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GLOBAL EXISTENCE AND ENERGY DECAY OF SOLUTIONS TO A PETROVSKY EQUATION WITH GENERAL NONLINEAR DISSIPATION AND SOURCE TERM

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Abstract. We consider the nonlinearly damped semilinear Petrovsky equation

$$u'' - \Delta_x^2 u + g(u') = b\ u|u|^{p-2} \quad \text{on} \quad \Omega \times [0, +\infty[$$

and prove the global existence of its solutions by means of the stable set method in $H_0^2(\Omega)$ combined with the Faedo–Galerkin procedure. Furthermore, we study the asymptotic behavior of solutions when the nonlinear dissipative term g does not necessarily have a polynomial growth near the origin.

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1. Introduction

We consider the initial boundary value problem

$$(P) \qquad \begin{cases} u'' - \Delta_x^2 u + g(u') = b\ u|u|^{p-2} & \text{in } \Omega \times [0, +\infty[, \\ u = \partial_\nu u = 0 & \text{on } \Gamma \times [0, +\infty[, \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) & \text{on } \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega = \Gamma$.

For the problem (P) when $g(s) = \delta|s|^{m-2}s$ $(m \geq 1)$, S. A. Messaoudi [7] obtained relations between m and p for which the global existence or alternatively finite time blow up takes place. More precisely, he showed that solutions with any initial data continue to exist globally in time if $m \geq p$ and blow up in finite time if $m < p$ and the initial energy is negative. To prove the global existence he used a new method introduced by Georgiev and Todorova [2] based on the fixed point theorem.

In [3], for a wave equation ($\Delta_x u$ instead of $\Delta_x^2 u$ in (P)) Ikehata by using the stable set method due to Sattinger [10] proved that a global solution exists with no relation between p and m, and Todorova [11] proved that an energy decay rate is $E(t) \leq (1+t)^{-2/(m-2)}$ for $t \geq 0$, for which she used the general method on energy decay introduced by Nakao [9]. Unfortunately, the methods used by Messaoudi and Todorova do not seem to be applicable to the case of more general functions g.

Our purpose in this paper is to give the global solvability in the class H_0^2 and the energy decay estimates of solutions to the problem (P) when $g(s)$ does not

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necessarily have a polynomial growth near zero and a source term of the form $b\ |y|^{p-2}y$ with a small parameter b. As proved in [4] and [11], a decay rate of the global solution depends on the polynomial growth near zero of $g(s)$.

We use some ideas from [6] (see also [1]) introduced in the study of decay rates of solutions to the wave equation $u_{tt} - \Delta_x u + g(u_t) = 0$ in $\Omega \times \mathbb{R}^+$. So, to obtain global decaying solutions to the problem (P), we use the argument combining the Galerkin approximation scheme (see [5]) with the concept of a stable set in H_0^2 and the method in [6] to derive a decay rate of the solution.

We conclude this section by stating our plan and giving some notations. In Section 2 we formulate some lemmas needed for our arguments. Sections 3 and 4 are devoted to the proof of the global existence and decay estimates for the problem (P).

Throughout this paper all the functions considered are real-valued. We omit the space variable x of $u(t,x)$, $u_t(t,x)$ and simply denote $u(t,x)$, $u_t(t,x)$ by $u(t)$, $u'(t)$, respectively, when no confusion arises. Let l be a number with $2 \leq l \leq \infty$. We denote by $\|\cdot\|_l$ the L^l norm over Ω. In particular, the L^2 norm is denoted $\|\cdot\|_2$. $(\cdot)$ denotes the usual L^2 inner product. We use the familiar function spaces H_0^2, H^4.

2. Preliminaries

Let us state the precise hypotheses on p and g.

(H1) Assume that

$$2 < p \leq \infty \ \ (n = 1,2,3,4) \quad \text{or} \quad 2 < p \leq \frac{2n-2}{n-4} \quad (n \geq 5). \tag{1}$$

(H2) g is an odd increasing C^1 function and

$$c_1|s| \leq |g(s)| \leq c_2|s|^r \quad \text{if} \quad |s| \geq 1 \quad \text{with} \quad 1 \leq r \leq \infty \quad (n = 1,2,3,4)$$
$$\text{or} \quad 1 \leq r \leq \frac{n+4}{n-4} \ \ (n \geq 5),$$

where c_1 and c_2 are positive constants.

We first state three well known lemmas that will be needed later.

