Georgian Mathematical Journal
Volume 13 (2006), Number 3, 501–514

ON NON-COMPACT OPERATORS IN WEIGHTED IDEAL AND SYMMETRIC FUNCTION SPACES

GIORGI ONIANI

Abstract. A resonance type theorem is proved, where conditions are given, which imply the non-compactness and a certain estimation of the measure of the non-compactness of operators in weighted ideal and symmetric function spaces. The application of the theorem to some concrete classes of operators is discussed.

2000 Mathematics Subject Classification: 42B25, 46E30, 28A15.
Key words and phrases: Operator, weighted spaces, compactness.

Introduction

The compactness of classical operators in weighted function spaces was investigated by various authors. In particular, for an ordinary Hardy–Littlewood maximal operator M (i.e. for a maximal operator corresponding to the differentiation basis of balls) D. E. Edmunds and A. Meskhi [2] proved that there exist no a.e. positive weights w and v for which M acts compactly from $L^p_w(\mathbb{R}^n)$ to $L^p_v(\mathbb{R}^n)$ $(p > 1)$, and estimated from below the measure of the non-compactness of M in terms of w, v and p. In [8] and [9], this result was extended to the Hardy–Littlewood maximal operators corresponding to quite general bases and to any symmetric space.

This paper presents extensions of the results of [8] and [9]. In particular, a resonance type theorem is proved, where conditions are found, which imply the non-compactness of operators for any pair of weights and the estimation from below of a non-compactness measure. The theorem is applied to some operators, among which detailed consideration is given to the Hardy–Littlewood and Fefferman–Stein maximal operators corresponding to differentiation bases and the majorant of partial sums of Fourier series with respect to orthonormal systems. Note that, as distinct from [2], [8], [9], here we consider not only the weighted spaces E_w generated by the measure with density equal to w, but also weighted spaces of a different type – $E(w)$ defined by multiplying functions by a weight w.

1. Definitions and the Notation

The definitions and facts connected with the ideal and symmetric spaces discussed in this paper are borrowed from [5].

1.1. **Weighted ideal and symmetric spaces.** Let (X, μ) be a measure space with σ-finite measure, and $\Delta(X, \mu)$ be the class of all μ-measurable and μ-a.e.

ISSN 1072-947X / $8.00 / © Heldermann Verlag *www.heldermann.de*

finite functions defined on X. A Banach function space E on (X,μ) is said to be *ideal* if

$$f\in\Delta(X,\mu),\quad g\in E,\quad |f|\le|g|\ \ \mu\text{-a.e.}\Rightarrow f\in E\ \text{ and }\ \|f\|_E\le\|g\|_E.$$

The class of all ideal spaces on (X,μ) is denoted by $\mathbb{I}(X,\mu)$. We call $E\in\mathbb{I}(X,\mu)$ *regular* if $\lim\limits_{\mu H\to 0,\,\chi_H\in E}\|\chi_H\|_E=0$.

Let S_μ be the σ-algebra on which μ is defined. For $H\in S_\mu$ and $S\subset S_\mu$ denote $S_\mu(H)=\{A\in S_\mu : A\subset H\}$ and $\Pi(S)=\{c\chi_A : c\in\mathbb{R},\ A\in S\}$.

Let $E\in\mathbb{I}(X,\mu)$ and $w\in\Delta(X,\mu)$ be a non-negative function with $\mu\{w>0\}>0$, i.e., w be a *weight* on (X,μ). If there is a set $H\subset S_\mu(\{w>0\})$ such that $\mu H>0$ and $\chi_H\in E$, then the weighted space $E(w)$ is defined as a space of all $f\in\Delta(X,\mu)$ such that $fw\in E$ and is equipped with the norm $\|f\|_{E(w)}=\|fw\|_E$. It is obvious that $E(w)\in\mathbb{I}(X,\mu)$.

A Banach function space E on (X,μ) is said to be *symmetric* if it is ideal and

$$f\in\Delta(X,\mu),\quad g\in E,\quad f\ \text{ is equimeasurable with }\ g\Rightarrow f\in E,\quad \|f\|_E=\|g\|_E.$$

For $E\in\mathbb{I}(X,\mu)$ and $H\in S_\mu$, set $E|_H=\{f|_H : f\in E\}$ and $\|f|_H\|_{E|_H}=\|f\chi_H\|_E$ $(f\in E)$.

