int_T^t(t-s)^\lambda s^{-5/2}ds\geq\frac{2\gamma_2}{3}\frac{1}{T^{3/2}},$$

and

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{H(t,T)}A_T(\kappa^{-\alpha}\rho^{-(\alpha+1)}|h|^{\alpha+1};t)\\ &\quad\leq C_2^{-\alpha}\lambda^{\alpha+1}\limsup_{t\to\infty}\frac{1}{(t-T)^\lambda}\int_T^t(t-s)^{\lambda-\alpha-1}s^{-\beta(n-1)+\alpha}ds\\ &\quad\leq C_2^{-\alpha}\lambda^{\alpha+1}\limsup_{t\to\infty}\frac{(t-T)^{\lambda-\alpha-1}}{(t-T)^\lambda}\int_T^t s^{-\beta(n-1)+\alpha}ds\\ &\quad\leq C_2^{-\alpha}\lambda^{\alpha+1}\limsup_{t\to\infty}\frac{1}{(t-T)^{\alpha+1}}\int_T^t s^{-3(\alpha+1)/2+\alpha}ds=0.\end{aligned}$$

Now, set

$$\phi_1(T)=\frac{2\gamma_2}{3}\frac{1}{T^{3/2}}\quad\text{and}\quad\phi_2(T)=0,$$

then

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} A_T(\kappa\,\rho^{-(1+1/\alpha)}[\phi_1 - \theta\phi_2]_+^{1+1/\alpha}; t)$$

$$\geq C_2(\frac{2\gamma_2}{3})^{1+1/\alpha} \liminf_{t\to\infty} \frac{1}{(t-T)^\lambda} \int_T^t (t-s)^\lambda\, s^{[2\beta(n-1)-5\alpha-3]/(2\alpha)} ds$$

$$\geq C_2(\frac{2\gamma_2}{3})^{1+1/\alpha} \liminf_{t\to\infty} \int_T^t s^{-1} ds = \infty.$$

Thus we can conclude that equation (3.2) is oscillatory by Theorem 2.2.

Remark 3.1. With an appropriate choice of functions H and ρ, it is possible to derive a number of oscillation criteria for equation (1.1). Defining, for example, for some integers $\lambda > 1$ and $\gamma \in \mathbb{R}$, the functions $H(t,s)$ and $\rho(t)$ by

$$H(t,s) = (t-s)^\lambda, \quad \rho(t) = t^\gamma, \ \ (t,s) \in D, \tag{3.3}$$

as direct consequences of Theorems 2.1 and 2.2, we can establish a number of oscillation criteria.

Of course, we are not limited only to a choice of H defined by (3.3), which has become standard and goes back to the well known Kamenev-type condition [6]. With a different choice of the function H, it is possible to derive from Theorems 2.1 and 2.2 other sets of oscillation criteria. Indeed, other possibilities are to choose the function H as follows:

$$H(t,s) = \left(\int_s^t \frac{du}{\psi(u)} \right)^\lambda, \quad t \geq s \geq t_0,$$

where $\lambda > 1$ and $\psi \in C[t_0,\infty),\mathbb{R}_+)$ satisfies the condition

$$\lim_{t\to\infty} \int_{t_0}^t \frac{du}{\psi(u)} = \infty.$$

Remark 3.2. It is straightforward to formulate and prove an analogue of our main results for the even order damped delay differential equation

$$(r(t)|x^{(n-1)}(t)|^{\alpha-1}x^{(n-1)}(t))' + p(t)|x^{(n-1)}(t)|^{\gamma-1}x^{(n-1)}(t)$$
$$+F(t, x[g_{01}(t)], \cdots, x[g_{0m}(t)], \cdots, x^{(n-1)}[g_{01}(t)], \cdots, x^{(n-1)}[g_{0m}(t)]) = 0, \tag{3.4}$$

(for its particular cases, we refer to [3],[4], [10], [12]) and to various forms of differential equations by making appropriate changes in the hypotheses.

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