Georgian Mathematical Journal
Volume 13 (2006), Number 3, 495–500

INVERSION OF AHLFORS AND GRUNSKY INEQUALITIES

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Abstract. We solve the old Kühnau's problem on the exact lower bound in the inverse inequality estimating the dilatation of a univalent function by its Grunsky norm and in the related Ahlfors inequality for Fredholm eigenvalues.

2000 Mathematics Subject Classification: Primary: 30C35, 30C62, 30F45; Secondary: 30F60, 31A10, 53A35.
Key words and phrases: Grunsky coefficient inequalities, Fredholm eigenvalues, quasiconformal reflection, subharmonic function, Strebel point, conformal metric, generalized Gaussian curvature.

1. The Ahlfors inequality for oriented quasiconformal Jordan curves (quasicircles) on the Riemann sphere $L \subset \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

$$\frac{1}{\rho_L} \leq q_L \tag{1}$$

is fundamental in the theory of Fredholm eigenvalues. Here q_L is the reflection coefficient of L and ρ_L is its (first nontrivial) Fredholm eigenvalue (see, e.g., [1], [2], [6], [9]).

It suffices to take the images $L = f^\mu(S^1)$ of the unit circle S^1 under quasiconformal self-maps of $\widehat{\mathbb{C}}$ with Beltrami coefficients $\mu(z) = \partial_{\bar{z}} f / \partial_z f$ supported in the unit disk $\Delta = \{z : \ |z| < 1\}$ and hydrodynamic normalization $f(z) = z + \mathrm{const} + O(|z|^{-1})$ at $z = \infty$. Then q_L equals a minimal dilatation $k(f^\mu) = \|\mu\|_\infty$ among such maps, and inequality (1.1) is reduced to the Grunsky inequality

$$\varkappa(f) := \sup_{\mathbf{x}} \Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn} x_m x_n \Big| \leq k(f), \tag{2}$$

where α_{mn} are the Grunsky coefficients of f defined by

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = - \sum_{m,n=1}^{\infty} \alpha_{mn} z^{-m} \zeta^{-n}, \quad (z, \zeta) \in (\Delta^*)^2,$$

and $\mathbf{x} = (x_n)$ runs over the unit sphere $S(l^2)$ of the Hilbert space l^2 with $\|\mathbf{x}\|^2 = \sum\limits_1^\infty |x_n|^2$, choosing the principal branch of logarithmic function (cf. [4], [9]). The quantity $\varkappa(f)$ is called the *Grunsky norm* of f. By Kühnau–Schiffer theorem $\varkappa(f)$ is reciprocal of ρ_L(see [10], [13]).

ISSN 1072-947X / $8.00 /

We denote the collection of univalent nonvanishing functions $f(z) = z + b_0 + b_1 z^{-1} + \dots$ by Σ and its subset of functions with quasiconformal extensions across S^1 by Σ^0.

The point is that for most of $f \in \Sigma$, we have in (1.2) the strict inequality $\varkappa(f) < k(f)$ (see, e.g., [8]). On the other hand, the functions with $\varkappa(f) = k(f)$ are crucial in many applications of the Grunsky inequality technique. Moreover, by theorem of Pommerenke and Zhuravlev, any $f \in \Sigma$ with $\varkappa(f) \le k < 1$ belongs to Σ^0 and has a k_1-quasiconformal extension with $k_1 = k_1(k) \ge k$ (see [11], [7], pp. 82–84). An explicit not sharp bound $k_1(k)$ is given in [10].

The important problem on the sharp estimation of the dilatation $k(f)$ by the Grunsky norm of f, or equivalently, by the Fredholm eigenvalue of $f(S^1)$ was first stated by Kühnau in 1981 and still remains open. Our main result is the following theorem which solves this problem and has many other applications.

Theorem 1. *For $f \in \Sigma^0$ we have the estimate*

$$k(f) \le \frac{3}{2\sqrt{2}} \varkappa(f) = 1.07\dots\varkappa(f) \tag{3}$$

(*similarly for Fredholm eigenvalues $\rho_{f(S^1)}$*), *which is asymptotically sharp as $\varkappa \to 0$. The equality holds for the map*

$$f_{3,t}(z) = \begin{cases} z(1 + t/z^3)^{2/3} & \text{if } \ |z| > 1, \\ z[1 + t(|z|/z)^3]^{2/3} & \text{if } \ |z| \le 1 \end{cases} \tag{4}$$

with $t = \mathrm{const} \in (0,1)$.

Note that the Beltrami coefficient of this map in the disk Δ is $\mu_3(z) = t|z|/z$.

