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EXISTENCE OF SOLUTIONS FOR FOURTH ORDER NONLOCAL BOUNDARY VALUE PROBLEMS

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Abstract. Uniqueness implies existence results are obtained for solutions of the fourth order ordinary differential equation, $y^{(4)} = f(x, y, y', y'', y''')$, satisfying 5-point, 4-point and 3-point nonlocal boundary conditions.

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1. Introduction

We are concerned with existence of solutions of nonlocal boundary value problems for the fourth order ordinary differential equation,

$$y^{(4)} = f(x, y, y', y'', y'''), \quad a < x < b, \tag{1.1}$$

where

(A) $f : (a, b) \times \mathbb{R}^4 \to \mathbb{R}$ is continuous.
(B) Solutions of initial value problems for (1.1) are unique and exist on (a, b).

In particular, we deal with "uniqueness implies existence" questions for solutions of (1.1) satisfying nonlocal 5-point boundary conditions,

$$y(x_1) = y_1,\ y(x_2) = y_2,\ y(x_3) = y_3,\ y(x_4) - y(x_5) = y_4, \tag{1.2}$$

$$y(x_1) - y(x_2) = y_1,\ y(x_3) = y_2,\ y(x_4) = y_3,\ y(x_5) = y_4, \tag{1.3}$$

where $a < x_1 < x_2 < x_3 < x_4 < x_5 < b$, as well as for solutions of (1.1) satisfying nonlocal 4-point boundary conditions given by

$$y(x_1) = y_1,\ y'(x_1) = y_2,\ y(x_2) = y_3,\ y(x_3) - y(x_4) = y_4, \tag{1.4}$$

$$y(x_1) - y(x_2) = y_1,\ y(x_3) = y_2,\ y(x_4) = y_3,\ y'(x_4) = y_4, \tag{1.5}$$

and

$$y(x_1) = y_1,\ y(x_2) = y_2,\ y'(x_2) = y_3,\ y(x_3) - y(x_4) = y_4, \tag{1.6}$$

$$y(x_1) - y(x_2) = y_1,\ y(x_3) = y_2,\ y'(x_3) = y_3,\ y(x_4) = y_4, \tag{1.7}$$

where $a < x_1 < x_2 < x_3 < x_4 < b$, and finally for solutions of (1.1) satisfying nonlocal 3-point boundary conditions given by

$$y(x_1) = y_1,\ y'(x_1) = y_2,\ y''(x_1) = y_3,\ y(x_2) - y(x_3) = y_4, \tag{1.8}$$

$$y(x_1) - y(x_2) = y_1,\ y(x_3) = y_2, y'(x_3) = y_3,\ y''(x_3) = y_4, \tag{1.9}$$

where $a < x_1 < x_2 < x_3 < b$, and in each case $y_1, y_2, y_3, y_4 \in \mathbb{R}$.

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In a recent paper, the authors [18] examined uniqueness relationships among solutions of the above 5-point, 4-point and 3-point boundary value problems. The motivation for that paper was that such results often imply the existence of solutions for boundary value problems; see, for example [1, 2, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19, 20, 21, 24, 25, 27, 29, 30, 42]. Indeed, that is the question which is dealt with in this paper.

The literature is vast on fourth order nonlinear boundary value problems, and we cite [3, 5, 26, 31, 32, 36, 37, 38, 39, 41, 43] as a list for just a few of these papers dealing with both theoretical issues as well as application models. In addition, nonlocal boundary value problems have received a good deal of research attention. For a brief overview of some research devoted to nonlocal boundary value problems, we suggest the list of papers [6, 7, 8, 17, 22, 28, 33, 34, 35, 40, 45, 46].