Lemma 2.1 (Sobolev–Poincaré inequality)**.** *Let q be a number with $2 \leq q < +\infty$ $(n = 1,2,3,4)$ or $2 \leq q \leq 2n/(n-4)$ $(n \geq 5)$, then there is a constant $C_* = C(\Omega, q)$ such that*

$$\|u\|_q \leq C_* \|\Delta u\|_2 \quad for \quad u \in H_0^2(\Omega). \tag{2}$$

We denote by c various positive constants which may be different at different occurrences.

Lemma 2.2 ([6])**.** *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing C^1 function such that*

$$\phi(0) = 0 \quad and \quad \phi(t) \to +\infty \quad as \quad t \to +\infty.$$

Assume that there exist $\sigma \geq 0$ and $\omega > 0$ such that

$$\int_S^{+\infty} E^{1+\sigma}(t)\phi'(t)\,dt \leq \frac{1}{\omega}E^{\sigma}(0)E(S), \quad 0 \leq S < +\infty.$$

Then

$$\begin{aligned} &E(t) \leq E(0)\left(\frac{1+\sigma}{1+\omega\sigma\phi(t)}\right)^{\frac{1}{\sigma}} && \forall t \geq 0, \quad \textit{if} \quad \sigma > 0, \\ &E(t) \leq cE(0)e^{1-\omega\phi(t)} && \forall t \geq 0, \quad \textit{if} \quad \sigma = 0. \end{aligned}$$

Remark 2.1. A 'weight function' $\phi(t)$ was sufficiently used by Martinez [6], and Mochizuki and Motai [8] to establish a decay rate of solutions to a hyperbolic PDE.

Lemma 2.3 ([6]). *There exists an increasing function $\phi : \mathbb{R}_+ \to \mathbb{R}$ such that ϕ is concave and $\phi(t) \to +\infty$ as $t \to +\infty$, $\phi'(t) \to 0$ as $t \to +\infty$, and*

$$\int_1^{+\infty} \phi'(t)\left(g^{-1}(\phi'(t))\right)^2\,dt < +\infty.$$

In order to state and prove our main results, we first introduce the following notation:

$$\begin{aligned} I(t) &= I(u(t)) = \|\Delta_x u(t)\|_2^2 - b\|u(t)\|_p^p, \\ J(t) &= J(u(t)) = \frac{1}{2}\|\Delta_x u(t)\|_2^2 - \frac{b}{p}\|u(t)\|_p^p, \\ E(t) &= E(u(t), u'(t)) = J(t) + \frac{1}{2}\|u'(t)\|_2^2. \end{aligned}$$

Then we can define the stable set as

$$H = \left\{w \in H_0^2(\Omega) \ \mid \ I(w) > 0\right\} \cup \{0\},$$

where we use w instead of $w(\cdot, t)$.

3. Global Existence

Throughout this section we assume $u_0 \in H^4(\Omega) \cap H$ and $u_1 \in H_0^2(\Omega) \cap L^{2r}(\Omega)$. We employ the Galerkin method to construct a global solution. Let $T > 0$ be fixed and denote by V_m the space generated by $\{w_1, w_2, \ldots, w_m\}$, where the set $\{w_m; m \in \mathbb{N}\}$ is a basis of L^2, H_0^2 and $H^4 \cap H_0^2$. We construct approximate solutions u_m $(m = 1, 2, 3, \ldots)$ in the form

$$u_m(t) = \sum_{j=1}^{m} g_{jm} w_j,$$

where g_{jm} $(j = 1, 2, \dots, m)$ are determined by the following ordinary differential equations:

$$(u''_m(t), w_j) + (\Delta_x u_m(t), \Delta_x w_j) + (g(u'_m(t)), w_j) \\ = (b|u_m(t)|^{p-2}u_m(t), w_j), \quad 1 \le j \le m, \tag{3}$$

$$u_m(0) = u_{0m} = \sum_{j=1}^{m} (u_0, w_j)w_j \to u_0 \quad \text{in} \quad H^4 \cap H_0^2 \quad \text{as} \quad m \to +\infty, \tag{4}$$

$$u'_m(0) = u_{1m} = \sum_{j=1}^{m} (u_1, w_j)w_j \to u_1 \quad \text{in} \quad H_0^2 \cap L^{2r} \quad \text{as} \quad m \to +\infty. \tag{5}$$

By virtue of the theory of ordinary differential equations, system (3)–(5) has a unique local solution which is extended to a maximal interval $[0, T_m[$ (with $0 < T_m \le +\infty$) by the Zorn lemma, since the nonlinear terms in (3) are locally Lipschitz continuous. Note that $u_m(t)$ is a C^2-function.

