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ALMOST EVERYWHERE CONVERGENCE OF (C,α)-MEANS OF QUADRATICAL PARTIAL SUMS OF DOUBLE VILENKIN–FOURIER SERIES

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Abstract. We prove that the maximal operator of the (C,α)-means of quadratical partial sums of double Vilenkin–Fourier series is of weak type (1,1). Moreover, the (C,α)-means $t_n^\alpha f$ of a function $f \in L^1$ converge a.e. to f as $n \to \infty$.

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1. Introduction

In 1939, for the two-dimensional trigonometric Fourier partial sums $S_{j,j}f$ Marcinkiewicz [8] proved that for all $f \in L\log L([0,2\pi]^2)$ the relation

$$t_n^1 f = \frac{1}{n}\sum_{j=0}^{n} S_{j,j}f \to f$$

holds a.e. as $n \to \infty$. Zhizhiashvili [13] improved this result and showed that for $f \in L([0,2\pi]^2)$ the (C,α) means

$$t_n^\alpha f = \frac{1}{A_n^\alpha}\sum_{j=0}^{n} A_{n-j}^{\alpha-1} S_{j,j}f$$

converge to f a.e. for any $\alpha > 0$. Dyachenko [3] proved this result for dimensions greater than 2. In papers [12, 6] by Weisz and Goginava one can find that the $(C,1)$ means $t_n^1 f$ of the double Walsh–Fourier series of a function $f \in L([0,1]^2)$ converges to f a.e. Recently, Gát [4] proved this result with respect to two-dimensional Vilenkin systems. The d-dimensional Walsh–Fourier case is discussed in [7]. The aim of this paper is to generalize the result of Zhizhiashvili [13] concerning the (C,α)-means with respect to two-dimensional (bounded) Vilenkin systems.

First, we give a brief introduction to the theory of Vilenkin systems. These orthonormal systems were introduced by N. Ya. Vilenkin in 1947 (see, e.g., [11, 1]) as follows.

Let $m := (m_k, k \in \mathbb{N})$ $(\mathbb{N} := \{0,1,\dots\}, \mathbb{P} := \mathbb{N} \setminus \{0\})$ be a sequence of integers, each of them not less than 2. Let Z_{m_k} denote the discrete cyclic group of order m_k. That is, Z_{m_k} can be represented by the set $\{0,1,\dots,m_k - 1\}$, with the group operation $\mod m_k$ addition. Since the group is discrete, every subset

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is open. The normalized Haar measure μ_k on Z_{m_k} is defined by $\mu_k(\{j\}) := 1/m_k$ ($j \in \{0, 1, \ldots, m_k - 1\}$). Let

$$G_m := \mathop{\times}_{k=0}^{\infty} Z_{m_k}.$$

Then every $x \in G_m$ can be represented by a sequence $x = (x_i, i \in \mathbb{N})$, where $x_i \in Z_{m_i}$ ($i \in \mathbb{N}$). The group operation on G_m (denoted by $+$) is the coordinate-wise addition (the inverse operation is denoted by $-$), the measure (denoted by μ), which is the normalized Haar measure, and the topology are respectively the product measure and the topology. Consequently, G_m is a compact Abelian group. If $\sup_{n\in\mathbb{N}} m_n < \infty$, then we call G_m a bounded Vilenkin group. If the generating sequence m is not bounded, then G_m is said to be an unbounded Vilenkin group. In this paper we discuss bounded Vilenkin groups only. A Vilenkin group is metrizable in the following way:

$$d(x, y) := \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{M_{i+1}} \quad (x, y \in G_m).$$

The topology induced by this metric, the product topology, and the topology given by intervals defined below, are the same. A base for the neighborhoods of G_m can be given by the intervals

$$I_0(x) := G_m, \quad I_n(x) := \{y = (y_i, i \in \mathbb{N}) \in G_m : y_i = x_i \text{ for } i < n\}$$

for $x \in G_m$, $n \in \mathbb{P}$. Let $0 = (0, i \in \mathbb{N}) \in G_m$ denote the null element of G_m and $I_n(0) := I_n$, $\overline{I}_n = G_m \backslash I_n$.

Furthermore, let $L^p(G_m)$ ($1 \le p \le \infty$) denote the usual Lebesgue spaces ($\|\cdot\|_p$ are the corresponding norms) on G_m, $_n$ the σ-algebra generated by the sets $I_n(x)$ ($x \in G_m$), and E_n the conditional expectation operator with respect to n ($n \in \mathbb{N}$).

Let $1 \le p \le +\infty$ be real. We say that an operator T is of type (p, p) if there exists an absolute constant $C > 0$ such that $\|Tf\|_p \le C\|f\|_p$ for all $f \in L^p$. T is said to be of weak type $(1, 1)$ if there exist an absolute constant $C > 0$ such that $\|Tf\|_{\text{weak-}L^1} \le C\|f\|_1$ for all $f \in L^1(G_m)$, where $\|f\|_{\text{weak-}L^1} = \sup_{\lambda>0} \lambda\mu(|f| > \lambda)$. It is known that the operator which maps a function f on the maximal function $f^* := \sup|E_n f|$ is of weak type $(1, 1)$, and of type (p, p) for all $1 < p \le \infty$ (see, e.g., [2]).

Let $M_0 := 1, M_{n+1} := m_n M_n$ ($n \in \mathbb{N}$) be the so-called generalized powers. Then each natural number n can be uniquely expressed as

$$n = \sum_{i=0}^{\infty} n_i M_i \quad (n_i \in \{0, 1, \ldots, m_i - 1\}, \ \ i \in \mathbb{N}),$$

where only a finite number of n_i's differs from zero. For $1 \le n \in \mathbb{N}$ we denote by $|n| := \max\{k \in \mathbb{N} : M_k \le n\}$ the order of a natural number n. In other words, $M_{|n|} \le n < M_{|n|+1}$. The generalized Rademacher functions are defined as

$$r_n(x) := \exp\left(2\pi\imath \frac{x_n}{m_n}\right) \quad (x \in G_m, \ \ n \in \mathbb{N}, \ \ \imath := \sqrt{-1}).$$

The n^{th} Vilenkin function is

$$\psi_n := \prod_{j=0}^{\infty} r_j^{n_j} \quad (n \in \mathbb{N}).$$

The system $\psi := (\psi_n : n \in \mathbb{N})$ is called a Vilenkin system. Each ψ_n is a character of G_m, and all the characters of G_m are of this form. Define the m-adic addition as

$$k \oplus n := \sum_{j=0}^{\infty} (k_j + n_j (\bmod m_j)) M_j \quad (k, n \in \mathbb{N}).$$

Then $\psi_{k \oplus n} = \psi_k \psi_n$, $\psi_n(x+y) = \psi_n(x)\psi_n(y)$, $\psi_n(-x) = \bar{\psi}_n(x)$, $|\psi_n| = 1$ $(k, n \in \mathbb{N},\ x, y \in G_m)$.