The spaces (X_1,μ_1) and (X_2,μ_2) are called *isomorphic* if there is a mapping $\omega : X_1\to X_2$ that, after neglecting appropriate zero measure sets from X_1 and X_2, becomes a bijective and measure preserving mapping whose inverse is also measure preserving. ω is called an *isomorphism* between (X_1,μ_1) and (X_2,μ_2).

Let E be a symmetric space on (X_1,μ_1), and (X_2,μ_2) be isomorphic to $(H,\mu_1|_{S_{\mu_1}(H)})$, where H is some μ_1-measurable set with $\mu_1 H>0$. Let $\omega : H\to X_2$ be an isomorphism between $(H,\mu_1|_{S_{\mu_1}(H)})$ and (X_2,μ_2). Denote

$$E(X_2,\mu_2)=\{f\in\Delta(X_2,\mu_2) : f\circ\omega\in E|_H\}.$$

It is easy to see that $E(X_2,\mu_2)$ with the norm $\|f\|_{E(X_2,\mu_2)}=\|f\circ\omega\|_{E|_H}$ $(f\in E(X_2,\mu_2))$ is a symmetric space on (X_2,μ_2).

For a measurable set $H\subset\mathbb{R}^n$ denote by $|\cdot|=|\cdot|_H$ the Lebesgue measure on H.

(X,μ) is called a *Lebesgue space* if it is isomorphic to $((0,\mu X),|\cdot|)$. As is known, for any set $H\subset\mathbb{R}^n$ with a positive measure, $(H,|\cdot|)$ is a Lebesgue space.

For a weight w on (X,μ) denote by μ_w the measure defined as follows: $\mu_w A=\int\limits_A w\,d\mu$ $(A\in S_\mu)$.

Let E be a symmetric space on $((0,\infty),|\cdot|)$, (X,μ) be a Lebesgue space, and w be a weight on (X,μ). From Rochlin's result (see [10, pp. 121–122]) it follows that (X,μ_w) is also a Lebesgue space. Thus, along with $E(X,\mu)$ we can consider its weighted variant $E(X,\mu_w)$.

Let $\mathbb{S}$ be the class of all symmetric spaces on $((0,\infty),|\cdot|)$. For a Lebesgue space (X,μ) denote $\mathbb{S}(X,\mu)=\{E(X,\mu) : E\in\mathbb{S}\}$. For $E_0\in\mathbb{S}$, $E=E_0(X,\mu)$ and a weight w on (X,μ) denote $E_w=E_0(X,\mu_w)$.

1.2. **A measure of non-compactness and a similar characteristic.** Let E_1 and E_2 be normed spaces. The class of all compact operators acting from E_1 to E_2 denote by $K(E_1, E_2)$. For an operator $T : E_1 \to E_2$ the number

$$\operatorname{dist}(T, K(E_1, E_2)) = \inf_{A \in K(E_1,E_2)} \sup_{\|x\|_{E_1} \le 1} \|T(x) - A(x)\|_{E_2}$$

is called a *measure of the non-compactness of* T (see, e.g., [2]).

For a measure space (X, μ) denote by $\overline{\Delta}(X, \mu)$ the class of all μ-measurable functions defined on X and by S^*_μ the class $\{A \in S_\mu : \mu A < \infty\}$.

Let (X, μ), (X, μ_1) and (X, μ_2) be measure spaces with $S_{\mu_1} = S_{\mu_2} = S_\mu$; $E_1 \in \mathbb{I}(X, \mu_1)$, $E_2 \in \mathbb{I}(X, \mu_2)$, and $T : \Pi(S^*_\mu) \to \overline{\Delta}(X, \mu)$ be a positively homogeneous operator. If T acts boundedly from $\Pi(S^*_\mu) \cap E_1$ to E_2, then denote by $\alpha(T, E_1, E_2)$ the supremum of all numbers $c \ge 0$ for which there is a sequence $\{H_k\} \subset S^*_\mu$ with the properties: $\{\chi_{H_k}\} \subset E_1$, $\|\chi_{H_k}\|_{E_1} > 0$ ($k \in \mathbb{N}$) and