In the next step, we obtain a priori estimates for the solution so that it can be extended outside $[0, T_m[$ to obtain one solution defined for all $t > 0$.

We can utilize a standard compactness argument for the limiting procedure and it suffices to derive some a priori estimates for u_m. But this procedure allows us to employ the energy method for a smooth solution $u(t)$ to the problem (P) (the results should be in fact applied to approximated solutions).

Remark 3.1. By multiplying the first equation of (P) by $u'(t)$, integrating over Ω, and using integration by parts and the boundary conditions we get

$$E'(t) = -\int_\Omega g(u'(t))u'(t)\,dx \le 0 \quad \forall t \in [0, T).$$

Lemma 3.1. *Assume that* **(H1)** *holds. Let $u(t)$ be a solution with the initial data $\{u_0, u_1\}$ satisfying $u_0 \in H$ and $u_1 \in L^2(\Omega)$. If $\{u_0, u_1\}$ satisfies*

$$\eta = 1 - b\, C_*^p \left(\frac{2p}{p-2} E(u_0, u_1)\right)^{(p-2)/2} > 0, \tag{6}$$

then $u(t) \in H$ for all $t \in [0, +\infty)$ and there exists a constant $M = M(\|\nabla_x u_0\|_2, \|u_1\|_2) > 0$ such that

$$\|\Delta_x u(t)\|_2^2 + \|u'(t)\|_2^2 \le M \quad \textit{for} \quad t \ge 0,$$

and

$$\int_0^t \int_\Omega g(u'(s))u'(s)\,ds \le M \quad \textit{for} \quad t \ge 0. \tag{7}$$

Proof. Since $I(u_0) > 0$, it follows from the continuity of $u(t)$ that

$$I(u(t)) \ge 0 \tag{8}$$

for some interval near $t = 0$. Let $t_{\max}$ be a maximal time (possibly $t_{\max} = T_m$), when (8) holds on $[0, t_{\max})$. On the other hand,

$$\begin{aligned} J(t) &= \frac{1}{2}\|\Delta_x u(t)\|_2^2 - \frac{b}{p}\|u(t)\|_p^p \\ &= \frac{p-2}{2p}\|\Delta_x u(t)\|_2^2 + \frac{1}{p} I(u(t)) \\ &\geq \frac{p-2}{2p}\|\Delta_x u(t)\|_2^2 \qquad \forall t \in [0, t_{\max}); \end{aligned}$$

hence

$$\begin{aligned} \|\Delta_x u(t)\|_2^2 &\leq \frac{2p}{p-2} J(t) \leq \frac{2p}{p-2} E(t) \\ &\leq \frac{2p}{p-2} E(u_0, u_1) \qquad \forall t \in [0, t_{\max}). \end{aligned} \tag{9}$$

Using $(2), (6)$, and (9), we deduce that

$$\begin{aligned} b\|u(t)\|_p^p &\leq b\, C_*^p \|\Delta_x u(t)\|_2^p = b\, C_*^p \|\Delta_x u(t)\|_2^{p-2} \|\Delta_x u(t)\|_2^2 \\ &\leq b\, C_*^p \left(\frac{2p}{p-2} E(u_0, u_1)\right)^{(p-2)/2} \|\Delta_x u(t)\|_2^2 \\ &< \|\Delta_x u(t)\|_2^2 \qquad \forall t \in [0, t_{\max}); \end{aligned} \tag{10}$$

Therefore we get

$$\|\Delta_x u(t)\|_2^2 - b\|u(t)\|_p^p > 0 \quad \text{on} \quad [0, t_{\max}).$$

This implies that we can take $t_{\max} = T_m$. Furthermore, by the fact that the energy is non-increasing we have

$$\begin{aligned} E(u_0, u_1) \geq E(t) &= \frac{1}{2}\|\Delta_x u(t)\|_2^2 - \frac{b}{p}\|u(t)\|_p^p + \frac{1}{2}\|u'(t)\|_2^2 \\ &= \frac{p-2}{2p}\|\Delta_x u(t)\|_2^2 + \frac{1}{p} I(u(t)) + \frac{1}{2}\|u'(t)\|_2^2 \\ &\geq \frac{p-2}{2p}\|\Delta_x u(t)\|_2^2 + \frac{1}{2}\|u'(t)\|_2^2 \text{ on } [0, t_{\max}), \end{aligned}$$

since $I(u(t)) \geq 0$, and hence

$$\|\Delta_x u(t)\|_2^2 + \|u'(t)\|_2^2 \leq C_1 E(u_0, u_1) \text{ on } [0, t_{\max}). \tag{11}$$