Set $A_n^\alpha := \frac{(1+\alpha)\dots(n+\alpha)}{n!}$ for any $n \in \mathbb{N}, \alpha \in \mathbb{R}$. It is known that $A_n^\alpha \sim n^\alpha$. Define the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels, the (C, α) means, kernels, and the Fejér means and kernels with respect to the Vilenkin system ψ as follows:

$$\begin{aligned}
\hat{f}(n) &:= \int_{G_m} f \bar{\psi}_n d\mu, \\
S_n f &:= \sum_{k=0}^{n-1} \hat{f}(k) \psi_k, \\
D_n &:= \sum_{k=0}^{n-1} \psi_k, \\
\sigma_n^\alpha f &= \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} S_k f, \\
K_n^\alpha &:= \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} D_k, \\
\sigma_n f &:= \sigma_n^1 f, \quad K_n := K_n^1 \quad (f \in L^1(G_m)).
\end{aligned}$$

It is well-known that

$$S_n f(y) = \int_{G_m} f(x) D_n(y-x) d\mu(x) \quad (n \in \mathbb{N},\ \ y \in G_m,\ \ f \in L^1(G_m)).$$

It is also well-known [1] that

$$\begin{aligned}
D_{M_n}(x) &= \begin{cases} M_n & \text{if } x \in I_n := I_n(0), \\ 0 & \text{if } x \notin I_n, \end{cases} \\
S_{M_n} f(x) &= M_n \int_{I_n(x)} f d\mu = E_n f(x) \quad (f \in L^1(G_m),\ \ n \in \mathbb{N}).
\end{aligned} \tag{1}$$

Next, we introduce some notation for the theory of two-dimensional Vilenkin systems. Let $\tilde{m}$ be a sequence like m. The relation between the sequences $(\tilde{m}_n)$ and $(\tilde{M}_n)$ is the same as between the sequences (m_n) and (M_n). The group $G_m \times G_{\tilde{m}}$ is called a two-dimensional Vilenkin group. The normalized Haar measure is denoted by μ as in the one-dimensional case. We also suppose that $m = \tilde{m}$, that is, $G_m \times G_{\tilde{m}} = G_m^2$.

The two-dimensional Fourier coefficients, the rectangular partial sums of the Fourier series, the Dirichlet kernels, the (C,α) means, the kernels of the (C,α) means with respect to a two-dimensional Vilenkin system are defined as follows:

$$\hat{f}(n_1,n_2) := \int\limits_{G_m^2} f(x,y)\bar{\psi}_{n_1}(y)\bar{\psi}_{n_2}(y)d\mu(x,y),$$

$$S_{n_1,n_2}f(u,v) := \sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1}\hat{f}(k_1,k_2)\psi_{k_1}(u)\psi_{k_2}(v),$$

$$D_{n_1,n_2}(x,y) := D_{n_1}(x)D_{n_2}(y),$$

$$t_n^\alpha f := \frac{1}{A_n^\alpha}\sum_{j=0}^{n} A_{n-j}^{\alpha-1}S_{j,j}f,$$

$$T_n^\alpha := \frac{1}{A_n^\alpha}\sum_{j=0}^{n} A_{n-j}^{\alpha-1}D_{j,j}.$$

It is also well-known that

$$t_n^\alpha f(u,v) = \int\limits_{G_m^2} f(x,y)T_n^\alpha(u-x,v-y)d\mu(x,y).$$

For the two-dimensional variable $(x,y)\in G_m^2$ we use the notation

$$\psi_n^1(x,y)=\psi_n(x),\quad D_n^1(x,y)=D_n(x),\quad K_n^{\alpha,1}(x,y)=K_n^\alpha(x),$$
$$\psi_n^2(x,y)=\psi_n(y),\quad D_n^2(x,y)=D_n(y),\quad K_n^{\alpha,2}(x,y)=K_n^\alpha(y)$$

for any $\alpha>0$ and $n\in\mathbb{N}$.

Set the maximal operator $t_*^\alpha f := \sup_{n\in\mathbb{N}} |t_n^\alpha f|$ for any $f\in L^1(G_m^2)$ and $\alpha>0$.

2. Main Results

Theorem 1. *Let $f\in L^1(G_m^2)$ and $\alpha>0$. Then*

$$\|t_*^\alpha f\|_{weak\text{-}L^1} \le C\,\|f\|_1 .$$

Corollary 1. *Let $f\in L^1(G_m^2)$ and $\alpha>0$. Then*

$$t_n^\alpha(f)\to f \quad a.e. \quad as \ \ n\to\infty.$$

3. Auxiliary results

Lemma 1. *Let $0 \le j < n_s M_s$ and $0 \le n_s < m_s$. Then*

$$D_{n_s M_s - j} = D_{n_s M_s} - \psi_{n_s M_s - 1} \overline{D}_j.$$

Proof. It is clear that

$$D_{n_s M_s} = D_{n_s M_s - j} + \sum_{k=n_s M_s - j}^{n_s M_s - 1} \psi_k = D_{n_s M_s - j} + \sum_{k=0}^{j-1} \psi_{n_s M_s - k - 1}.$$

Consequently,

$$\begin{aligned} \psi_{n_s M_s - k - 1}(x) &= \psi_{(n_s - 1) M_s + (m_{s-1} - 1) M_{s-1} + \cdots + (m_0 - 1) M_0 - k}(x) \\ &= \psi_{(n_s - k_s - 1) M_s + (m_{s-1} - k_{s-1} - 1) M_{s-1} + \ldots (m_0 - k_0 - 1) M_0}(x) \\ &= \psi_{(n_s - 1) M_s + (m_{s-1} - 1) M_{s-1} + \ldots (m_0 - 1) M_0}(x) \bar{\psi}_k(x) \\ &= \psi_{n_s M_s - 1}(x) \bar{\psi}_k(x). \end{aligned}$$