$$\|Tf_k - Tf_m\|_{E_2} \ge c \quad (k \ne m),$$

where $f_k = \chi_{H_k}/\|\chi_{H_k}\|_{E_1}$ ($k \in \mathbb{N}$). If T does not act boundedly from $\Pi(S^*_\mu) \cap E_1$ to E_2, then we will mean that $\alpha(T, E_1, E_2) = \infty$. It is easy to verify that for an operator $M : E_1 \to E_2$ with $M|_{\Pi(S^*_\mu)\cap E_1} = T$ we have the inequality

$$\operatorname{dist}(M, K(E_1, E_2)) \ge \frac{1}{2}\, \alpha(T, E_1, E_2).$$

1.3. **Some classes of operators.** For a ball $B \subset \mathbb{R}^n$ and $\varepsilon > 0$ denote by $V(B, \varepsilon)$ the set of all $x \in \mathbb{R}^n$ for which there are $y \in B$ and $i \in \overline{1, n}$ such that $|x_i - y_i| < \varepsilon$.

Let $X \subset \mathbb{R}^n$, $|X| > 0$ and T be a positively homogeneous operator defined on a subspace $\Omega \subset \Delta(X, |\cdot|)$ and taking values on $\overline{\Delta}(X, |\cdot|)$. We assume that $\Pi(S^*_{|\cdot|}) \subset \Omega$. We will write that

1) $T \in \mathbb{T}_1(c_1, c_2)$ where $c_1 > 0$, $0 < c_2 \le 1$, if for any $H \in S_{|\cdot|}$, $|H| > 0$ and any ball B with $|B \cap H| > 0$ there are $\varepsilon_0 > 0$ and sets $H_\varepsilon \in S_{|\cdot|}(B \cap H)$ ($0 < \varepsilon \le \varepsilon_0$) such that $H_{\varepsilon'} \subset H_\varepsilon$ if $0 < \varepsilon' < \varepsilon \le \varepsilon_0$, $|H_\varepsilon| = \varepsilon$ and $|\{|T\chi_{H_\varepsilon}| \ge c_1\} \cap H_\varepsilon| \ge c_2|H_\varepsilon|$ ($0 < \varepsilon \le \varepsilon_0$);

2) $T \in \mathbb{T}_2(c_1, c_2)$ where $c_1 > 0$, $0 < c_2 \le 1$, if for any $H \in S^*_{|\cdot|}$ there is $H^* \in S_{|\cdot|}(H)$ with $|H^*| > 0$ such that $|\{|T\chi_{H^*}| \ge c_1\} \cap H^*| \ge c_2|H^*|$;

3) $T \in \mathbb{T}_3$ if for any ball $B \subset \mathbb{R}^n$ with $B \cap X \ne \varnothing$ and $\varepsilon > 0$

$$\lim_{\delta\to 0} \sup \bigl\{|T\chi_H(x)| : H \in S_{|\cdot|}(B \cap X),\ \operatorname{diam} H < \delta,\ x \in X \setminus V(B, \varepsilon)\bigr\} = 0;$$

4) $T \in \mathbb{T}_4$ if for any ball $B \subset \mathbb{R}^n$ with $B \cap X \ne \varnothing$ and $\varepsilon > 0$

$$\lim_{\delta\to 0} \sup \bigl\{|T\chi_H(x)| : H \in S_{|\cdot|}(B \cap X), \operatorname{diam} H < \delta,\ x \in X \setminus V(B, \varepsilon)\bigr\} < \infty.$$

1.4. **Some more notation.** Let (X, μ) be a measure space, $\mathbb{W}(X, \mu)$ be the collection of all pairs of weights (w, v) on (X, μ) for which $\mu\{wv > 0\} > 0$, and let for $E \in \mathbb{I}(X, \mu)$, $\mathbb{W}_E(X, \mu)$ be the collection of all pairs of weights on (X, μ) for which there is a set $H \in S_\mu(\{wv > 0\})$ such that $\mu H > 0$ and

$\chi_{_H} \in E(w)\cap E(v)$. For $(w,v)\in \mathbb{W}(X,\mu)$ and $E\in \mathbb{I}(X,\mu)$, $(w,v)\in \mathbb{W}_E(X,\mu)$ denote

$$c_{w,v} = \operatorname*{ess\,sup}_{\{wv>0\}} \frac{v}{w}\,,$$

$$c_{w,v,E} = \sup\Big\{ \operatorname*{ess\,sup}_{H} \frac{v}{w} \;:\; H\in S_\mu(\{wv>0\}),\;\; \mu H>0,\;\; \chi_{_H}\in E(w)\cap E(v)\Big\}.$$

It is easy to see that if $X\subset \mathbb{R}^n$, $|X|>0$ and E is a symmetric space on $(X,|\cdot|)$, then $c_{w,v,E}=c_{w,v}$.