2. Uniqueness Results for Nonlocal Problems and Existence Results for Conjugate Problems

In this section, we will state a result from [18] concerning uniqueness relationships among solutions of (1.1) satisfying the 5-point, 4-point and 3-point nonlocal boundary conditions. In addition, we will also make use of a uniqueness implies existence result for conjugate boundary value problems for (1.1). Conjugate boundary value problems for (1.1) involve, respectively, 4-point conjugate boundary conditions of the form,

$$y(x_1) = y_1,\ y(x_2) = y_2,\ y(x_3) = y_3,\ y(x_4) = y_4,$$

$a < x_1 < x_2 < x_3 < x_4 < b$, along with 3-point conjugate boundary conditions of the forms,

$$\begin{aligned} &y(x_1) = y_1,\ y'(x_1) = y_2,\ y(x_2) = y_3,\ y(x_3) = y_4,\\ &y(x_1) = y_1,\ y(x_2) = y_2,\ y'(x_2) = y_3,\ y(x_3) = y_4,\\ &y(x_1) = y_1,\ y(x_2) = y_2,\ y(x_3) = y_3,\ y'(x_3) = y_4,\end{aligned}$$

$a < x_1 < x_2 < x_3 < b$, as well as 2-point conjugate boundary conditions of the forms,

$$\begin{aligned} &y(x_1) = y_1,\ y'(x_1) = y_2,\ y''(x_1) = y_3,\ y(x_2) = y_4,\\ &y(x_1) = y_1,\ y'(x_1) = y_2,\ y(x_2) = y_3,\ y'(x_2) = y_4,\\ &y(x_1) = y_1,\ y(x_2) = y_2,\ y'(x_2) = y_3,\ y''(x_2) = y_4\end{aligned}$$

$a < x_1 < x_2 < b$, and in each case $y_1, y_2, y_3, y_4 \in \mathbb{R}$.

A part of the arguments for this paper relies on a "uniqueness implies existence" result of Hartman [11] and Klaasen [27] for conjugate boundary value problems for nth order differential equations. In the context of fourth order equation (1.1), we now state that theorem.

Theorem 2.1. *Assume that conditions* (A) *and* (B) *are satisfied. Assume that solutions of* 4*-point conjugate boundary value problems for* (1.1) *are unique* *on* (a, b) *when they exist. Then for* $k \in \{2, 3, 4\}$*, each* k*-point conjugate boundary value problem for* (1.1) *has a unique solution on* (a, b)*.*

We next state a theorem which summarizes the authors' results [18] concerning uniqueness relationships among the nonlocal boundary value problems for (1.1).

Theorem 2.2. *Assume conditions* (A) *and* (B) *are satisfied. Then solutions of both* (1.1), (1.2) *and* (1.1), (1.3) *are unique when they exist, if and only if solutions of* (1.1) *satisfying each of* $(1.j)$*,* $j = 4, \dots, 9$*, are unique when they exist.*

3. Uniqueness of 5-Point Implies Existence of 5-Point, 4-Point and 3-Point

In this section, we show that uniqueness of solutions of 5-point nonlocal boundary value problems for (1.1) implies existence of solutions for each of 5-point, 4-point and 3-point nonlocal boundary value problems. In addition to hypotheses (A) and (B), we will state as an hypothesis the uniqueness conditions on the 5-point nonlocal problems (1.1), (1.2) and (1.1), (1.3):

(C) Given $a < x_1 < x_2 < x_3 < x_4 < x_5 < b$, if $y(x)$ and $z(x)$ are two solutions of (1.1) satisfying

$$y(x_1) = z(x_1),\ y(x_2) = z(x_2),\ y(x_3) = z(x_3),\ y(x_4) - y(x_5) = z(x_4) - z(x_5),$$

then $y(x) = z(x),\ a < x < b$.

(D) Given $a < x_1 < x_2 < x_3 < x_4 < x_5 < b$, if $y(x)$ and $z(x)$ are two solutions of (1.1) satisfying

$$y(x_1) - y(x_2) = z(x_1) - z(x_2),\ y(x_3) = z(x_3),\ y(x_4) = z(x_4),\ y(x_5) = z(x_5),$$

then $y(x) = z(x),\ a < x < b$.

Remarks. **(a)** We note that, under assumption (C) or (D), in conjunction with Theorem 2.2, solutions of 4-point and 3-point nonlocal boundary value problems for (1.1) are also unique, when they exist.