Lemma 1 is proved. □

Lemma 2. *Let $\alpha \in (0,1)$ and $n := n^{(A)} = n_A M_A + \cdots + n_0 M_0$. Then in the one-dimensional case*

$$|K_n^\alpha| \le \frac{c(\alpha)}{n^\alpha} \sum_{i=0}^{A} \left\{ \sum_{p=1}^{i} M_p^{\alpha-1} \sum_{j=M_{p-1}}^{M_p - 1} |K_j| + M_i^\alpha |K_{M_i - 1}| + M_i^\alpha D_{M_i} \right\}.$$

Proof. It is evident that

$$\sum_{j=1}^{n} A_{n-j}^{\alpha-1} D_j = \sum_{j=1}^{n_A M_A} A_{n-j}^{\alpha-1} D_j + \sum_{j=n_A M_A + 1}^{n} A_{n-j}^{\alpha-1} D_j = I + II. \tag{2}$$

Since [5] for $r \in \{0, \ldots, m_A - 1\}$

$$D_{j + r M_A} = \left(\sum_{q=0}^{r-1} \psi_{M_A}^q \right) D_{M_A} + \psi_{M_A}^r D_j,$$

then for I we write

$$\begin{aligned} I = {} & \sum_{r=0}^{n_A - 1} \sum_{j=1}^{M_A} A_{n-j-rM_A}^{\alpha-1} \left(\sum_{q=0}^{r-1} \psi_{M_A}^q \right) D_{M_A} \\ & + \sum_{r=0}^{n_A - 1} \left(\sum_{j=0}^{M_A - 1} A_{(n_A - r - 1) M_A + n^{(A-1)} + j}^{\alpha-1} D_{M_A - j} \right) \psi_{M_A}^r = I_1 + I_2. \end{aligned}$$

It is evident that

$$|I_1| \le c(\alpha) M_A^\alpha D_{M_A}. \tag{3}$$

Using Lemma 1, for I_2 we obtain

$$|I_2| \le c(\alpha) \left\{ M_A^\alpha D_{M_A} + \sum_{r=0}^{n_A - 1} \left| \sum_{j=1}^{M_A - 1} A_{(n_A - r - 1) M_A + n^{(A-1)} + j}^{\alpha-1} \overline{D}_j \right| \right\}. \tag{4}$$

Since

$$D_{j+n_AM_A}=D_{n_AM_A}+\psi_{n_AM_A}D_j,$$

we write

$$|II|\le c(\alpha)\left\{M_A^{\alpha}D_{M_A}+n^{(A-1)}\left|K^{\alpha}_{n^{(A-1)}}\right|\right\}. \tag{5}$$

Combining (2)–(5), we obtain

$$n\left|K_n^{\alpha}\right|\le c(\alpha)\left\{M_A^{\alpha}D_{M_A}+\sum_{r=0}^{n_A-1}\left|\sum_{j=1}^{M_A-1}A^{\alpha-1}_{(n_A-r-1)M_A+n^{(A-1)}+j}\overline{D}_j\right|\right.$$
$$\left.+\,n^{(A-1)}\left|K^{\alpha}_{n^{(A-1)}}\right|\right\}.$$

Iterating this inequality, we obtain

$$n\left|K_n^{\alpha}\right|\le c(\alpha)\left\{\sum_{i=0}^{A}M_i^{\alpha}D_{M_i}+\sum_{i=0}^{A}\sum_{r=0}^{n_i-1}\left|\sum_{j=1}^{M_i-1}A^{\alpha-1}_{(n_i-r-1)M_i+n^{(i-1)}+j}\overline{D}_j\right|\right\}.$$

Applying Abel's transformation, we write

$$\sum_{j=1}^{M_i-1}A^{\alpha-1}_{(n_i-r-1)M_i+n^{(i-1)}+j}\overline{D}_j=-\sum_{j=1}^{M_i-2}A^{\alpha-2}_{(n_i-r-1)M_i+n^{(i-1)}+j}j\overline{K}_j$$
$$+A^{\alpha-1}_{(n_i-r)M_i+n^{(i-1)}-1}\left(M_i-1\right)K_{M_i-1},$$

consequently,

$$n\left|K_n^{\alpha}\right|\le c(\alpha)\left\{\sum_{i=0}^{A}M_i^{\alpha}D_{M_i}+\sum_{i=0}^{A}\sum_{p=1}^{i}M_p^{\alpha-1}\sum_{j=M_{p-1}}^{M_p-1}\left|K_j\right|+\sum_{i=0}^{A}M_i^{\alpha}\left|K_{M_i-1}\right|\right\}.$$

Lemma 2 is proved. □

Lemma 3. *Let $A\ge k$. Then*

$$\int_{\bar{I}_k}\sup_{n\ge A}\left|K_{M_n}\right|\le c\frac{M_k}{M_A}.$$

Proof. Since [9]

$$\left|K_{M_A}(x)\right|\le cM_s\sum_{x_s=1}^{m_s-1}\mathbf{1}_{I_n(0)+e_sx_s}(x),\ \ x\in I_s\backslash I_{s+1},\ \ s=0,\dots,n-1,$$

where $\mathbf{1}_E$ is the characteristic function of a set E and $e_s:=(0,\dots,0,1,0,\dots)\in G$, the s-th coordinate of which is 1 and the rest are zeros, we obtain

$$\int_{\bar{I}_k}\sup_{n\ge A}\left|K_{M_n}\right|\le\sum_{n=A}^{\infty}\int_{\bar{I}_k}\left|K_{M_n}\right|=\sum_{n=A}^{\infty}\sum_{s=0}^{k-1}\int_{I_{s+1}\backslash I_s}\left|K_{M_n}\right|$$

$$\leq c\sum_{n=A}^{\infty}\sum_{s=0}^{k-1}M_s\sum_{x_s=1}^{m_s-1}\int\limits_{I_{s+1}\setminus I_s}\mathbf{1}_{I_n(0)+e_sx_s}\leq c\sum_{n=A}^{\infty}\frac{1}{M_n}\sum_{s=0}^{k-1}M_s\leq c\frac{M_k}{M_n}.$$

Lemma 3 is proved. □

Lemma 4. *Let $A\geq k$. Then*

$$\int\limits_{\overline{I}_k}\sup_{n\geq M_A}|K_n|\leq c\frac{M_k\left(A-k+1\right)}{M_A}.$$