2. *The proof* of this theorem consists of several independent stages which will be outlined below.

$\mathbf{1^0}$. It suffices to establish the assertion of Theorem 1 for $f \in \Sigma^0$ having Teichmüller extremal quasiconformal extensions onto Δ, i.e., with the Beltrami coefficient $\mu_f(z) = k|\varphi(z)|/\varphi(z)$, where $k = \mathrm{const} \in (0,1)$ and φ is integrable holomorphic function in Δ. This means that f is represented in the universal Teichmüller space $\mathbf{T}$ by a *Strebel point* $[f]$. Such points are dense in $\mathbf{T}$ (see [3], [14]).

Recall that $\mathbf{T}$ is the space of quasisymmetric homeomorphisms of the unit circle S^1 factorized by Möbius maps. It inherits a complex Banach structure factorizing the ball of conformal structures

$$\mathrm{Belt}(\Delta)_1 = \{\mu \in L_\infty(\mathbb{C}) : \ \mu|\Delta^* = 0, \ \|\mu\| < 1\}$$

on Δ so that $\mu, \nu \in \mathrm{Belt}(\Delta)_1$ are equivalent if the corresponding maps $w^\mu, w^\nu \in \Sigma^0$ coincide on S^1. The equivalence classes are in the one-to-one correspondence with the Schwarzian derivatives $S_f := (w''/w')' - w''/w')^2/2$ of $f \in \Sigma^0$ in the complementary disk $\Delta^* = \{z \in \widehat{\mathbb{C}} : |z| > 1\}$.

For elements $\mu \in \mathrm{Belt}(\Delta)_1$ we define

$$\langle \mu, \varphi \rangle_\Delta = \iint\limits_{\Delta} \mu_0^*(z; f)\psi(z)dxdy, \quad \varphi \in L_1(\Delta),$$

and put $\mu^*(z) = \mu(z)/\|\mu\|_\infty$ so that $\|\mu^*\|_\infty = 1$, and $\|\varphi\|_1 := \|\varphi\|_{L_1(\Delta)}$.

2^0. We first prove

Theorem 2. *For every function $f \in \Sigma^0$ with a unique extremal extension f^{μ_0} to Δ, we have the sharp bound*

$$k(f^{\mu_0}) \le \frac{1}{\alpha(f^{\mu_0})} \min_{|t|=1} \varkappa(f^{t\mu_0}) \tag{5}$$

with

$$\alpha(f^{\mu_0}) = \sup_{\psi \in A_1^2, \|\varphi\|_{A_1}=1} |\langle \mu_0^*, \varphi \rangle_\Delta|, \tag{6}$$

where

$$A_1^2 = \{\varphi \in L_1(\Delta) : \ \varphi = \psi^2, \ \psi \text{ is holomorphic}\}.$$

Proof of Theorem 2 is geometric and relies on certain deep properties of conformal (semi)metrics $ds = \lambda(z)|dz|$ on the disk Δ with $\lambda(z) \ge 0$ of *negative integral curvature* bounded from above. The curvature is understood in the supporting sense of Ahlfors or, more generally, in the potential sense of Royden (see, e.g., [12]). For such metrics we have

Lemma 3 ([12])**.** *If a circularly symmetric conformal metric $\lambda(|z|)|dz|$ in Δ has curvature at most -4 in the potential sense, then $\lambda(r) \ge a(1 - a^2r^2)$, where $a = \lambda(0)$.*

On the extremal disk

$$\Delta(\mu_0^*) = \{\phi_{\mathbf{T}}(t\mu_0^*) : \ t \in \Delta\} \subset \mathbf{T},$$

where $\phi_{\mathbf{T}}$ denotes the projection $\mathrm{Belt}(\Delta)_1 \to \mathbf{T}$, the infinitesimal Kobayashi-Teichmüller metric $\lambda_{\mathcal{K}}$ of $\mathbf{T}$ is isometrically equivalent to hyperbolic metric $ds = |dz|/(1 - |z|^2)$ on Δ of curvature -4.