Let $X\subset\mathbb{R}^n$, $|X|>0$ and E be a symmetric space on $(X,|\cdot|)$. The function $\varphi_E(t)=\|\chi_{(0,t)}\|_E$ $(t\in(0,|X|)$ is called the *fundamental function* of E. φ_E has the properties: $\varphi_E(0)=0$, φ_E is positive and increasing on $(0,|X|)$ and $\varphi_E(t)/t$ is decreasing on $(0,|X|)$. Denote

$$\overline{\psi}_E(c)=\varlimsup_{t\to 0+}\frac{\varphi_E(ct)}{\varphi_E(t)}\,,\quad \underline{\psi}_E(c)=\varliminf_{t\to 0+}\frac{\varphi_E(ct)}{\varphi_E(t)}\quad (c>0).$$

By the properties of φ_E for $0<c_1<c_2$ and $0<c_2t<|X|$ we have

$$\frac{c_1}{c_2}\le \frac{\varphi_E(c_1t)}{\varphi_E(t)}\Big/\frac{\varphi_E(c_2t)}{\varphi_E(t)}\le 1,$$

wherefrom we obtain the continuity of $\overline{\psi}_E$ and $\underline{\psi}_E$. Moreover, the properties of φ_E readily imply that $\overline{\psi}_E$ and $\underline{\psi}_E$ are increasing on $(0,\infty)$, $1\le\underline{\psi}_E(c)\le\overline{\psi}_E(c)\le c$ for $c\ge 1$ and $c\le\underline{\psi}_E(c)\le\overline{\psi}_E(c)\le 1$ for $0<c<1$. Note that if the space E is not regular, then $\overline{\psi}_E(c)=\underline{\psi}_E(c)=1$ $(c>0)$ and if $E=L^p(X,|\cdot|)$, then $\varphi_E(t)=t^{\frac{1}{p}}$ $(0<t<|X|)$ and $\overline{\psi}_E(c)=\underline{\psi}_E(c)=c^{\frac{1}{p}}$ $(c>0)$.

2. The Main Theorem

Let $X\subset\mathbb{R}^n$, $|X|>0$. In the sequel, for simplicity, we write X instead of $(X,|\cdot|)$. The classes of all regular spaces $E\in\mathbb{S}(X)$ and $E\in\mathbb{I}(X)$ are denoted by $\mathbb{S}_r(X)$ and $\mathbb{I}_r(X)$, respectively.

Theorem 1. *Let $X\subset\mathbb{R}^n$, $|X|>0$, T be a positively homogeneous operator defined on a subspace of $\Delta(X)$ containing $\Pi(S^*_{|\cdot|})$ and taking the values on $\overline{\Delta}(X)$, $c_1>0$ and $0<c_2\le 1$. Then the following assertions are valid:*

1) *If $E\in\mathbb{S}_r(X)$ and $T\in\mathbb{T}_1(c_1,c_2)\cap\mathbb{T}_4$, then for any $(w,v)\in\mathbb{W}(X)$*

$$\alpha(T,E_w,E_v)\ge c_1\underline{\psi}_E(c_2)\overline{\psi}_E(c_{w,v}).$$

2) *If $E\in\mathbb{S}_r(X)$ and $T\in\mathbb{T}_2(c_1,c_2)$, then for any $(w,v)\in\mathbb{W}(X)$*

$$\alpha(T,E_w,E_v)\ge c_1\underline{\psi}_E(c_2)\underline{\psi}_E(c_{w,v}).$$

3) *If $E\in\mathbb{S}\setminus\mathbb{S}_r(X)$ and $T\in\mathbb{T}_2(c_1,c_2)\cap\mathbb{T}_3$, then for any $(w,v)\in\mathbb{W}(X)$*

$$\alpha(T,E_w,E_v)\ge c_1.$$

4) *If $E\in\mathbb{S}_r(X)$ and $T\in\mathbb{T}_2(c_1,c_2)$, then for any $(w,v)\in\mathbb{W}(X)$*