Proof. Since

$$|K_n|\leq c\sum_{j=0}^{A}\frac{M_j}{M_A}\left|K_{M_j}\right|\quad\text{for}\quad M_A\leq n<M_{A+1},$$

by Lemma 3 and the fact that [1]

$$\sup_{n\geq 1}\int\limits_{G_m}|K_n|<\infty, \tag{6}$$

we obtain

$$\int\limits_{\overline{I}_k}\sup_{n\geq M_A}|K_n|\leq\sum_{v=A}^{\infty}\sum_{j=0}^{v}\frac{M_j}{M_v}\int\limits_{\overline{I}_k}\left|K_{M_j}\right|$$

$$=\sum_{v=A}^{\infty}\sum_{j=0}^{k}\frac{M_j}{M_v}\int\limits_{\overline{I}_k}\left|K_{M_j}\right|+\sum_{v=A}^{\infty}\sum_{j=k+1}^{v}\frac{M_j}{M_v}\int\limits_{\overline{I}_k}\left|K_{M_j}\right|$$

$$\leq c\left\{\frac{M_k}{M_A}+\sum_{v=A}^{\infty}\frac{M_k\left(v-k+1\right)}{M_v}\right\}\leq c\frac{M_k\left(A-k+1\right)}{M_A}.$$

Lemma 4 is proved. □

Lemma 5. *Let $\alpha\in(0,1)$ and $A\geq k$. Then*

$$\int\limits_{\overline{I}_k}\sup_{n\geq M_A}|K_n^{\alpha}|\leq c\left(\alpha\right)\frac{A-k+1}{\left(M_A/M_k\right)^{\alpha}}.$$

Proof. From Lemma 2 we get

$$\int\limits_{\overline{I}_k}\sup_{n\geq M_A}|K_n^{\alpha}|\leq c\left(\alpha\right)\int\limits_{\overline{I}_k}\sup_{N\geq A}\frac{1}{M_N^{\alpha}}\sum_{i=0}^{N}\sum_{p=1}^{i}M_p^{\alpha-1}\sum_{j=M_{p-1}}^{M_p-1}\left|K_j\right|$$

$$+\,c\left(\alpha\right)\int\limits_{\overline{I}_k}\sup_{N\geq A}\frac{1}{M_N^{\alpha}}\sum_{i=0}^{N}M_i^{\alpha}\left|K_{M_i-1}\right|$$

$$+c(\alpha)\int\limits_{\bar{I}_k}\sup_{N\geq A}\frac{1}{M_N^\alpha}\sum_{i=0}^{N}M_i^\alpha D_{M_i}=I+II+III. \tag{7}$$

From (1), (6) and Lemma 4 we obtain

$$III\leq c(\alpha)\sum_{N=A}^{\infty}\frac{1}{M_N^\alpha}\sum_{i=0}^{k}M_i^\alpha\int\limits_{\bar{I}_k}D_{M_i}\leq c(\alpha)\frac{M_k^\alpha}{M_A^\alpha}, \tag{8}$$

$$\begin{aligned}
II&\leq c(\alpha)\sum_{N=A}^{\infty}\frac{1}{M_N^\alpha}\sum_{i=0}^{N}M_i^\alpha\int\limits_{\bar{I}_k}|K_{M_i-1}|\\
&=c(\alpha)\sum_{N=A}^{\infty}\frac{1}{M_N^\alpha}\sum_{i=0}^{k}M_i^\alpha\int\limits_{\bar{I}_k}|K_{M_i-1}|\\
&\quad+c(\alpha)\sum_{N=A}^{\infty}\frac{1}{M_N^\alpha}\sum_{i=k+1}^{N}M_i^\alpha\int\limits_{\bar{I}_k}|K_{M_i-1}|\\
&\leq c(\alpha)\frac{M_k^\alpha}{M_A^\alpha}+c(\alpha)\sum_{N=A}^{\infty}\frac{1}{M_N^\alpha}\sum_{i=k+1}^{N}M_i^\alpha\frac{(i-k)M_k}{M_i}\leq c(\alpha)\frac{M_k^\alpha}{M_A^\alpha}, \qquad (9)\\
I&\leq c(\alpha)\int\limits_{\bar{I}_k}\sup_{N\geq A}\frac{1}{M_N^\alpha}\sum_{i=0}^{k}\sum_{p=1}^{i}M_p^{\alpha-1}\sum_{j=M_{p-1}}^{M_p-1}|K_j|\\
&\quad+c(\alpha)\int\limits_{\bar{I}_k}\sup_{N\geq A}\frac{1}{M_N^\alpha}\sum_{i=k+1}^{N}\sum_{p=1}^{k}M_p^{\alpha-1}\sum_{j=M_{p-1}}^{M_p-1}|K_j|\\
&\quad+c(\alpha)\int\limits_{\bar{I}_k}\sup_{N\geq A}\frac{1}{M_N^\alpha}\sum_{i=k+1}^{N}\sum_{p=k+1}^{i}M_p^{\alpha-1}\sum_{j=M_{p-1}}^{M_p-1}|K_j|\\
&=I_1+I_2+I_3; \qquad (10)\\
I_1&\leq c(\alpha)\frac{M_k^\alpha}{M_A^\alpha}, \qquad (11)\\
I_2&\leq c(\alpha)\frac{M_k^\alpha}{M_A^\alpha}(A-k+1), \qquad (12)\\
I_3&\leq c(\alpha)\sum_{N=A}^{\infty}\frac{1}{M_N^\alpha}\sum_{i=k+1}^{N}\sum_{p=k+1}^{i}M_p^\alpha\int\limits_{\bar{I}_k}\sup_{l\geq M_{p-1}}|K_l|\\
&\leq c(\alpha)\sum_{N=A}^{\infty}\frac{1}{M_N^\alpha}\sum_{i=k+1}^{N}\sum_{p=k+1}^{i}M_p^\alpha\frac{p-k}{(M_p/M_k)}
\end{aligned}$$

$$\leq c(\alpha)\sum_{N=A}^{\infty}\frac{N-k+1}{(M_N/M_k)^{\alpha}}\leq c(\alpha)\frac{A-k+1}{(M_A/M_k)^{\alpha}}. \tag{13}$$