(b) We next note that, under either assumption (C) or (D), solutions of 4-point "conjugate" boundary value problems for (1.1) are unique, when they exist. That is, if $y(x)$ and $z(x)$ are both solutions of (1.1) such that, for some points $a < t_1 < t_2 < t_3 < t_4 < b$, $y(t_i) = z(t_i)$, $i = 1, 2, 3, 4$, then by the Intermediate Value Theorem, there exist $t_1 < \tau_1 < \tau_2 < t_2 < t_3 < \sigma_1 < \sigma_2 < t_4$ such that, both $y(\tau_1) - y(\tau_2) = z(\tau_1) - z(\tau_2)$, $y(t_i) = z(t_i)$, $i = 2, 3, 4$, and $y(t_i) = z(t_i)$, $i = 1, 2, 3$, $y(\sigma_1) - y(\sigma_2) = z(\sigma_1) - z(\sigma_2)$. Namely, if either (C) or (D) holds, then $y(x) = z(x)$.

(c) As a follow-up to (b), if either (A), (B) and (C), or (A), (B) and (D) are assumed, then Theorem 2.1 implies that each k-point "conjugate" boundary value problem for (1.1), $k = 2, 3, 4$, has a unique solution.

Fundamental to the results of this section is the role of continuous dependence of solutions on boundary conditions. This continuous dependence arises somewhat from applications of the Brouwer Theorem on Invariance of Domain [44] in conjunction with continuous dependence of solutions on initial conditions. We present such a continuous dependence result for solutions of (1.1), (1.2). The proof is rather standard in the context of uniqueness properties of solutions with respect to both initial conditions and boundary conditions. So we will omit the details of the proof, but we suggest [2] and [23] as good references for typical arguments used in the proof.

Theorem 3.1. *Assume* (A), (B) *and* (C), *and let* $z(x)$ *be an arbitrary solution of* (1.1). *Then, for any* $a < x_1 < x_2 < x_3 < x_4 < x_5 < b$ *and* $a < c < x_1$, *and* $x_5 < d < b$, *and given any* $\epsilon > 0$, *there exists* $\delta(\epsilon, [c,d]) > 0$, *so that* $|x_i - t_i| < \delta$, $1 \le i \le 5$, $|z(x_i) - y_i| < \delta$, $i = 1,2,3$, *and* $|z(x_4) - z(x_5) - y_4| < \delta$ *imply that* (1.1) *has a solution* $y(x)$ *with*

$$y(t_i) = y_i, \quad i = 1,2,3,$$
$$y(t_4) - y(t_5) = y_4,$$

and $|y^{(i-1)}(x) - z^{(i-1)}(x)| < \epsilon$ *on* $[c,d]$, $i = 1,2,3,4$.

Also fundamental to our existence arguments will be applications of a precompactness condition on bounded sequences of solutions of (1.1). This precompactness condition is due to Jackson and Schrader; see Agarwal [1].

Theorem 3.2. *Assume that with respect to* (1.1), *conditions* (A) *and* (B) *hold. In addition, assume that solutions of* 4*-point conjugate boundary value problems are unique. If* $\{y_k(x)\}$ *is sequence of solutions of* (1.1) *for which there exists an interval* $[c,d] \subset (a,b)$ *and there exists an* $M > 0$ *such that* $|y_k(x)| < M$, *for all* $x \in [c,d]$ *and for all* $k \in \mathbb{N}$, *then there exists a subsequence* $\{y_{k_j}(x)\}$ *such that, for* $i = 0,1,2,3$, $\{y^{(i)}_{k_j}(x)\}$ *converges uniformly on each compact subinterval of* (a,b).

We now present a sequence of theorems exhibiting that uniqueness of solutions of (1.1), (1.2) implies existence of solutions for each of (1.1), (1.j), $j = 2,4,6,8$. Of course, dual results can be established in terms of uniqueness of solutions of (1.1), (1.3) implying the existence of solutions of (1.1), (1.j), $j = 3,5,7,9$.

Theorem 3.3. *Assume hypotheses* (A), (B) *and* (C) *are satisfied with respect to equation* (1.1). *Then, given* $a < x_1 < x_2 < x_3 < x_4 < x_5 < b$ *and* $y_1, y_2, y_3, y_4 \in \mathbb{R}$, *there exists a unique solution of* (1.1), (1.2) *on* (a,b).

Proof. Let $a < x_1 < x_2 < x_3 < x_4 < x_5 < b$ and $y_1, y_2, y_3, y_4 \in \mathbb{R}$ be selected. We note from Remark (c) that 4-point, 3-point, as well as 2-point, conjugate boundary value problems for (1.1) have unique solutions.