$$\alpha(T,E(w),E(v))\ge c_1\underline{\psi}_E(c_2)c_{w,v}.$$

5) *If* $E \in \mathbb{S} \setminus \mathbb{S}_r(X)$ *and* $T \in \mathbb{T}_2(c_1, c_2) \cap \mathbb{T}_3$, *then for any* $(w, v) \in \mathbb{W}(X)$

$$\alpha(T, E(w), E(v)) \geq c_1 c_{w,v}.$$

6) *If* $E \in \mathbb{I}_r(X)$ *and* $T \in \mathbb{T}_2(c_1, 1)$, *then for any* $(w, v) \in \mathbb{W}_E(X)$

$$\alpha(T, E(w), E(v)) \geq c_1\, c_{w,v,E}.$$

7) *If* $E \in \mathbb{I} \setminus \mathbb{I}_r(X)$ *and* $T \in \mathbb{T}_2(c_1, 1) \cap \mathbb{T}_3$, *then for any* $(w, v) \in \mathbb{W}_E(X)$

$$\alpha(T, E(w), E(v)) \geq c_1\; c_{w,v,E}.$$

3. Auxiliary Propositions

Lemma 1. *Let* $E_1 \in \mathbb{I}(X, \mu_1)$ *and* $E_2 \in \mathbb{I}(X, \mu_2)$. *If*

1) $S_{\mu_1} = S_{\mu_2}$;

2) E_1 *is a regular;*

3) *there is a set* $H \subset X$ *with* $\chi_H \in E_1 \cap E_2$, $\mu_1 H > 0$, $\mu_2 H > 0$ *and* $C_1 > 0$ *such that* $\|\chi_A\|_{E_2} \geq C_1 \|\chi_A\|_{E_1}$ *if* $A \in S_{\mu_1}(H)$; *and* $\mu_1(A) > 0$ *if* $A \in S_{\mu_1}(H)$, $\mu_2 A > 0$;

4) *the restriction of* μ_1 *on* $S_{\mu_1}(H)$ *is nonatomic;*

5) T *is a positively homogeneous operator acting from* $\Pi(S_{\mu_1}(H))$ *to* E_2;

6) *there is* $C_2 > 0$ *such that for any* $A \in S_{\mu_1}(H)$ *with* $\mu_1 A > 0$ *there is* $A^* \in S_{\mu_1}(A)$ *with* $\mu_1 A^* > 0$ *for which* $\|T\chi_{A^*} \cdot \chi_{A^*}\|_{E_2} \geq C_2 \|\chi_{A^*}\|_{E_2}$,

then there is a sequence $\{A_k\} \subset S_{\mu_1}(H)$ *with* $\mu_1 A_k > 0$ $(k \in \mathbb{N})$ *such that*

$$\varliminf_{k,m\to\infty,\, k\neq m} \frac{\|Tf_k - Tf_m\|_{E_2}}{C_1 C_2} \geq 1$$

where $f_k = \chi_{A_k} / \|\chi_{A_k}\|_{E_1}$.

Proof. By the conditions of the lemma we can choose $A_1 \subset S_{\mu_1}(H)$ such that $\mu_1 A_1 > 0$ and $\|T\chi_{A_1} \cdot \chi_{A_1}\|_{E_2} \geq C_2 \|\chi_{A_1}\|_{E_2}$. Assume that the sets $A_1 \supset \cdots \supset A_k$ with $A_m \in S_{\mu_1}(H)$, $\mu_1 A_m > 0$ and $\|T\chi_{A_m} \cdot \chi_{A_m}\|_{E_2} \geq c_2 \|\chi_{A_m}\|_{E_2}$ $(m \in \overline{1,k})$ are chosen. Denote $f_m = \chi_{A_m} / \|\chi_{A_m}\|_{E_1}$ $(m \in \overline{1,k})$. Let $\beta_k > 0$ be such that

$$\mu_2\left(A_k \cap \bigcap_{m=1}^{k} \{|Tf_m| < \beta_k\}\right) > 0.$$

Due to the conditions of the lemma we can choose

$$A_{k+1} \subset A_k \cap \bigcap_{m=1}^{k} \{|Tf_m| < \beta_k\}$$

so that $A_{k+1} \in S_{\mu_1}(H)$, $\mu_1 A_{k+1} > 0$, $\|T\chi_{A_{k+1}} \cdot \chi_{A_{k+1}}\|_{E_2} \geq C_2 \|\chi_{A_{k+1}}\|_{E_2}$ and $1/\|\chi_{A_{k+1}}\|_{E_1} > 2^k \beta_k$.