Further, the Grunsky coefficients of $f \in \Sigma^0$ allows us to construct the holomorphic maps

$$\widetilde{h}_{\mathbf{x}}(t) := h_{\mathbf{x}}(\varphi_t) = \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn}(\varphi_t)\ x_m x_n : \ \Delta \to \Delta,$$

where $\varphi_t = S_{f^{t\mu_0^*}}$ and again $\mathbf{x} = (x_n) \in S(l^2)$. Then $\sup\{|\widetilde{h}_{\mathbf{x}}(t)| : \ \mathbf{x} \in S(l^2)\} = \varkappa(f^{\mu_0^*})$. Pull-backing the hyperbolic metric to $\Delta(\mu_0^*)$ by applying these maps, we get the conformal metrics

$$\lambda_{\widetilde{h}_{\mathbf{x}}}(t) := \widetilde{h}_{\mathbf{x}}^*(\lambda_{\mathrm{hyp}}) = |\widetilde{h}_{\mathbf{x}}'(t)|/(1 - |\widetilde{h}_{\mathbf{x}}(t)|^2)$$

of Gaussian curvature -4 at noncritical points. Take their upper envelope $\widetilde{\lambda}_{\varkappa}(t) = \sup\{\lambda_{\widetilde{h}_{\mathbf{x}}}(t) : \ \mathbf{x} \in S(l^2)\}$ and pass to the upper semicontinuous regularization

$$\lambda_{\varkappa}(t) = \limsup_{t' \to t} \widetilde{\lambda}_{\varkappa}(t').$$

This yields a logarithmically subharmonic metric on Δ whose curvature in the supporting and in the potential sense both are less than or equal -4. Its circular mean

$$\mathcal{M}[\lambda_{\varkappa}](|t|) = (2\pi)^{-1} \int\limits_0^{2\pi} \lambda_{\varkappa}(re^{i\theta}) d\theta$$

is a circularly symmetric metric with curvature also at most -4 in the potential sense.

To calculate the value of $\mathcal{M}[\lambda_{\varkappa}](0)$, one can apply the standard variational method to the maps $f^{\mu} \in \Sigma^0$ and to their Grunsky coefficients, which yields

$$\mathcal{M}[\lambda_{\varkappa}](0) = \lambda_{\varkappa}(0) = \alpha(f^{\mu_0}). \tag{7}$$

Further, applying Lemma 3, we get

$$\mathcal{M}[\lambda_{\varkappa}](r) \geq \frac{\alpha(f_0)}{1 - \alpha(f_0)^2 r^2}$$

and, integrating both sides of this inequality over a radial segment $[0, \varrho]$ with $\varrho = \|\mu_0\|_{\infty}$,

$$\int\limits_0^{\varrho} \mathcal{M}[\lambda_{\varkappa}](r) dr \geq \tanh^{-1}[\alpha(f^{\mu_0})\varrho] = \tanh^{-1}[\alpha(f^{\mu_0}) k(f^{\varrho\mu_0^*})]$$

$$= \tanh^{-1}[\alpha(f^{\mu_0}) k(f^{\mu_0})].$$

On the other hand, since the disk $\Delta(\mu_0^*)$ is geodesic, we have

$$\int\limits_0^{t} \lambda_{\varkappa(f^{\mu_0})}(t)|dt| = \tanh^{-1}[\varkappa(f^{\mu_0})].$$

Using these relations, one obtains the desired estimates (5), (6).

3^0. To get (3), we have to estimate (6) from below. To this end, we apply the following important result.

Lemma 4 ([5], [6]). *The equality $\varkappa(f) = k(f)$ for a function $f \in \Sigma$ holds if and only if f is the restriction to Δ^* of a quasiconformal self-map w^{μ_0} of $\widehat{\mathbb{C}}$ with Beltrami coefficient μ_0 satisfying the condition*

$$\sup |\langle \mu_0, \varphi \rangle_{\Delta}| = \|\mu_0\|_{\infty},$$

where the supremum is taken over holomorphic functions $\varphi \in A_1^2(\Delta)$ with $\|\varphi\|_1 = 1$.

If, in addition, the class $[f]$ *contains a frame map (is a Strebel point), then* μ_0 *is of the form*

$$\mu_0(z) = \|\mu_0\|_\infty |\psi_0(z)|/\psi_0(z) \quad \text{with } \psi_0 \in A_1^2 \text{ in } \Delta. \tag{8}$$

For analytic curves $f(S^1)$, the equality (8) is given also in [10].

In view of this lemma and Theorem 2, we can restrict ourselves in the proof of (3) to finding a minimal value of the functionals $l_\mu(\psi) = |\langle \mu^*, \varphi \rangle_\Delta|$ on the set $\{\varphi \in A_1^2, \|\varphi\|_1 = 1\}$ for $\mu^* = |\psi|/\psi$ defined by integrable holomorphic functions in Δ of the form

$$\psi(z) = z^m(c_0 + c_1 z + \cdots), \quad m = 1, 3, 5, \dots \ .$$

A long complicate evaluation yields that this minimum equals $\frac{2\sqrt{2}}{3}$ and is attained on the map (4).

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(Received 25.07.2006)

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