Let $z(x)$ be the solution of (1.1) satisfying the 4-point conjugate boundary conditions at $x_2,\ x_3,\ x_4$ and x_5,

$$z(x_2) = y_2,\ z(x_3) = y_3,\ z(x_4) = y_4,\ z(x_5) = 0.$$

Observe that $z(x_4) - z(x_5) = y_4$. Next, define the set

$$S = \{u(x_1) |\ u(x) \text{ is a solution of equation (1.1) satisfying}$$

$$u(x_2) = z(x_2),\ u(x_3) = z(x_3),\ u(x_4) - u(x_5) = z(x_4) - z(x_5)\}.$$

We observe first that S is nonempty, since $z(x_1) \in S$.

Next, choose $s_0 \in S$. Then, there is a solution $u_0(x)$ of (1.1) satisfying

$$\begin{aligned} u_0(x_1) &= s_0, \\ u_0(x_2) &= z(x_2), \\ u_0(x_3) &= z(x_3), \\ u_0(x_4) - u_0(x_5) &= z(x_4) - z(x_5). \end{aligned}$$

By the continuous dependence theorem, Theorem 3.1, there exists a $\delta > 0$ such that, for each $0 \le |s - s_0| < \delta$, there is a solution $u_s(x)$ of (1.1) satisfying

$$u_s(x_1) = s,\ u_s(x_2) = u_0(x_2) = z(x_2),\ u_s(x_3) = u_0(x_3) = z(x_3),$$

and

$$u_s(x_4) - u_s(x_5) = u_0(x_4) - u_0(x_5) = z(x_4) - z(x_5),$$

or in other words, $s \in S$; in particular, $(s_0 - \delta, s_0 + \delta) \subset S$, and so S is an open subset of $\mathbb{R}$.

The remainder of the argument is devoted to showing that S is also a closed subset of $\mathbb{R}$. To that end, we assume for the purpose of contradiction that S is not closed. Then there exists an $r_0 \in \overline{S} \backslash S$ and a strictly monotone sequence $\{r_k\} \subset S$ such that $\lim_{n\to\infty} r_k = r_0$.

We may assume, without loss of generality, that $r_k \uparrow r_0$. By the definition of S, we denote, for each $k \in \mathbb{N}$, by $u_k(x)$ the solution of equation (1.1) satisfying

$$\begin{aligned} u_k(x_1) = r_k,\ u_k(x_2) = z(x_2), \\ u_k(x_3) = z(x_3),\ u_k(x_4) - u_k(x_5) = z(x_4) - z(x_5). \end{aligned}$$

By hypothesis (C), we have for each $k \in \mathbb{N}$,

$$u_k(x) < u_{k+1}(x) \text{ on } (a, x_2).$$

We now claim that $\{u_k(x)\}$ is not uniformly bounded above on each compact subinterval of (a, x_1) and (x_1, x_2). Suppose there exists a subinterval $[c, d] \subset (a, x_1)$ so that $\{u_k(x)\}$ is uniformly bounded above on $[c, d]$. That is, there exists $H > 0$ so that $u_k(x) \le H$, for all $c \le x \le d$ and $k \ge 1$. In particular,

$$u_1(x) \le u_k(x) \le H \text{ for all } c \le x \le d \text{ and } k \ge 1.$$

But by the precompactness condition of Theorem 3.2, there exists a subsequence $\{u_{k_j}(x)\}$ such that $\{u_{k_j}^{(i)}(x)\}$ converges uniformly on each compact subinterval of (a, b), $i = 0, 1, 2, 3$.

Suppose $u_{k_j}^{(i)}(x) \to v^{(i)}(x)$ uniformly on the compact interval $[x_1, x_5]$, for $i = 0, 1, 2, 3$. Then $v(x)$ is a solution of equation (1.1) satisfying

$$\begin{aligned} v(x_1) = \lim_{j\to\infty} u_{k_j}(x_1) = \lim_{j\to\infty} r_{k_j} = r_0, \\ v(x_2) = z(x_2),\ v(x_3) = z(x_3), \end{aligned}$$

and

$$v(x_4) - v(x_5) = \lim_{j\to\infty} (u_{k_j}(x_4) - u_{k_j}(x_5)) = z(x_4) - z(x_5).$$

Therefore $r_0 \in S$. But this is contradictory to the assumption $r_0 \notin S$. So $\{u_k(x)\}$ is not uniformly bounded above on each compact subinterval of (a, x_1). The argument relative to (x_1, x_2) is exactly analogous.