These estimates imply that the (approximated) solution $u(t)$ exists globally in $[0, +\infty)$. This ends the proof of Lemma 3.1. □

Estimate (11) yields

$$\Delta_x u_m \text{ is bounded in } L^\infty_{loc}(0, \infty; L^2). \tag{12}$$

Lemma 3.2. *There exists $K_1 > 0$ such that $\|g(u'_m)\|_{L^{\frac{r+1}{r}}(\Omega\times[0,T])} \leq K_1$ for all $m \in \mathbb{N}$.*

Proof. If we define

$$A_m = \{(x,t) \in Q \backslash |u'_m(x,t)| \leq 1\}$$

and

$$B_m = \{(x,t) \in Q \backslash |u'_m(x,t)| > 1\},$$

where $Q = \Omega \times [0,T]$, then from (**H2**):

$$\begin{aligned}
&\int_0^T \int_\Omega |g(u'_m(x,t))|^{\frac{r+1}{r}}\, dx\, dt \\
&\quad = \int_{A_m}\int |g(u'_m(x,t))|^{\frac{r+1}{r}}\, dx\, dt + \int_{B_m}\int |g(u'_m(x,t))|^{\frac{r+1}{r}}\, dx\, dt \\
&\quad \leq \int_0^T \int_\Omega \sup_{|s|\leq 1} |g(s)|^{\frac{r+1}{r}}\, dx\, dt + c_2 \int_{B_m}\int |g(u'_m(x,t))||u'_m(x,t)|\, dx\, dt.
\end{aligned}$$

Hence, by (7), we have

$$\int_0^T \int_\Omega |g(u'_m(x,t))|^{\frac{r+1}{r}}\, dx\, dt \leq |Q| \sup_{|s|\leq 1} |g(s)|^{\frac{r+1}{r}} + c_2\, M \quad \text{for} \quad m \in \mathbb{N}$$

which completes the proof. Here $|Q|$ denotes the Lebesgue measure in $\mathbb{R}^{n+1}$. □

Lemma 3.3. *There exists a constant M' such that*

$$\|u''_m(t)\|_2 + \|\Delta_x u'_m(t)\|_2 \leq M'$$

for all $m \in \mathbb{N}$.

Proof. From (3) we obtain

$$\begin{aligned}
\|u''_m(0)\|_2 &\leq \|\Delta_x^2 u_{0m}\|_2 + \|g(u_{1m})\|_2 + \|f(u_{0m})\|_2 \\
&\leq \|\Delta_x^2 u_{0m}\|_2 + \|g(u_{1m})\|_2 + k_1 \|\Delta_x u_{0m}\|_2^{p-1},
\end{aligned}$$

where we set $f(u) = bu|u|^{p-2}$. Using the Gagliardo–Nirenberg inequality, we have

$$\|f(u_{0m})\|_2 \leq c\|\Delta_x^2 u_{0m}\|_2^{p-1}.$$

Since $g(u_{1m})$ is bounded in $L^2(\Omega)$ by (**H2**), from (4) and (5) we obtain

$$\|u''_m(0)\|_2 \leq C.$$

Differentiating (3) with respect to t, we get

$$(u'''_m(t) + \Delta_x^2 u'_m(t) + u''_m(t) g'(u'_m) - u'_m f'(u_m), w_j) = 0.$$

Multiplying it by $2g''_{jm}(t)$ and summing over j from 1 to m give

$$\frac{d}{dt}\left(\|u''_m(t)\|_2^2 + \|\Delta_x u'_m(t)\|_2^2\right) + 2\int_\Omega {u''}^2_m(t) g'(u'_m(t))\,dx \le 2b(p-1)\int_\Omega |u''_m(t)||u'_m(t)||u_m(t)|^{p-2}\,dx. \quad (13)$$

Next, we are are going to analyze the term on the right-hand side of (13). Making use of the generalized Hölder inequality, observing that $\frac{p-2}{2(p-1)} + \frac{1}{2(p-1)} + \frac{1}{2} = 1$, using Lemmas 2.1 and 3.1 we conclude that

$$\begin{aligned}\left|\int_\Omega u''_m(t)u'_m(t)f'(u_m(t))\,dx\right| &\le b(p-1)\|u_m(t)\|_{2(p-1)}^{p-2}\|u'_m(t)\|_{2(p-1)}\|u''_m(t)\|_2 \\ &\le C_1\|\Delta_x u_m(t)\|_2^{p-2}\|\Delta_x u'_m(t)\|_2\|u''_m(t)\|_2 \\ &\le C_2\left(\|\Delta_x u'_m(t)\|_2^2 + \|u''_m(t)\|_2^2\right), \end{aligned} \quad (14)$$

where C_1 and C_2 are positive constants independent of m and $t \in [0,T]$.