Combining (7)–(13), we complete the proof of Lemma 5. □

Lemma 6. *Let* $\alpha\in(0,1)$ *and* $n=n_AM_A+\cdots+n_0M_0$. *Then*

$$|T_n^{\alpha}|\leq c(\alpha)\sum_{i=1}^{10}B_i,$$

where

$$B_1=\frac{1}{n^{\alpha}}\sum_{s=0}^{A+1}\sum_{r=0}^{s-1}M_r^{\alpha-1}\sum_{j=M_r}^{M_{r+1}-1}\left|T_j^1\right|,$$

$$B_2=\frac{1}{n^{\alpha}}\sum_{s=0}^{A}M_s^{\alpha}\left|T_{n_sM_s}^1\right|,$$

$$B_3=\frac{1}{n^{\alpha}}\sum_{s=0}^{A+1}|D_{n_sM_s}^1|\sum_{r=0}^{s-1}M_r^{\alpha-1}\sum_{j=M_r}^{M_{r+1}-1}\left|K_j^{1,2}\right|,$$

$$B_4=\frac{1}{n^{\alpha}}\sum_{s=0}^{A}|D_{n_sM_s}^1|M_s^{\alpha}\left|K_{n_sM_s}^{1,2}\right|,$$

$$B_5=\frac{1}{n^{\alpha}}\sum_{s=0}^{A+1}|D_{n_sM_s}^2|\sum_{r=0}^{s-1}M_r^{\alpha-1}\sum_{j=M_r}^{M_{r+1}-1}\left|K_j^{1,1}\right|,$$

$$B_6=\frac{1}{n^{\alpha}}\sum_{s=0}^{A}|D_{n_sM_s}^2|M_s^{\alpha}\left|K_{n_sM_s}^{1,1}\right|,$$

$$B_7=\frac{1}{n^{\alpha}}\sum_{s=0}^{A}M_s^{\alpha}|D_{n_sM_s,n_sM_s}|,$$

$$B_8=\frac{1}{n^{\alpha}}\sum_{s=1}^{A}|D_{n_sM_s}^2|A_{n^{(s-1)}}^{\alpha}\left|K_{n^{(s-1)}}^{\alpha,1}\right|,$$

$$B_9=\frac{1}{n^{\alpha}}\sum_{s=1}^{A}|D_{n_sM_s}^1|A_{n^{(s-1)}}^{\alpha}\left|K_{n^{(s-1)}}^{\alpha,2}\right|,$$

$$B_{10}\leq c(\alpha).$$

Proof. It is evident that

$$A_n^{\alpha}T_n^{\alpha}=\sum_{j=1}^{n_AM_A}A_{n-j}^{\alpha-1}D_{j,j}+\sum_{j=n_AM_A+1}^{n}A_{n-j}^{\alpha-1}D_{j,j}$$

$$=\sum_{j=0}^{n_AM_A-1}A_{n^{(A-1)}+j}^{\alpha-1}D_{n_AM_A-j,n_AM_A-j}+\sum_{j=1}^{n^{(A-1)}}A_{n^{(A-1)}-j}^{\alpha-1}D_{j+n_AM_A,j+n_AM_A}.$$

Since (see Lemma 1)

$$D_{n_AM_A-j,n_AM_A-j}=D_{n_AM_A,n_AM_A}-D^1_{n_AM_A}\psi^2_{n_AM_A-1}\overline{D}^2_j$$
$$-D^2_{n_AM_A}\psi^1_{n_AM_A-1}\overline{D}^1_j+\psi^1_{n_AM_A-1}\psi^2_{n_AM_A-1}\overline{D}_{j,j}$$

and

$$D_{j+n_AM_A,j+n_AM_A}=D_{n_AM_A,n_AM_A}+D^1_{n_AM_A}\psi^2_{n_AM_A}D^2_j$$
$$+D^2_{n_AM_A}\psi^1_{n_AM_A}D^1_j+\psi^1_{n_AM_A}\psi^2_{n_AM_A}D_{j,j},$$

we get

$$A^\alpha_nT^\alpha_n=A^\alpha_{n^{(A)}-1}D_{n_AM_A,n_AM_A}$$
$$-\left(\sum_{j=0}^{n_AM_A-1}A^{\alpha-1}_{n^{(A-1)}+j}\overline{D}^2_j\right)D^1_{n_AM_A}\psi^2_{n_AM_A-1}$$
$$-\left(\sum_{j=0}^{n_AM_A-1}A^{\alpha-1}_{n^{(A-1)}+j}\overline{D}^1_j\right)D^2_{n_AM_A}\psi^1_{n_AM_A-1}$$
$$+\left(\sum_{j=0}^{n_AM_A-1}A^{\alpha-1}_{n^{(A-1)}+j}\overline{D}_{j,j}\right)\psi^1_{n_AM_A-1}\psi^2_{n_AM_A-1}$$
$$+A^\alpha_{n^{(A-1)}}K^{\alpha,2}_{n^{(A-1)}}D^1_{n_AM_A}\psi^2_{n_AM_A}+A^\alpha_{n^{(A-1)}}K^{\alpha,1}_{n^{(A-1)}}D^2_{n_AM_A}\psi^1_{n_AM_A}$$
$$+\psi^1_{n_AM_A}\psi^2_{n_AM_A}A^\alpha_{n^{(A-1)}}T^\alpha_{n^{(A-1)}}.$$

Consequently,

$$A^\alpha_n\left|T^\alpha_n\right|\le A^\alpha_{n^{(A)}-1}\left|D_{n_AM_A,n_AM_A}\right|+\left|\sum_{j=0}^{n_AM_A-1}A^{\alpha-1}_{n^{(A-1)}+j}\overline{D}^2_j\right|\left|D^1_{n_AM_A}\right|$$
$$+\left|\sum_{j=0}^{n_AM_A-1}A^{\alpha-1}_{n^{(A-1)}+j}\overline{D}^1_j\right|D^2_{n_AM_A}+\left|\sum_{j=0}^{n_AM_A-1}A^{\alpha-1}_{n^{(A-1)}+j}\overline{D}_{j,j}\right|$$
$$+A^\alpha_{n^{(A-1)}}\left|K^{\alpha,2}_{n^{(A-1)}}\right|\left|D^1_{n_AM_A}\right|+A^\alpha_{n^{(A-1)}}\left|K^{\alpha,1}_{n^{(A-1)}}\right|\left|D^2_{n_AM_A}\right|+A^\alpha_{n^{(A-1)}}\left|T^\alpha_{n^{(A-1)}}\right|.$$