Next let $w(x)$ be the solution of the 3-point conjugate boundary problem for equation (1.1) satisfying,

$$w(x_1) = r_0,\ w'(x_1) = 0,\ w(x_2) = y_2,\ w(x_3) = y_3.$$

It follows that, for some K large, there exists points $a < \tau_1 < x_1 < \tau_2 < x_2$ so that

$$u_K(\tau_1) = w(\tau_1),\ u_K(\tau_2) = w(\tau_2).$$

Also,

$$u_K(x_2) = z(x_2) = w(x_2),\ u_K(x_3) = z(x_3) = w(x_3).$$

By uniqueness of solutions of 4-point conjugate boundary value problems for (1.1), we have $u_K \equiv w$. However,

$$w(x_1) = r_0 > r_K = u_K(x_1),$$

which gives a contradiction. Thus S is also a closed subset of $\mathbb{R}$.

In summary, S is a nonempty subset of $\mathbb{R}$ that is both open and closed. Thus, we have $S = \mathbb{R}$. By choosing $y_1 \in S$, there is a corresponding solution $y(x)$ of equation (1.1) such that

$$y(x_1) = y_1,\ y(x_2) = z(x_2) = y_2,\ y(x_3) = z(x_3) = y_3,$$
$$y(x_4) - y(x_5) = z(x_4) - z(x_5) = y_4.$$

This completes the proof. □

We now turn to existence of solutions for 4-point and 3-point nonlocal boundary value problems for (1.1). We first address the 4-point problems.

Theorem 3.4. *Assume* (A), (B) *and* (C) *are satisfied with respect to* (1.1). *Given points* $a < x_1 < x_2 < x_3 < x_4 < b$, *and* $y_1, y_2, y_3, y_4 \in \mathbb{R}$, *there exists a unique solution of* (1.1), (1.4) *on* (a, b).

Proof. Let $a < x_1 < x_2 < x_3 < x_4 < b$, and $y_1, y_2, y_3, y_4 \in \mathbb{R}$ be selected. Also, fix $a < \tau < x_1$. We repeat that 4-point, 3-point, as well as 2-point, conjugate boundary value problems for (1.1) have unique solutions.

Let $z(x)$ be the solution of the nonlocal 5-point boundary value problem (1.1), (1.2) obtained in Theorem 3.3 and satisfying,

$$z(\tau) = 0,\ z(x_1) = y_1,\ z(x_2) = y_3,\ z(x_3) - z(x_4) = y_4.$$

This time, define the set

$$S = \{u'(x_1) |\ u(x) \text{ is a solution of (1.1) satisfying}$$
$$u(x_1) = z(x_1), u(x_2) = z(x_2), u(x_3) - u(x_4) = z(x_3) - z(x_4)\}.$$

Again S is nonempty since $z'(x_1) \in S$.

Next, choose $s_0 \in S$. Then, there is a solution $u_0(x)$ of (1.1) satisfying

$$\begin{aligned} u_0(x_1) &= z(x_1), \\ u_0'(x_1) &= s_0, \\ u_0(x_2) &= z(x_2), \\ u_0(x_3) - u_0(x_4) &= z(x_3) - z(x_4). \end{aligned}$$

Due to the uniqueness of solutions of 4-point nonlocal problems, such solutions of 4-point nonlocal problems depend continuously on boundary conditions. So, there exists a $\delta > 0$ such that, for each $0 \le |s - s_0| < \delta$, there is a solution $u_s(x)$ of (1.1) satisfying

$$u_s(x_1) = u_0(x_1) = z(x_1),\ u_s'(x_1) = s,\ u_s(x_2) = u_0(x_2) = z(x_2),$$

and

$$u_s(x_3) - u_s(x_4) = u_0(x_3) - u_0(x_4) = z(x_3) - z(x_4),$$

or in other words, $s \in S$; in particular, $(s_0 - \delta, s_0 + \delta) \subset S$, and so S is an open subset of $\mathbb{R}$.