Denote $\alpha_k = 1/\|\chi_{A_k}\|_{E_1}$ $(k \in \mathbb{N})$. For the constructed sequence $\{A_k\}$ we have

$$\begin{aligned}
\|Tf_k - Tf_m\|_{E_2} &\geq \|(Tf_k - Tf_m)\chi_{A_k}\|_{E_2} \geq \|Tf_k \cdot \chi_{A_k}\|_{E_2} - \|Tf_m \cdot \chi_{A_k}\|_{E_2} \\
&\geq C_2 \alpha_k \|\chi_{A_k}\|_{E_2} - \beta_{k-1} \|\chi_{A_k}\|_{E_2} \geq C_2\big(1 - 1/C_2 2^{k-1}\big)\alpha_k \|\chi_{A_k}\|_{E_2} \\
&\geq C_2\big(1 - 1/C_2 2^{k-1}\big)\alpha_k C_1 \|\chi_{A_k}\|_{E_1} = \big(1 - 1/C_2 2^{k-1}\big) C_1 C_2
\end{aligned}$$

for any $k, m \in \mathbb{N}$, $k > m$. The lemma is proved. $\square$

Lemma 2. *Let $E_1 \in \mathbb{I}(X, \mu_1)$ and $E_2 \in \mathbb{I}(X, \mu_2)$. If*

1) $S_{\mu_1} = S_{\mu_2}$*;*

2) *X is a separable quasimetric space and $B \in S_{\mu_1}$ for any ball $B \subset X$;*

3) *there is a set $H \subset X$ with $\chi_H \in E_1 \cap E_2$, $\mu_1 H > 0$, $\mu_2 H > 0$ and $C_1 > 0$ such that $\|\chi_A\|_{E_2} \geq C_1 \|\chi_A\|_{E_1}$ if $A \in S_{\mu_1}(H)$;*

4) *T is a positively homogeneous operator acting from $\Pi(S_{\mu_1}(H))$ to E_2;*

5) *there is $C_2 > 0$ such that for any $A \in S_{\mu_1}(H)$ with $\mu_1 A > 0$ there is $A^* \in S_{\mu_1}(A)$ with $\mu_1 A^* > 0$ for which $\|T\chi_{A^*} \cdot \chi_{A^*}\|_{E_2} \geq C_2 \|\chi_{A^*}\|_{E_2}$;*

6) *there is a sequence $\{H_k\} \subset S_{\mu_1}(H)$ with $\mu_1 H_k > 0$ $(k \in \mathbb{N})$, $H_k \cap H_m = \varnothing$ $(k \neq m)$ such that for any $k \in \mathbb{N}$*

$$\lim_{\delta \to 0} \sup \Big\{ |T\chi_P(x)| : P \in S_{\mu_1}(H_k), \ \ \operatorname{diam} P < \delta, \ \ x \in \bigcup_{m \neq k} H_m \Big\} = 0,$$

then there is a sequence $\{A_k\} \subset S_{\mu_1}(H)$ with $\mu_1 A_k > 0$ $(k \in \mathbb{N})$ such that

$$\varliminf_{k,m \to \infty,\, k \neq m} \frac{\|Tf_k - Tf_m\|_{E_2}}{C_1 C_2} \geq 1$$

where $f_k = \chi_{A_k} / \|\chi_{A_k}\|_{E_1}$ $(k \in \mathbb{N})$.

Proof. For any $k \in \mathbb{N}$ we can choose $\delta_k > 0$ such that (see 6)) if $A \in S_{\mu_1}(H_k)$ and $\operatorname{diam} A < \delta_k$, then

$$\sup \Big\{ |T\chi_A(x)| : x \in \bigcup_{m \neq k} H_m \Big\} < \frac{1}{k}\,.$$

By virtue of 2) it is easy to see that there is a sequence $A_k \in S_{\mu_1}(H_k)$ with $\mu_1 A_k > 0$ and $\operatorname{diam} A_k < \delta_k$ $(k \in \mathbb{N})$. For any $k \in \mathbb{N}$ let us consider (see 5)) the set $A_k^* \subset S_{\mu_1}(A_k)$, $\mu_1 A_k^* > 0$ for which $\|T\chi_{A_k^*} \cdot \chi_{A_k^*}\|_{E_2} \geq C_2 \|\chi_{A_k^*}\|_{E_2}$. Denote $f_k = \chi_{A_k^*} / \|\chi_{A_k^*}\|_{E_1}$, $\alpha_k = 1/\|\chi_{A_k^*}\|_{E_1}$.