Iterating this inequality, we obtain

$$A^\alpha_n\left|T^\alpha_n\right|\le\sum_{s=1}^{A}\Bigg\{A^\alpha_{n^{(s)}-1}\left|D_{n_sM_s,n_sM_s}\right|+\left|\sum_{j=0}^{n_sM_s-1}A^{\alpha-1}_{n^{(s-1)}+j}\overline{D}^2_j\right|\left|D^1_{n_sM_s}\right|$$
$$+\left|\sum_{j=0}^{n_sM_s-1}A^{\alpha-1}_{n^{(s-1)}+j}\overline{D}^1_j\right|\left|D^2_{n_sM_s}\right|+\left|\sum_{j=0}^{n_sM_s-1}A^{\alpha-1}_{n^{(s-1)}+j}\overline{D}_{j,j}\right|$$
$$+A^\alpha_{n^{(s-1)}}\left|K^{\alpha,2}_{n^{(s-1)}}\right|\left|D^1_{n_sM_s}\right|+A^\alpha_{n^{(s-1)}}\left|K^{\alpha,1}_{n^{(s-1)}}\right|\left|D^2_{n_sM_s}\right|\Bigg\}+A^\alpha_{n^{(0)}}\left|T^\alpha_{n^{(0)}}\right|. \tag{14}$$

Applying Abel's transformation, we write

$$\left|\sum_{j=0}^{n_sM_s-1}A^{\alpha-1}_{n^{(s-1)}+j}\overline{D}_{j,j}\right|$$

$$\leq c(\alpha)\left\{\sum_{j=0}^{n_sM_s-2}\left(n^{(s-1)}+j\right)^{\alpha-2}j\left|T_j^1\right|+\left(n^{(s)}\right)^{\alpha-1}\left(n^{(s)}-1\right)\left|T_{n_sM_s-1}^1\right|\right\}$$

$$\leq c(\alpha)\left\{\sum_{r=0}^{s}\sum_{j=M_r}^{M_{r+1}-1}\left(n^{(s-1)}+j\right)^{\alpha-2}j\left|T_j^1\right|+\left(n^{(s)}\right)^{\alpha}\left|T_{n_sM_s-1}^1\right|\right\}$$

$$\leq c(\alpha)\left\{\sum_{r=0}^{s}M_r^{\alpha-1}\sum_{j=M_r}^{M_{r+1}-1}\left|T_j^1\right|+\left(n^{(s)}\right)^{\alpha}\left|T_{n_sM_s-1}^1\right|\right\}. \qquad (15)$$

Analogously,

$$\left|\sum_{j=0}^{n_sM_s-1}A_{n^{(s-1)}+j}^{\alpha-1}\overline{D}_j^l\right|$$

$$\leq c(\alpha)\left\{\sum_{r=0}^{s}M_r^{\alpha-1}\sum_{j=M_r}^{M_{r+1}-1}\left|K_j^{1,l}\right|+\left(n^{(s)}\right)^{\alpha}\left|T_{n_sM_s-1}^1\right|\right\},\quad l=1,2. \qquad (16)$$

Combining (14)–(16) we complete the proof of Lemma 6. □

Corollary 2. *Let* $\alpha\in(0,1)$. *Then*

$$\sup_{n\geq 1}\int_{G_m^2}\left|T_n^{\alpha}\right|<\infty.$$

Proof. Since [1], [4]

$$\sup_{n\geq 1}\int_{G_m^2}\left|T_n^1\right|<\infty \qquad (17)$$

and

$$\sup_{n\geq 1}\int_{G_m}\left|K_n^{\alpha}\right|<\infty,\quad \alpha>0,$$

from Lemma 6 we obtain the proof of Corollary 2. □

Lemma 7. *Let* $\alpha\in(0,1)$. *Then*

$$\int_{I_k\times I_k}\sup_{n\geq M_k}\left|T_n^{\alpha}\right|\leq c(\alpha)<\infty.$$

Proof. Since in [4] one can find

$$\int_{I_k\times I_k}\sup_{n\geq M_k}\left|T_n^1\right|\leq c<\infty,$$

by (17) we can write

$$\int_{I_k\times I_k}\sup_{n\geq M_k}B_1\leq c(\alpha)\int_{I_k\times I_k}\sup_{A\geq k}\frac{1}{M_A^{\alpha}}\sum_{s=0}^{A+1}\sum_{r=0}^{s-1}M_r^{\alpha-1}\sum_{j=M_r}^{M_{r+1}-1}\left|T_j^1\right|$$

$$\leq c(\alpha)\int\limits_{I_k\times I_k}\sup_{A\geq k}\frac{1}{M_A^{\alpha}}\sum_{s=0}^{k}\sum_{r=0}^{s-1}M_r^{\alpha-1}\sum_{j=M_r}^{M_{r+1}-1}\left|T_j^1\right|$$

$$+c(\alpha)\int\limits_{I_k\times I_k}\sup_{A\geq k}\frac{1}{M_A^{\alpha}}\sum_{s=k+1}^{A+1}\sum_{r=0}^{k-1}M_r^{\alpha-1}\sum_{j=M_r}^{M_{r+1}-1}\left|T_j^1\right|$$

$$+c(\alpha)\int\limits_{I_k\times I_k}\sup_{A\geq k}\frac{1}{M_A^{\alpha}}\sum_{s=k+1}^{A+1}\sum_{r=k}^{s-1}M_r^{\alpha-1}\sum_{j=M_r}^{M_{r+1}-1}\left|T_j^1\right|$$

$$\leq c(\alpha)\frac{1}{M_k^{\alpha}}\sum_{s=0}^{k}\sum_{r=0}^{s-1}M_r^{\alpha-1}\sum_{j=M_r}^{M_{r+1}-1}\int\limits_{I_k\times I_k}\left|T_j^1\right|$$

$$+c(\alpha)\frac{1}{M_k^{\alpha}}\sum_{r=0}^{k-1}M_r^{\alpha-1}\sum_{j=M_r}^{M_{r+1}-1}\int\limits_{I_k\times I_k}\left|T_j^1\right|$$

$$+c(\alpha)\int\limits_{I_k\times I_k}\sup_{A\geq k}\frac{1}{M_A^{\alpha}}\sum_{s=k+1}^{A+1}\sum_{r=k}^{s-1}M_r^{\alpha}\sup_{l\geq M_k}\left|T_l^1\right|\leq c(\alpha)<\infty. \tag{18}$$

