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ON THE CONVERGENCE OF CONJUGATE TRIGONOMETRIC POLYNOMIALS

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Abstract. Sufficient conditions are found, under which for $f \in C([-\pi,\pi])$ or $f \in L([-\pi,\pi])$ the convergence of a sequence of trigonometric polynomials in the norms of these spaces implies the convergence of their conjugates in the same norms.

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Let X denote the space $C(T)$ or $L(T)$, $T=[-\pi,\pi]$,

$$\widetilde{X} = \{f : f \in X, \ \ \widetilde{f} \in X\},$$

where as usual $\widetilde{f}$ denotes the conjugate function of f. By $E_n(f)_X$ we denote the best approximation of the function f by trigonometric polynomials of order $\le n$ in the norm of the space X.

Let $f \in \widetilde{X}$ and let the sequence of trigonometric polynomials $\{T_n(X)\}$ converge to the function f with respect to the norm of the space X, i.e.,

$$\lim_{n\to\infty} \|f - T_n\|_X = 0. \tag{1}$$

The question arises about the least rate of convergence to zero of the sequence $\{\|f-T_n\|_X\}$ which yields

$$\lim_{n\to\infty} \|\widetilde{f} - \widetilde{T}_n\|_X = 0. \tag{2}$$

In the case when $f \in L_p(T)$,

$$\lim_{n\to\infty} \|f - T_n\|_{L_p} = 0 \ \Rightarrow \ \lim_{n\to\infty} \|\widetilde{f} - \widetilde{T}_n\|_X = 0.$$

This follows from the Riesz theorem (see [1, Ch. 8, §14]).

Consider also the case where as approximating polynomials of functions we take the triangular means of their Fourier series. Let the triangular matrix $\Lambda = (\lambda_{n,k})$, $\lambda_{n,k}=0$ if $k>n$, be given. The linear means of functions f and $\widetilde{f}$ formed by the matrix Λ have, respectively, the following form

$$U_n(f,\Lambda,x) = \frac{a_0}{2}\,\lambda_{n,0} + \sum_{k=1}^{n} \lambda_{n,k}(a_k \cos kx + b_k \sin kx), \tag{3}$$

$$U_n(\widetilde{f},\Lambda,x) = \sum_{k=1}^{n} \lambda_{n,k}(-b_k \cos kx + a_k \sin kx), \tag{4}$$

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where a_k and b_k are the Fourier coefficients of the function f.

There naturally arises a question what conditions the elements of the matrix Λ should satisfy so that the convergence

$$\lim_{n\to\infty} \|f - U_n(f,\Lambda)\|_X = 0$$

yield the convergence

$$\lim_{n\to\infty} \|\widetilde{f} - U_n(\widetilde{f},\Lambda)\|_X = 0.$$

This paper deals with the above problem.

Let (see [2, Ch. 3, §3])

$$K_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \left(1 - \frac{k}{n+1}\right) \cos kt = \frac{2}{n+1} \left(\frac{\sin(n+1)\frac{t}{2}}{2\sin\frac{t}{2}} \right)^2,$$

$$\widetilde{K}_n(t) = \sum_{k=1}^{n} \left(1 - \frac{k}{n+1}\right) \sin kt = \frac{(n+1)\sin nt - \sin(n+1)t}{(n+1)\bigl(2\sin\frac{t}{2}\bigr)^2},$$

be, respectively, the Fejer and the conjugate Fejer kernel. It is known (see [2, Ch. 3, §3] that

$$\int_T K_n(t)\,dt = O(1). \tag{5}$$

Besides, it is easy to prove that there exist positive absolute constants A_1 and A_2 such that

$$A_1 \ln(n+1) \le \int_T |\widetilde{K}_n(t)|\,dt \le A_2 \ln(n+1), \quad n = 1, 2, \dots. \tag{6}$$