Combining (13) and (14) we deduce

$$\frac{d}{dt}\left(\|u''_m(t)\|_2^2 + \|\Delta_x u'_m(t)\|_2^2\right) + 2\int_\Omega {u''}^2_m g'(u'_m)\,dx \le C_2\left(\|u''_m(t)\|_2^2 + \|\Delta_x u'_m(t)\|_2^2\right),$$

Integrating the last inequality over $(0,t)$ and applying Gronwall's lemma, we obtain

$$\|u''_m(t)\|_2^2 + \|\Delta_x u'_m(t)\|_2^2 \le e^{C_2 T}\left(\|u''_m(0)\|_2^2 + \|\Delta_x u'_m(0)\|_2^2\right)$$

for all $t \in \mathbb{R}_+$. Therefore we conclude that

$$u''_m \text{ is bounded in } L^\infty_{loc}(0,\infty;L^2), \quad (15)$$

$$\Delta_x u'_m \text{ is bounded in } L^\infty_{loc}(0,\infty;L^2). \quad (16)$$

Furthermore, we claim that

$$u'_m \text{ is precompact in } L^2_\infty(0,\infty;L^2). \quad (17)$$

Indeed, it follows from (15) and (16) that

$$u'_m \text{ is bounded in } L^\infty_{loc}(0,\infty;H_0^2)$$

and

$$u''_m(t) \text{ is bounded } L^\infty_{loc}(0,\infty;L^2(\Omega)). \quad (18)$$

Applying a compactness argument, (17) follows. □

Applying the Dunford–Pettis theorem we conclude from (12), Lemma 3.2, (15) and (16) replacing, if needed, the sequence u_m with a subsequence that

$$u_m \to u \quad \text{weak-star in} \quad L^\infty_{loc}(0,\infty;H_0^2), \quad (19)$$

$$u'_m \to u' \quad \text{weak-star in} \quad L^\infty_{loc}(0,\infty;H_0^2),$$

$$u''_m \to u'' \ \text{ weak-star in } \ L^\infty_{loc}(0,\infty;L^2), \tag{20}$$

$$g(u'_m) \to \chi \ \text{ weak-star in } \ L^{\frac{q+1}{q}}(\Omega\times(0,T))$$

for suitable functions $u \in L^\infty(0,T;H_0^2)$ and $\chi \in L^{\frac{q+1}{q}}(\Omega\times(0,T))$ for all $T \geq 0$. We have to show that u is a solution of (P).

Lemma 3.4. *For each $T > 0$, $g(u') \in L^1(Q)$ and $\|g(u')\|_{L^1(Q)} \leq K_1$, where K_1 is obtained in Lemma* 3.2.

Proof. By (**H2**) and (17) we have

$$g(u'_m(x,t)) \to g(u'(x,t)) \ \text{ a.e. in } \ Q,$$

$$0 \leq g(u'_m(x,t))u'_m(x,t) \to g(u'(x,t))u'(x,t) \ \text{ a.e. in } \ Q.$$

Hence, by (7) and Fatou's lemma we have

$$\int_0^T\int_\Omega u'(x,t)g(u'(x,t))\,dx\,dt \leq K \quad \text{for} \quad T>0. \tag{21}$$

Now, using (21), the proof follows similarly to Lemma 3.2. □

Lemma 3.5. *$g(u'_m) \to g(u')$ in $L^1(\Omega\times(0,T))$.*

Proof. Let $E \subset \Omega\times[0,T]$ and set

$$E_1 = \left\{(x,t)\in E;\ g(u'_m(x,t)) \leq \frac{1}{\sqrt{|E|}}\right\}, \quad E_2 = E\setminus E_1,$$

where $|E|$ is the measure of E. If $M(r) := \inf\{|s|;\ s\in\mathbb{R} \ \text{ and } \ |g(s)|\geq r\}$, then

$$\int_E |g(u'_m)|\,dxdt \leq \sqrt{|E|} + \left(M\left(\frac{1}{\sqrt{|E|}}\right)\right)^{-1}\int_{E_2}|u'_m g(u'_m)|\,dxdt.$$