Analogously, we obtain

$$\int\limits_{I_k\times I_k}\sup_{n\geq M_k}B_2<\infty. \tag{19}$$

We have

$$\int\limits_{\overline{I_k\times I_k}}\sup_{n\geq M_k}B_3=\int\limits_{\overline{I}_k\times\overline{I}_k}\sup_{n\geq M_k}B_3+\int\limits_{\overline{I}_k\times I_k}\sup_{n\geq M_k}B_3+\int\limits_{I_k\times\overline{I}_k}\sup_{n\geq M_k}B_3. \tag{20}$$

It is evident that

$$\int\limits_{\overline{I}_k\times I_k}\sup_{n\geq M_k}B_3\leq c(\alpha)\int\limits_{\overline{I}_k\times I_k}\sup_{A\geq k}\frac{1}{M_A^{\alpha}}\sum_{s=0}^{A+1}\left(\sum_{n_s=0}^{m_s-1}\left|D_{n_sM_s}^1\right|\right)\sum_{r=0}^{s-1}M_r^{\alpha-1}\sum_{j=M_r}^{M_{r+1}-1}\left|K_j^{1,2}\right|$$

$$\leq c(\alpha)\int\limits_{\overline{I}_k\times I_k}\sup_{A\geq k}\frac{1}{M_A^{\alpha}}\sum_{s=0}^{k}\left(\sum_{n_s=0}^{m_s-1}\left|D_{n_sM_s}^1\right|\right)\sum_{r=0}^{s-1}M_r^{\alpha-1}\sum_{j=M_r}^{M_{r+1}-1}\left|K_j^{1,2}\right|$$

$$\leq c(\alpha)<\infty. \tag{21}$$

Analogously,

$$\int\limits_{\overline{I}_k\times\overline{I}_k}\sup_{n\geq M_k}B_3<\infty. \tag{22}$$

We write

$$\int\limits_{I_k\times\overline{I}_k}\sup_{n\geq M_k}B_3\leq c(\alpha)\int\limits_{I_k\times\overline{I}_k}\sup_{A\geq k}\frac{1}{M_A^{\alpha}}\sum_{s=0}^{A+1}\left(\sum_{n_s=0}^{m_s-1}\left|D^1_{n_sM_s}\right|\right)\sum_{r=0}^{s-1}M_r^{\alpha-1}\sum_{j=M_r}^{M_{r+1}-1}\left|K_j^{1,2}\right|$$

$$\leq c(\alpha)\int\limits_{I_k\times\overline{I}_k}\sup_{A\geq k}\frac{1}{M_A^{\alpha}}\sum_{s=0}^{k}\left(\sum_{n_s=0}^{m_s-1}\left|D^1_{n_sM_s}\right|\right)\sum_{r=0}^{s-1}M_r^{\alpha-1}\sum_{j=M_r}^{M_{r+1}-1}\left|K_j^{1,2}\right|$$

$$+c(\alpha)\int\limits_{I_k\times\overline{I}_k}\sup_{A\geq k}\frac{1}{M_A^{\alpha}}\sum_{s=k+1}^{A+1}\left(\sum_{n_s=0}^{m_s-1}\left|D^1_{n_sM_s}\right|\right)\sum_{r=0}^{k-1}M_r^{\alpha-1}\sum_{j=M_r}^{M_{r+1}-1}\left|K_j^{1,2}\right|$$

$$+c(\alpha)\int\limits_{I_k\times\overline{I}_k}\sup_{A\geq k}\frac{1}{M_A^{\alpha}}\sum_{s=k+1}^{A+1}\left(\sum_{n_s=0}^{m_s-1}\left|D^1_{n_sM_s}\right|\right)\sum_{r=k}^{s-1}M_r^{\alpha-1}\sum_{j=M_r}^{M_{r+1}-1}\left|K_j^{1,2}\right|$$

$$=: M+N+R. \tag{23}$$

It is evident that

$$M\leq c(\alpha)<\infty, \tag{24}$$

$$N\leq c(\alpha)\sum_{A=k}^{\infty}\frac{M_k^{\alpha}}{M_A^{\alpha}}\sum_{s=k+1}^{A+1}\sum_{n_s=0}^{m_s-1}\int\limits_{I_k}\left|D^1_{n_sM_s}\right|$$

$$\leq c(\alpha)\sum_{A=k}^{\infty}\frac{A-k+1}{\left(M_A/M_k\right)^{\alpha}}\leq c(\alpha)<\infty. \tag{25}$$

Using Lemma 4 and (1), for R we get

$$R\leq c(\alpha)\sum_{A=k}^{\infty}\frac{1}{M_A^{\alpha}}\sum_{s=k+1}^{A+1}\sum_{n_s=0}^{m_s-1}\sum_{r=k}^{s-1}M_r^{\alpha}\int\limits_{I_k\times\overline{I}_k}\left|D^1_{n_sM_s}\right|\sup_{l\geq M_r}\left|K_l^{1,2}\right|$$

$$\leq c(\alpha)\sum_{A=k}^{\infty}\frac{1}{M_A^{\alpha}}\sum_{s=k+1}^{A+1}\sum_{r=k}^{s-1}M_r^{\alpha}\frac{r-k+1}{(M_r/M_k)}$$

$$\leq c(\alpha)\sum_{A=k}^{\infty}\frac{M_k^{\alpha}}{M_A^{\alpha}}\sum_{s=k+1}^{A+1}\sum_{r=k}^{s-1}\frac{r-k+1}{\left(M_r/M_k\right)^{1-\alpha}}$$

$$\leq c(\alpha)\sum_{A=k}^{\infty}\frac{M_k^{\alpha}}{M_A^{\alpha}}(A-k+1)\leq c(\alpha)<\infty. \tag{26}$$

Combining (23)–(26), we obtain

$$\int\limits_{I_k\times\overline{I}_k}\sup_{n\geq M_k}B_3<\infty. \tag{27}$$

After substituting (21), (22) and (27) into (20), we have

$$\int\limits_{\overline{I_k\times I_k}} \sup_{n\geq M_k} B_3 < \infty. \tag{28}$$

Analogously, we obtain

$$\int\limits_{\overline{I_k\times I_k}} \sup_{n\geq M_k} B_j < \infty, \quad j=4,5,6,7. \tag{29}$$