As in the proof of Theorem 3.3, the remainder of the argument is devoted to showing S is also a closed subset of $\mathbb{R}$. Again, we assume, for contradiction, that S is not closed. Then, there is an $r_0 \in \overline{S} \backslash S$ and a strictly monotone sequence $\{r_k\} \subset S$ such that $\lim_{k \to \infty} r_k = r_0$. Again, we may assume $r_k \uparrow r_0$.

By the definition of S, we denote, for each $k \in \mathbb{N}$, by $u_k(x)$ the solution of (1.1) satisfying

$$\begin{gathered} u_k(x_1) = z(x_1),\ u_k'(x_1) = r_k,\ u_k(x_2) = z(x_2), \\ u_k(x_3) - u_k(x_4) = z(x_3) - z(x_4). \end{gathered}$$

By uniqueness of solutions of (1.1), (1.2), we have

$$u_k(x) > u_{k+1}(x) \quad \text{on} \quad (a, x_1), \text{ and } u_k(x) < u_{k+1}(x) \quad \text{on} \quad (x_1, x_2).$$

We claim that $\{u_k(x)\}$ is not uniformly bounded below on each compact subinterval of (a, x_1) and is not uniformly bounded above on each compact subinterval of (x_1, x_2). Suppose there exists a subinterval $[c, d] \subset (a, x_1)$ so that $\{u_k(x)\}$ is uniformly bounded below on $[c, d]$. That is, there exists $H > 0$ so that $u_k(x) \ge H$ for all $c \le x \le d$ and $k \ge 1$. In particular,

$$u_1(x) \ge u_k(x) \ge H \ \text{ for all } \ c \le x \le d \ \text{ and } \ k \ge 1.$$

By the precompactness condition of Theorem 3.2, there exists a subsequence $\{u_{k_j}(x)\}$ such that $\{u_{k_j}^{(i)}(x)\}$ converges uniformly on each compact subinterval of (a, b), $i = 0, 1, 2, 3$.

Suppose $u_{k_j}^{(i)}(x) \to v^{(i)}(x)$ uniformly on the compact interval $[x_1, x_4]$, where $i = 0, 1, 2, 3$. Then $v(x)$ is a solution of equation (1.1) satisfying

$$v(x_1) = z(x_1),\ v'(x_1) = \lim_{j \to \infty} u_{k_j}'(x_1) = r_0,\ v(x_2) = z(x_2),$$

and

$$v(x_3) - v(x_4) = \lim_{j \to \infty} (u_{k_j}(x_3) - u_{k_j}(x_4)) = z(x_3) - z(x_4).$$

Therefore $r_0 \in S$, which is a contradiction to the assumption $r_0 \notin S$. So $\{u_k(x)\}$ is not uniformly bounded below on each compact subinterval of (a, x_1). The argument that $\{u_k(x)\}$ is not uniformly bounded above on each compact subinterval on (x_1, x_2) is completely analogous.

Next let $w(x)$ be the solution of the 3-point conjugate boundary problem for equation (1.1) satisfying,

$$w(x_1) = z(x_1),\ w'(x_1) = r_0,\ w(x_2) = z(x_2),\ w(x_3) = z(x_3).$$

It follows that, for some K large, there exists points $a < \tau_1 < x_1 < \tau_2 < x_2$ so that

$$u_K(\tau_1) = w(\tau_1),\ u_K(\tau_2) = w(\tau_2).$$

Also,

$$u_K(x_1) = z(x_1) = w(x_1),\ u_K(x_2) = z(x_2) = w(x_2).$$

By uniqueness of solutions of 4-point conjugate boundary value problems for (1.1), we have $u_K \equiv w$. However,

$$w'(x_1) = r_0 > r_K = u'_K(x_1),$$

which is a contradiction. Thus S is also a closed subset of $\mathbb{R}$.