Let

$$V_n(t) = 2K_{2n-1}(t) - K_{n-1}(t),$$
$$\widetilde{V}_n(t) = 2\widetilde{K}_{2n-1}(t) - \widetilde{K}_{n-1}(t)$$

be, respectively, the de la Vallée-Poussin kernel and the conjugate de la Vallée-Poussin kernel. From (5) and (6) we have

$$\begin{aligned} &\int_T |V_n(t)|\,dt = O(1), \\ &\int_T |\widetilde{V}_n(t)|\,dt = O\left(\ln(n+1)\right). \end{aligned} \tag{7}$$

The expressions

$$\tau_n(f,x) = \frac{1}{\pi} \int_T f(x+t) V_n(t)\,dt,$$

$$\widetilde{\tau}_n(f,x) = \frac{1}{\pi} \int_T f(x+t) \widetilde{V}_n(t)\,dt$$

are said to be, respectively, the de la Vallée-Poussin means and the conjugate de la Vallée-Poussin means.

Theorem (de la Vallé'-Poussin (see [2, Ch. 3, §13]).

$$\|f - \tau_n\|_X \le 4E_n(f)_X. \tag{8}$$

It should be noted that in [2, Ch. 3, §13] inequality (8) is proved for $X = C(T)$. The proof of inequality (8) when $X = L(T)$ is similar.

We have the following statement.

Theorem 1. a) *Let $f \in \widetilde{X}$. Then for any trigonometric polynomial $T_n(x)$ of degree $\le n$ the inequality*

$$\|\widetilde{f} - \widetilde{T}_n\|_X = O\left(E_n(\widetilde{f})_X + \|f - T_n\|_X \ln(n+1)\right) \tag{9}$$

holds.

b) *For any function $f \in \widetilde{X}$ there exists a sequence of trigonometric polynomials $\{T_n(x)\}$ such that*

$$\|f - T_n\|_X = O\left(E_n(f)_X + \frac{1}{\ln(n+1)}\right),$$

$$\|\widetilde{f} - \widetilde{T}_n\|_X \ge 1 - 4E_n(\widetilde{f})_X.$$

Proof. a) Since

$$\tau_n(T_n, x) = T_n(x)$$

and

$$\widetilde{\tau}_n(f, x) = \tau_n(\widetilde{f}, x),$$

we have

$$\widetilde{f}(x) - \widetilde{T}_n(x) = \widetilde{f}(x) - \tau_n(\widetilde{f}, x) + \widetilde{\tau}_n(f - T_n, x). \tag{10}$$

In view of the de la Vallée-Poussin theorem

$$\|\widetilde{f} - \tau_n(\widetilde{f})\|_X \le 4E_n(\widetilde{f})_X. \tag{11}$$

Moreover, we have

$$\widetilde{\tau}_n(f - T_n, x) = \frac{1}{\pi} \int\limits_T [f(x+t) - T_n(x+t)]\, \widetilde{V}_n(t)\, dt.$$

Hence, using (7), we get

$$\|\widetilde{\tau}_n(f - T_n)\|_X = O\left(\|f - T_n\|_X \ln(n+1)\right). \tag{12}$$

From (10)–(12) we get the validity of inequality (9).

b) First, let $X = C(T)$. For any $f \in \widetilde{C}(T)$ let us assume that

$$T_n(x) = \tau_n(f, x) + \frac{1}{\ln(n+1)} \sum_{k=1}^{n} \frac{\sin kx}{k}.$$

Since there exists an absolute constant $A_3 > 1$ such that

$$\left\| \sum_{k=1}^{n} \frac{\sin kx}{k} \right\|_C \le A_3$$

for all $n \in \mathbb{N}$ and $x \in T$, we have

$$\|f-T_n\|_C \le \|f-\tau_n(f)\|_C + \frac{1}{\ln(n+1)}\left\|\sum_{k=1}^{n}\frac{\sin kx}{k}\right\|_C = O\left(E_n(f)_C + \frac{1}{\ln(n+1)}\right),$$

it is clear that

$$\widetilde{T}_n(x) = \tau_n(\widetilde{f},x) - \frac{1}{\ln(n+1)}\sum_{k=1}^{n}\frac{\cos kx}{k}.$$