For B_9 we write

$$\int\limits_{\overline{I_k\times I_k}} \sup_{n\geq M_k} B_9 = \int\limits_{\overline{I}_k\times\overline{J}_k} \sup_{n\geq M_k} B_9 + \int\limits_{\overline{I}_k\times I_k} \sup_{n\geq M_k} B_9 + \int\limits_{I_k\times\overline{I}_k} \sup_{n\geq M_k} B_9. \tag{30}$$

Using Lemma 2, we have

$$\begin{aligned}
\int\limits_{\overline{I}_k\times I_k} \sup_{n\geq M_k} B_9 &\leq c(\alpha) \int\limits_{\overline{I}_k\times I_k} \sup_{A\geq k} \frac{1}{M_A^\alpha} \sum_{s=1}^{A} \left(\sum_{n_s=0}^{m_s-1} \left|D^1_{n_sM_s}\right|\right) \sup_{1\leq n\leq M_s} A_n^\alpha \left|K_n^{\alpha,2}\right| \\
&\leq c(\alpha) \frac{1}{M_k^\alpha} \sum_{s=1}^{k} \int\limits_{G_m} \sup_{1\leq n\leq M_s} A_n^\alpha \left|K_n^{\alpha,2}\right| \\
&\leq c(\alpha) \frac{1}{M_k^\alpha} \sum_{s=1}^{k} \sum_{i=0}^{s} \left\{ \sum_{p=1}^{i} M_p^{\alpha-1} \sum_{j=M_p-1}^{M_p-1} \int\limits_{G_m} \left|K_j^{1,2}\right| \right. \\
&\left. + M_i^\alpha \int\limits_{G_m} \left|K^{1,2}_{M_i-1}\right| + M_i^\alpha \int\limits_{G_m} \left|D_{M_i}\right| \right\} \leq c(\alpha) < \infty.
\end{aligned} \tag{31}$$

Analogously, we obtain

$$\int\limits_{I_k\times\overline{I}_k} \sup_{n\geq M_k} B_9 < \infty. \tag{32}$$

By Lemmas 2, 5 and (1) we have

$$\begin{aligned}
\int\limits_{I_k\times\overline{I}_k} \sup_{n\geq M_k} B_9 &\leq c(\alpha) \int\limits_{I_k\times\overline{I}_k} \sup_{A\geq k} \frac{1}{M_A^\alpha} \left(\sum_{s=k}^{A} \sum_{n_s=0}^{m_s-1} \left|D^1_{n_sM_s}\right|\right) \sum_{a=0}^{A} M_a^\alpha \sup_{|q|=a} \left|K_q^{\alpha,2}\right| \\
&+ c(\alpha) \int\limits_{I_k\times\overline{I}_k} \sup_{A\geq k} \frac{1}{M_A^\alpha} \sum_{s=1}^{k-1} \left|D^1_{n_sM_s}\right| A^\alpha_{n^{(s-1)}} \left|K^{\alpha,2}_{n^{(s-1)}}\right| \\
&\leq c(\alpha) \sum_{A=k}^{\infty} \frac{1}{M_A^\alpha} \sum_{a=0}^{k-1} \int\limits_{I_k\times\overline{I}_k} \sum_{s=k}^{A} \left(\sum_{n_s=0}^{m_s-1} \left|D^1_{n_sM_s}\right|\right) M_a^\alpha \sup_{|q|=a} \left|K_q^{\alpha,2}\right|
\end{aligned}$$

$$+c(\alpha)\sum_{A=k}^{\infty}\frac{1}{M_A^{\alpha}}\sum_{a=k}^{A}\int\limits_{I_k\times\overline{I}_k}\sum_{s=k}^{A}\left(\sum_{n_s=0}^{m_s-1}\left|D^1_{n_sM_s}\right|\right)M_a^{\alpha}\sup_{|q|=a}\left|K_q^{\alpha,2}\right|$$

$$+c(\alpha)\int\limits_{I_k\times\overline{I}_k}\frac{1}{M_k^{\alpha}}\sum_{s=1}^{k-1}\left(\sum_{n_s=0}^{m_s-1}\left|D^1_{n_sM_s}\right|\right)\sup_{1\le l\le M_s}A_l^{\alpha}\left|K_l^{\alpha,2}\right|$$

$$\le c(\alpha)\left\{\sum_{A=k}^{\infty}\frac{A-k+1}{M_A^{\alpha}}\sum_{a=0}^{k-1}M_a^{\alpha}+\sum_{A=k}^{\infty}\frac{A-k+1}{M_A^{\alpha}}\sum_{a=k}^{A}M_a^{\alpha}\frac{a-k+1}{(M_a/M_k)^{\alpha}}\right.$$

$$\left.+\frac{1}{M_k^{\alpha}}\sum_{s=1}^{k-1}\sum_{i=0}^{s}\left\{\sum_{p=1}^{i}M_p^{\alpha}+M_i^{\alpha}\right\}\right\}\le c(\alpha)<\infty. \tag{33}$$

By virtue of (30)–(33) we have

$$\int\limits_{\overline{I_k\times I_k}}\sup_{n\ge M_k}B_9\le c(\alpha)<\infty. \tag{34}$$

Analogously, we get

$$\int\limits_{\overline{I_k\times I_k}}\sup_{n\ge M_k}B_8\le c(\alpha)<\infty. \tag{35}$$

Combining (18), (19), (28), (29), (34) and (35), by Lemma 6 we complete the proof of Lemma 7. □

Proof of Theorem 1. For $\alpha\ge 1$ Theorem 1 is proved in [4]. Hence it can be assumed that $0<\alpha<1$. As a consequence of Corollary 2 we have that the maximal operator $t_*^{\alpha}f:=\sup_{n\in\mathbb{N}}|t_n^{\alpha}f|$ is of type (∞,∞). Since this sublinear operator is quasi-local (this is what Lemma 4 means), then by standard arguments (see, e.g., the book [10]) it follows that it is of weak type $(1,1)$. That is, the proof of Theorem 1 is complete. □

Proof of Corollary 1. The set of Vilenkin polynomials is dense in $L^1(G_m^2)$, so by the well-known density argument we have that $t_n^{\alpha}f\to f$ a.e. for all integrable two-variable functions f. We remark that the Marcinkiewicz interpolation theorem (see, e.g., [10]) also gives that the maximal operator t_*^{α} is of type (p,p) for all $1<p\le\infty$. The proof of Corollary 1 is complete. □

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