Applying (7) we deduce that $\sup_m \int_E |g(u'_m)|\,dxdt \to 0$ as $|E|\to 0$. From Vitali's convergence theorem we deduce that $g(u'_m)\to g(u')$ in $L^1(\Omega\times(0,T))$, hence

$$g(u'_m) \to g(u') \text{ weak star in } L^{\frac{r+1}{r}}(Q),$$

and this implies that

$$\int_0^T\int_\Omega g(u'_m)v\,dx\,dt \to \int_0^T\int_\Omega g(u')v\,dx\,dt \ \text{ for all } \ v\in L^{r+1}(0,T;H_0^2) \tag{22}$$

as $m\to+\infty$. Using the compactness of H_0^2 in L^2, we see that

$$\int_0^T\int_\Omega b|u_m|^{p-2}u_m v\,dx\,dt \to \int_0^T\int_\Omega b|u|^{p-2}uv\,dx\,dt \ \text{ for all } \ v\in L^{r+1}(0,T;H_0^2) \tag{23}$$

as $m \to +\infty$. It follows at once from (18), (19), (20), (22) and (23) that for each fixed $v \in L^{r+1}(0,T;H_0^2)$

$$\int_0^T \int_\Omega (u_m'' + \Delta_x^2 u_m + g(u_m') - b|u_m|^{p-2}u_m)v\,dx\,dt$$

$$\to \int_0^T \int_\Omega (u'' + \Delta_x^2 u + g(u') - b|u|^{p-2}u)v\,dx\,dt$$

as $m \to +\infty$.

Hence

$$\int_0^T \int_\Omega (u'' + \Delta_x^2 u + g(u') - b|u|^{p-2}u)v\,dx\,dt = 0, \quad v \in L^{r+1}(0,T;H_0^2).$$

Thus the problem (P) admits a global weak solution u such that $u \in W^{1,\infty}(0,T;H_0^2(\Omega)) \cap W^{2,\infty}(0,T;L^2(\Omega))$.

The uniqueness of this solution is a consequence of the monotonicity of g and that f is a locally Lipschitz function. □

4. Asymptotic Behavior

Before stating and proving the decay result, we start with

Lemma 4.1. *Suppose that* (2) *holds and* $u_0 \in H$ *and* $u_1 \in L^2(\Omega)$ *satisfy* (6)*. Then*

$$b\|\ u(t)\|_p^p \le (1-\eta)\|\Delta_x u(t)\|_2^2$$

Proof. It suffices to rewrite (10) as

$$b\|u(t)\|_p^p \le \left\{1 - \left[1 - b\,C_*^p \left(\frac{2p}{p-2}E(u_0,u_1)\right)^{(p-2)/2}\right]\right\}\|\Delta_x u(t)\|_2^2.$$

Theorem 4.1. *Suppose that* (1) *holds and* $u_0 \in H$ *and* $u_1 \in L^2(\Omega)$ *satisfy* (6)*. Then the solution satisfies the decay estimates*

$$E(t) \le c\left(G^{-1}\left(\frac{1}{t}\right)\right)^2,$$

where $G(s) = sg(s)$*. If, in addition,* $s \mapsto g(s)/s$ *is non-decreasing on* $[0,\mu]$ *for some* $\mu > 0$*, then we have*

$$E(t) \le c\left(g^{-1}\left(\frac{1}{t}\right)\right)^2.$$

Examples.

- 1) If $g(s) = e^{-1/s^p}$ for $0 < s < 1,\ p > 0$, then we have

$$E(t) \le \frac{c}{(\ln t)^{2/p}}.$$

- 2) If $g(s) = e^{-e^{1/s}}$ for $0 < s < 1$, then we have

$$E(t) \leq \frac{c}{(\ln(\ln t))^2}.$$

Proof of Theorem 4.1. We multiply the first equation of (P) by $E\phi' u$, where ϕ is a function satisfying all the hypotheses of Lemma 2.3. We obtain

$$\begin{aligned}
0 &= \int_S^T E\phi' \int_\Omega u(u'' - \Delta_x^2 u + g(u') - b|u|^{p-2}u)\,dx\,dt \\
&= \left[E\phi' \int_\Omega uu'\,dx\right]_S^T - \int_S^T (E'\phi' + E\phi'') \int_\Omega uu'\,dx\,dt - 2\int_S^T E\phi' \int_\Omega u'^2\,dx\,dt \\
&\quad + \int_S^T E\phi' \int_\Omega \left(u'^2 + |\Delta_x u|^2 - \frac{2b}{p}|u|^p\right) dx\,dt + \int_S^T E\phi' \int_\Omega ug(u')\,dx\,dt \\
&\quad + \int_S^T E\phi' \int_\Omega b\left(\frac{2}{p} - 1\right)|u|^p\,dx\,dt.
\end{aligned}$$