Therefore

$$\begin{aligned}\|\widetilde{f}-\widetilde{T}_n\|_C &= \left\|\widetilde{f}-\tau_n(\widetilde{f}) - \frac{1}{n+1}\sum_{k=1}^{n}\frac{\cos kx}{k}\right\|_C \\ &\ge \frac{1}{\ln(n+1)}\left\|\sum_{k=1}^{n}\frac{\cos kx}{k}\right\|_C - \|\widetilde{f}-\tau_n(\widetilde{f})\|_C. \end{aligned} \tag{13}$$

Using (11) and the fact that

$$\left\|\sum_{k=1}^{n}\frac{\cos kx}{k}\right\|_C = \sum_{k=1}^{n}\frac{1}{k} > \ln(n+1)$$

from (13) we get

$$\|\widetilde{f}-\widetilde{T}_n\|_C \ge 1 - 4E_n(\widetilde{f})_C.$$

Consider now the case where $X = L(T)$. Assume that

$$T_n(x) = \tau_n(f,x) + \frac{1}{A_1\ln(n+1)}K_n(x),$$

where $K_n(x)$ is the Fejer kernel and A_1 is the positive absolute constant from inequality (6). Using (7) and (8) we have

$$\begin{aligned}\|f-T_n\|_L &\le \|f-\tau_n(f)\|_L + \frac{\|K_n\|_L}{A_1\ln(n+1)} = O\left(E_n(f)_L + \frac{1}{\ln(n+1)}\right),\\ \|\widetilde{f}-\widetilde{T}_n\|_L &= \left\|\widetilde{f}-\tau_n(\widetilde{f}) - \frac{\widetilde{K}_n}{A_1\ln(n+1)}\right\|_L \\ &\ge \frac{1}{A_1}\|\widetilde{K}_n\|_L - 4E_n(\widetilde{f})_L = 1 - 4E_n(\widetilde{f})_L.\end{aligned}$$

Thus the validity of b) is proved. Note that in b) the order of $T_n(x)$ is $2n-1$. □

Theorem 1 yields

Corollary 1. *Let* $f \in \widetilde{X}$. *Then*

a) *if*

$$\lim_{n\to\infty}\|f-T_n\|_X \ln n = 0, \tag{14}$$

then

$$\lim_{n\to\infty}\|\widetilde{f}-\widetilde{T}_n\|_X = 0;$$

b) *for any function* $f \in \widetilde{X}$ *satisfying the condition*

$$E_n(f)_X = O\left(\frac{1}{\ln n}\right)$$

there exists a sequence of trigonometric polynomials such that

$$\|f - T_n\|_X = O\left(\frac{1}{\ln n}\right) \tag{15}$$

and

$$\varliminf_{n\to\infty} \|\widetilde{f} - \widetilde{T}_n\|_X \ge 1.$$

Thus for (1) to yield (2) it is sufficient to fulfill condition (14). The condition b) of the corollary shows that it is impossible to weaken condition (14), i.e., to replace condition (14) by condition (15).

Let s be some natural number, let $p_i \ge 1$, $i = 1, \dots, s$, be finite numbers and

$$P_{n,k}(s) = \prod_{i=1}^{s} \left(1 - \frac{k}{n + p_i}\right). \tag{16}$$

We denote by $U_n(f, P(s)\cdot\Lambda, x)$ (see (3), (4)) the triangular linear means of the function f formed by the matrix $P(s)\cdot\Lambda = (P_{n,k}(s)\lambda_{n,k})$. We have