In summary, S is a nonempty subset of $\mathbb{R}$ that is both open and closed. We have $S = \mathbb{R}$. By choosing $y_2 \in S$, there is a corresponding solution $y(x)$ of equation (1.1) such that

$$y(x_1) = z(x_1) = y_1,\ y'(x_1) = y_2,\ y(x_2) = z(x_2) = y_3,$$
$$y(x_3) - y(x_4) = z(x_3) - z(x_4) = y_4.$$

This completes the proof. □

In the subsequent existence results, the arguments involved in the proofs follow closely along the lines of those just presented. We will omit most of the details of the proofs, and in the spirit of the preceding results, we will provide only the corresponding function $z(x)$, the set S and the function $w(x)$.

Theorem 3.5. *Assume* (A), (B) *and* (C) *are satisfied with respect to* (1.1). *Given points* $a < x_1 < x_2 < x_3 < x_4 < b$, *and* $y_1, y_2, y_3, y_4 \in \mathbb{R}$, *there exists a unique solution of* (1.1), (1.6) *on* (a, b).

Proof. Let $a < x_1 < x_2 < x_3 < x_4 < b$, and $y_1, y_2, y_3, y_4 \in \mathbb{R}$ be selected. Also, fix $x_1 < \tau < x_2$.

Let $z(x)$ be the solution of the nonlocal 5-point boundary value problem (1.1), (1.2) obtained in Theorem 3.3 and satisfying,

$$z(x_1) = y_1,\ z(\tau) = 0,\ z(x_2) = y_2,\ z(x_3) - z(x_4) = y_4.$$

This time, define the set

$$S = \{u'(x_2)|\ u(x) \text{ is a solution of (1.1) satisfying}$$
$$u(x_1) = z(x_1), u(x_2) = z(x_2), u(x_3) - u(x_4) = z(x_3) - z(x_4)\}.$$

S is a nonempty open subset of $\mathbb{R}$. Making use of the solution $w(x)$ of the 3-point conjugate boundary problem for equation (1.1) satisfying,

$$w(x_1) = z(x_1),\ w(x_2) = z(x_2),\ w'(x_2) = r_0,\ w(x_3) = z(x_3),$$

it follows that S is also a closed subset of $\mathbb{R}$, and so $S = \mathbb{R}$. Choosing $y_3 \in S$ gives rise to the desired solution of (1.1), (1.6). □

Theorem 3.6. *Assume* (A), (B) *and* (C) *are satisfied with respect to* (1.1). *Given points* $a < x_1 < x_2 < x_3 < b$, *and* $y_1, y_2, y_3, y_4 \in \mathbb{R}$, *there exists a unique solution of* (1.1), (1.8) *on* (a, b).

Proof. Let $a < x_1 < x_2 < x_3 < b$, and $y_1, y_2, y_3, y_4 \in \mathbb{R}$ be selected. Also, fix $a < \tau < x_1$.

Let $z(x)$ be the solution of the 4-point nonlocal boundary value problem (1.1), (1.6) obtained in Theorem 3.5 and satisfying,

$$z(\tau) = 0,\ z(x_1) = y_1,\ z'(x_1) = y_2,\ z(x_2) - z(x_3) = y_4.$$

Now, define the set

$$S = \{u''(x_1) |\ u(x) \text{ is a solution of (1.1) satisfying}$$
$$u(x_1) = z(x_1),\ u'(x_1) = z'(x_1),\ u(x_2) - u(x_3) = z(x_2) - z(x_3)\}.$$

S is a nonempty open subset of $\mathbb{R}$. Making use of the solution $w(x)$ of the 2-point conjugate boundary problem for equation (1.1) satisfying,

$$w(x_1) = z(x_1),\ w'(x_1) = z'(x_1),\ w''(x_1) = r_0,\ w(x_3) = z(x_3),$$

it follows that S is also a closed subset of $\mathbb{R}$, and so $S = \mathbb{R}$. Choosing $y_3 \in S$ gives rise to the desired solution of (1.1), (1.8). □

There is a list of dual uniqueness implies existence results for (1.1) with respect to solutions satisfying conditions $(1.j)$, $j = 3, 5, 7, 9$. We state this in one inclusive theorem without proof.

Theorem 3.7. *Assume* (A), (B) *and* (D) *are satisfied with respect to* (1.1). *Then, for each* $j = 3, 5, 7, 9$, *there exists a unique solution of* (1.1), $(1.j)$ *on* (a, b).

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