Since

$$\begin{aligned}
b\left(1 - \frac{2}{p}\right)\int_\Omega |u|^p\,dx &\leq (1-\eta)\frac{p-2}{p}\int_\Omega |\Delta_x u|^2\,dx \\
&\leq (1-\eta)\frac{p-2}{p}\frac{2p}{p-2}E(t) \\
&= 2(1-\eta)E(t),
\end{aligned}$$

we deduce that

$$\begin{aligned}
2\eta\int_S^T E^2\phi'\,dt &\leq -\left[E\phi' \int_\Omega uu'\,dx\right]_S^T + \int_S^T (E'\phi' + E\phi'') \int_\Omega uu'\,dx\,dt \\
&\quad + 2\int_S^T E\phi' \int_\Omega u'^2\,dx\,dt - \int_S^T E\phi' \int_\Omega ug(u')\,dx\,dt + \int_S^T E\phi' \int_\Omega ug(u')\,dx\,dt \\
&\leq -\left[E\phi' \int_\Omega uu'\,dx\right]_S^T + \int_S^T (E'\phi' + E\phi'') \int_\Omega uu'\,dx\,dt \\
&\quad + 2\int_S^T E\phi' \int_\Omega u'^2\,dx\,dt + c(\varepsilon)\int_S^T E\phi' \int_{|u'|\leq 1} g(u')^2\,dx\,dt
\end{aligned}$$

$$+\,\varepsilon\int_S^T E\phi'\int_{|u'|\le 1} u^2\,dx\,dt+\int_S^T E\phi'\int_{|u'|>1} ug(u')\,dx\,dt$$

for every $\varepsilon > 0$. Moreover, using the Hölder inequality, Lemma 2.1 and the Young inequality, we obtain

$$\begin{aligned}
&\int_S^T E\phi'\int_{|u'|>1} ug(u')\,dx\,dt\\
&\quad\le\int_S^T E\phi'\Big(\int_\Omega |u|^{r+1}\,dx\Big)^{\frac{1}{(r+1)}}\Big(\int_{|u'|>1}|g(u')|^{\frac{(r+1)}{r}}\,dx\Big)^{\frac{r}{(r+1)}}\,dt\\
&\quad\le c\int_S^T E^{\frac32}\phi'\Big(\int_{|u'|>1} u'g(u')\,dx\Big)^{\frac{r}{(r+1)}}\,dt\le\int_S^T\phi' E^{\frac32}(-E')^{\frac{r}{(r+1)}}\,dt\\
&\quad\le c\int_S^T\phi'(E^{\frac32-\frac{r}{r+1}})\Big((-E')^{\frac{r}{(r+1)}}E^{\frac{r}{r+1}}\Big)\,dt\le c(\varepsilon')\int_S^T\phi'(-E'E)\,dt\\
&\qquad+\varepsilon'\int_S^T\phi' E^{(r+1)\left(\frac32-\frac{r}{(r+1)}\right)}\,dt\\
&\quad\le c(\varepsilon')E(S)^2+\varepsilon' E(0)^{\frac{(r-1)}{2}}\int_S^T\phi' E^2\,dt.
\end{aligned}$$

Choosing ε and ε' small enough, we deduce that

$$\begin{aligned}
\int_S^T E^2\phi'\,dt&\le-\Big[E\phi'\int_\Omega uu'\,dx\Big]_S^T+\int_S^T(E'\phi'+E\phi'')\int_\Omega uu'\,dx\,dt\\
&\qquad+c\int_S^T E\phi'\int_\Omega u'^2\,dx\,dt\\
&\le cE(S)+c\int_S^T E\phi'\int_\Omega u'^2\,dx\,dt.
\end{aligned}$$