Theorem 2. *Let* $f \in \widetilde{X}$ *and*

$$\lim_{n\to\infty} \|U_n(f, \Lambda) - f\|_X = 0. \tag{17}$$

If for some natural number r *and finite numbers* $p_i \ge 1$, $i = 1, \dots, r$,

$$\lim_{n\to\infty} \|U_n(\widetilde{f}, P(r)\cdot\Lambda) - \widetilde{f}\|_X = 0, \tag{18}$$

then

$$\lim_{n\to\infty} \|U_n(\widetilde{f}, \Lambda) - \widetilde{f}\|_X = 0. \tag{19}$$

Proof. First we prove that (17) implies the validity of the equalities

$$\lim_{n\to\infty} \|U_n(f, P(s)\cdot\Lambda) - f\|_X = 0, \quad s = 1, \dots, r. \tag{20}$$

From (17) it follows that for any $\varepsilon > 0$ there exists $N_\varepsilon = N$ such that for any $n > N$ we have

$$\|U_n(f, \Lambda) - U_N(f, \Lambda)\|_X < \frac{\varepsilon}{2}.$$

Hence from Bernstein's inequality we get

$$\frac{1}{n}\|\widetilde{U}_n'(f, \Lambda) - \widetilde{U}_N'(f, \Lambda)\|_X < \frac{\varepsilon}{2}, \quad n > N.$$

Besides, from the last two inequalities we have

$$\begin{aligned} \frac{1}{n}\|\widetilde{U}_n'(f, \Lambda)\|_X &\le \frac{1}{n}\|\widetilde{U}_n'(f, \Lambda) - \widetilde{U}_N'(f, \Lambda)\|_X + \frac{1}{n}\|\widetilde{U}_N'(f, \Lambda)\|_X \\ &\le \frac{\varepsilon}{2} + \frac{1}{n}\|\widetilde{U}_N'(f, \Lambda)\|_X. \end{aligned}$$

This inequality and (17) imply

$$\lim_{n\to\infty} \frac{1}{n+p_1}\, \|\widetilde{U}'_n(f,\Lambda)\|_X = 0. \tag{21}$$

Since

$$\frac{1}{n+p_1}\, \widetilde{U}'(f,\Lambda,x) = \frac{1}{n+p_1} \sum_{k=1}^{n} k\lambda_{n,k}(a_k \cos kx + b_k \sin kx),$$

from the last two inequalities and (17) we have

$$\lim_{n\to\infty} \|U_n(f, P(1)\cdot\Lambda) - f\|_X = 0. \tag{22}$$

Thus we get the validity of equality (20) where $s = 1$. Starting now from (22) we get the validity of (20) when $s = 2$ in the same way as we have obtained (22) from (17). Going on with the given argument we prove the validity of (20).

Let us show now that when $s = r-1$, (18) and (20) imply the validity of the equality

$$\lim_{n\to\infty} \|U_n(\widetilde{f}, P(r-1)\cdot\Lambda) - \widetilde{f}\|_X = 0. \tag{23}$$

From equality (20), assuming $s = r-1$ and using the Bernstein inequality like in the case of (21), we get

$$\lim_{n\to\infty} \frac{1}{n+p_r}\, \|U'_n(f, P(r-1)\cdot\Lambda)\|_X = 0.$$

Taking into consideration that

$$U'_n(f, P(r-1), x) = \sum_{k=1}^{n} k\lambda_{n,k} P(r-1)(-b_k \cos kx + a_k \sin kx),$$

from the last two inequalities and (18) we have

$$\lim_{n\to\infty} \|U_n(\widetilde{f}, P(r-1)\cdot\Lambda) - \widetilde{f}\|_X = 0. \tag{24}$$

Now starting from this equality and equality (20), when $s = r-2$, in the same way as from equality (20) when $s = r-1$ and equality (18) we have obtained (24), we can obtain the validity of

$$\lim_{n\to\infty} \|U_n(\widetilde{f}, P(r-2)\cdot\Lambda) - \widetilde{f}\|_X = 0.$$

Going on with the given reasoning, we get the validity of equality (19). □

Let $\sigma_n^{(\alpha)}(f,x)$ be Césaro means of order α of the function f.

Corollary 2. *If $\alpha \in (-\infty, 0]$, $\alpha \neq -1, -2, \dots,$ and*

$$\lim_{n\to\infty} \|f - \sigma_n^{(\alpha)}(f)\|_X = 0, \tag{25}$$

then

$$\lim_{n\to\infty} \|\widetilde{f} - \sigma_n^{(\alpha)}(\widetilde{f})\|_X = 0.$$

For $\alpha = 0$ this assertion is proved in [1, Ch. 8, §19, §22], while for $\alpha > -1$ in [4].

Proof. For any $\alpha \in (-\infty; 0]$ one can find natural r such that $\alpha+r > 0$. Therefore we have

$$\lim_{n\to\infty} \|\widetilde{f} - \sigma_n^{(\alpha+r)}(\widetilde{f})\|_X = 0. \tag{26}$$

Assume

$$\lambda_{n,k} = \frac{A_{n-k}^{(\alpha)}}{A_n^{(\alpha)}}, \quad p_i = \alpha + i.$$

In this case the means formed by the matrix Λ are Césaro means of order α, i.e.,

$$U_n(f, \Lambda, x) = \sigma_n^{(\alpha)}(f, x).$$

Besides, in this case

$$P_{n,k}(r) = \prod_{i=1}^{r} \left(1 - \frac{k}{n+\alpha+i}\right) = \prod_{i=1}^{r} \frac{n+\alpha+i-k}{n+\alpha+i}.$$

It is easy to verify that

$$\frac{A_{n-k}^{(\alpha)}}{A_n^{(\alpha)}} P_{n,k}(r) = \frac{A_{n-k}^{(\alpha+r)}}{A_n^{(\alpha+r)}}.$$

Therefore

$$U_n(\widetilde{f}, P(r) \cdot \Lambda, x) = \sigma_n^{(\alpha+r)}(\widetilde{f}, x).$$

Consequently, in the considered case conditions (17) and (18) of Theorem 2 become (25) and (26). Thus from Theorem 2 we get that the corollary is valid. □

Theorem 3. *Let $f \in \widetilde{X}$ and let equality* (17) *be fulfilled. If there exists an absolute constant A such that*

$$|\lambda_{n,k}| < A, \tag{27}$$

for any fixed k

$$\lim_{n\to\infty} \lambda_{n,k} = 1 \tag{28}$$

and for some natural r

$$\sum_{k=0}^{n-2} (k+1)|\Delta^2 \lambda_{n,k} P_{n,k}(r)| = O(1), \tag{29}$$

then equality (19) *be satisfied.*

Proof. By applying twice the Abel transformation we have

$$U_n(\widetilde{f}, P(r)\cdot\Lambda, x) - \widetilde{f}(x) = \sum_{k=1}^{n} \lambda_{n,k} P_{n,k}(r)(-b_k \sin kx) + a_k \cos kx) - \widetilde{f}(x)$$

$$= \sum_{k=1}^{n-1} \left(s_k(\widetilde{f}, x) - \widetilde{f}(x)\right) \Delta\lambda_{n,k} P_{n,k}(r) + \left(s_n(\widetilde{f}, x) - \widetilde{f}(x)\right) \lambda_{n,n} P_{n,n}(r)$$

$$= \sum_{k=1}^{n-2} (k+1) \left(\sigma_k(\widetilde{f}, x) - \widetilde{f}(x)\right) \Delta^2\lambda_{n,k} P_{n,k}(r)$$

$$+ (n-1) \left(\sigma_{n-1}(\widetilde{f}, x) - \widetilde{f}(x)\right) \Delta\lambda_{n,n-1} P_{n,n-1}(r)$$

$$+ \left(s_n(\widetilde{f}, x) - \widetilde{f}(x)\right) \lambda_{n,n} P_{n,n}(r).$$