Majorizing the last term of the above inequality, we have

$$\int_S^T E\phi'\int_\Omega u'^2\,dx\,dt=\int_S^T E\phi'\int_{\Omega_1}u'^2\,dx\,dt+\int_S^T E\phi'\int_{\Omega_2}u'^2\,dx\,dt$$

$$+\int_S^T E\phi' \int_{\Omega_3} u'^2\,dx\,dt,$$

where, for $t \geq 1$,

$$\begin{aligned}\Omega_1 &:= \{x \in \Omega,\ \ |u'| \leq h(t)\},\\ \Omega_2 &:= \{x \in \Omega,\ \ h(t) < |u'| \leq h(1)\},\\ \Omega_3 &:= \{x \in \Omega,\ \ |u'| > h(1)\},\end{aligned}$$

and $h(t) := g^{-1}(\phi'(t))$, which is a positive non-increasing function satisfying $h(t) \to 0$ as $t \to +\infty$. Since

- $$\begin{aligned}\int_S^T E\phi' \int_{\Omega_1} u'^2\,dx\,dt &\leq c\int_S^T E(t)\phi'(t)\Big(\int_{\Omega_1} h(t)^2\,ds\Big)\,dt\\ &\leq cE(S)\int_S^T \phi'(t)(g^{-1}(\phi'(t)))^2\,dt\\ &\leq cE(S),\end{aligned}$$
- for $x \in \Omega_2$ we have $\phi'(t) = g(h(t)) \leq |g(u')|$ (as g is non-decreasing) and hence
$$\begin{aligned}\int_S^T E\phi' \int_{\Omega_2} u'^2\,dx\,dt &\leq \int_S^T E \int_{\Omega_2} |g(u')|u'^2\,dx\,dt\\ &\leq h(1)\int_S^T E\int_{\Omega_2} u'g(u')\,dx\,dt\\ &\leq \frac{h(1)}{2}E(S)^2,\end{aligned}$$
- for $s \geq h(1)$ (as $g(s) \geq cs$) we have
$$\begin{aligned}\int_S^T E\phi' \int_{\Omega_3} u'^2\,dx\,dt &\leq c\int_S^T E\phi' \int_{\Omega} u'g(u')\,dx\,dt\\ &\leq c\int_S^T E(-E')\,dx\,dt\\ &\leq cE(S)^2,\end{aligned}$$

then we deduce that

$$\int_S^T E^2\phi'\,dt \leq cE(S),$$

and, thanks to Lemma 2.2, we obtain

$$E(t) \leq \frac{c}{\phi(t)} \qquad \forall t \geq 1.$$

Let s_0 be such that $g(\frac{1}{s_0}) \leq 1$; since g is non-decreasing, we have

$$\psi(s) \leq 1 + (s-1)\frac{1}{g\left(\frac{1}{s}\right)} \leq s\frac{1}{g\left(\frac{1}{s}\right)} = \frac{1}{G\left(\frac{1}{s}\right)} \quad \forall s \geq s_0,$$

hence

$$s \leq \phi\left(\frac{1}{G\left(\frac{1}{s}\right)}\right)$$

and

$$\frac{1}{\phi(t)} \leq \frac{1}{s} \quad \text{with} \quad t := \frac{1}{G\left(\frac{1}{s}\right)}.$$

Thus

$$\frac{1}{\phi(t)} \leq G^{-1}\left(\frac{1}{t}\right).$$

Now define $H(s) := \dfrac{g(s)}{s}$, where H is non-decreasing, $H(0) = 0$, then we use the function $h(t) := H^{-1}(\phi'(t))$. On Ω_2 there holds

$$\phi'(t)u'^2 \leq |H(u')|u'^2 = u'g(u'),$$

and the same calculations as above with

$$\phi^{-1}(t) = 1 + \int_1^t \frac{1}{H\left(\frac{1}{s}\right)}\, ds$$

yield

$$E(t) \leq c\left(g^{-1}\left(\frac{1}{t}\right)\right)^2. \qquad \square$$

Remark 4.1. We can extend all the results obtained above to the case $p = 2$. But we need some modification of Lemma 3.1, in that case the smallness of $|\Omega|$ plays an essential role in our argument. Indeed,

$$\begin{aligned} J(u(t)) &\geq \frac{1}{2}\|\Delta_x u\|_2^2 - \frac{1}{2}bC_*^2\|\Delta_x u\|_2^2 \\ &\geq \frac{1}{2}(1 - bC_*^2)\|\Delta_x u\|_2^2. \end{aligned}$$

If the condition $b\ C_*^2 < 1$ is fulfilled, then we find a similar result to Lemma 3.1. This condition implies that $|\Omega|$ is small in some sense.

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