Hence taking into account that

$$\begin{gathered} \|s_n(\widetilde{f}) - \widetilde{f}\|_X = O(\ln n), \\ \|\sigma_n(\widetilde{f}) - \widetilde{f}\|_X = o(1), \\ P_{n,n}(r) = O\left(\frac{1}{n}\right), \quad P_{n,n-1}(r) = O\left(\frac{1}{n}\right) \end{gathered} \tag{30}$$

and (27), (28), we get

$$\lim_{n\to\infty} \|U_n(\widetilde{f}, P(r)\cdot\Lambda) - \widetilde{f}\|_X \le \lim_{n\to\infty} \sum_{k=1}^{n} \|\sigma_k(\widetilde{f}) - \widetilde{f}\|_X (k+1)\Delta^2\lambda_{n,k} P_{n,k}(r). \tag{31}$$

It is well known (see [3, Ch. 5, §3.5]) that if for every k

$$\lim_{n\to\infty} (k+1)\Delta^2\lambda_{n,k} P_{n,k}(r) = 0 \tag{32}$$

and equality (29) is satisfied, then from (30) we get that

$$\lim_{n\to\infty} \sum_{k=1}^{n-2} \|\sigma_k(\widetilde{f}) - \widetilde{f}\|_X (k+1)\Delta^2\lambda_{n,k} P_{n,k}(r) = 0.$$

Therefore to complete the proof of the theorem (see (31)) it is sufficient to prove the validity of equality (32) for every fixed k. Since

$$\Delta\lambda_{n,k} P_{n,k}(r) = P_{n,k}(r)\Delta\lambda_{n,k} + \lambda_{n,k}\Delta P_{n,k}(r),$$

from (16), (27) and (28) we get

$$\lim_{n\to\infty} \Delta\lambda_{n,k} P_{n,k}(r) = 0$$

for any k. From the above equality follows the validity of (32). □

Corollary 3. *Let $f \in \widetilde{X}$ and let condition* (17) *be satisfied. If Λ is a regular matrix and for some natural r*

$$\sum_{k=0}^{n-2}(k+1)P_{n,k}(r)|\Delta^2\lambda_{n,k}| = O(1), \tag{33}$$

then equality (19) *holds.*

Proof. Since

$$\begin{aligned}\Delta^2\lambda_{n,k}P_{n,k}(r) &= \lambda_{n,k}\Delta^2 P_{n,k}(r) + \Delta P_{n,k-1}(r)\Delta\lambda_{n,k} \\ &\quad + \Delta\lambda_{n,k+1}\Delta P_{n,k+1} + P_{n,k}\Delta^2\lambda_{n,k} \\ &= I^{(1)}_{n,k} + I^{(2)}_{n,k} + I^{(3)}_{n,k} + I^{(4)}_{n,k}. \end{aligned} \tag{34}$$

It can be easily verified that

$$\Delta^2 P_{n,k}(r) = O\left(\frac{1}{n^2}\right).$$

This and (27) imply

$$\sum_{k=0}^{n-2}(k+1)|I^{(1)}_{n,k}| = O\left(\frac{1}{n^2}\right). \tag{35}$$

From the regularity of the matrix Λ we get

$$\sum_{k=0}^{n-2}|\Delta\lambda_{n,k}| = O(1).$$

Hence in view of

$$\Delta P_{n,k}(r) = O\left(\frac{1}{n}\right),$$

we obtain

$$\sum_{k=0}^{n-2}(k+1)|I^{(2)}_{n,k}| = O(1), \quad \sum_{k=0}^{n-2}(k+1)|I^{(3)}_{n,k}| = O(1). \tag{36}$$

Now from (33)–(36) we get (29) and thus the validity of the corollary. □

It is interesting to know whether the corollary without (33) will hold.

The main results of this paper were announced in [5].

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