Suppose $\inf_{k \in \mathbb{N}} \|\chi_{A_k^*}\|_{E_1} = a > 0$. Due to the construction of $\{A_k^*\}$, for $m < k$ we have

$$\begin{aligned}
\|Tf_m - Tf_k\|_{E_2} &\geq \|(Tf_m - Tf_k)\chi_{A_m^*}\|_{E_2} \geq \|Tf_m \cdot \chi_{A_m^*}\|_{E_2} - \|Tf_k \cdot \chi_{A_m^*}\|_{E_2} \\
&\geq \alpha_m C_2 \|\chi_{A_m^*}\|_{E_2} - \alpha_k \|T\chi_{A_k^*} \cdot \chi_{A_m^*}\|_{E_2} \\
&\geq \alpha_m C_2 \|\chi_{A_m^*}\|_{E_2} - \frac{1}{a}\,\frac{1}{k}\,\alpha_k \|\chi_{A_k^*}\|_{E_2} \\
&\geq C_2 \bigg(1 - \frac{1}{C_2 a \|\chi_H\|_{E_1} k}\bigg) \alpha_m C_1 \|\chi_{A_m^*}\|_{E_1} \\
&= \bigg(1 - \frac{1}{C_2 a \|\chi_H\|_{E_1} k}\bigg) C_1 C_2.
\end{aligned}$$

Let now $\inf_{k \in \mathbb{N}} \|\chi_{A_k^*}\|_{E_1} = 0$. Choose a sequence of indexes $k(1) < k(2) < \cdots$ such that $\alpha_{k(p+1)} \geq \alpha_{k(p)}$ $(p \in \mathbb{N})$. By the construction of $\{A_k^*\}$, for $p > q$ we have

$$\begin{aligned}\|Tf_{k(p)} - Tf_{k(q)}\|_{E_2} &\geq \|(Tf_{k(p)} - Tf_{k(q)})\chi_{A^*_{k(p)}}\|_{E_2} \\ &\geq \|Tf_{k(p)} \cdot \chi_{A^*_{k(p)}}\|_{E_2} - \|Tf_{k(q)} \cdot \chi_{A^*_{k(p)}}\|_{E_2} \\ &\geq C_2\alpha_{k(p)}\|\chi_{A^*_{k(p)}}\|_{E_2} - \frac{\alpha_{k(q)}}{k(q)}\|\chi_{A^*_{k(p)}}\|_{E_2} \\ &\geq C_2\left(1 - \frac{1}{C_2k(q)}\right)\alpha_{k(p)}\|\chi_{A^*_{k(p)}}\|_{E_2} \\ &\geq C_2\left(1 - \frac{1}{C_2k(q)}\right)\alpha_{k(p)}C_1\|\chi_{A^*_{k(p)}}\|_{E_1} = \left(1 - \frac{1}{C_2k(q)}\right)C_1C_2.\end{aligned}$$

The lemma is proved. □

Lemma 3. *For any $H \subset \mathbb{R}^n$, $|H| > 0$ there are sequences of balls $\{B_k\}$ and of positive numbers $\{\varepsilon_k\}$ such that for any $k \in \mathbb{N}$*
1) $|B_k \cap H| > 0$,
2) $V(B_k, \varepsilon_k) \cap \bigcup_{m \neq k} B_m = \varnothing$.

Proof. Without loss of generality assume that H is bounded. Obviously, we can choose a density point x_1 from H, a ball B_1 with center at x_1 and $\varepsilon_1 > 0$ so that $|H \setminus V(B_1, \varepsilon_1)| > 0$. Since x_1 is a density point, we have $|B_1 \cap H| > 0$. Suppose we have constructed the balls $B_1, \dots, B_k$ and positive numbers $\varepsilon_1, \dots, \varepsilon_k$ with the following properties: $|B_i \cap H| > 0$, $V(B_i, \varepsilon_i) \cap \bigcup_{j \in \overline{1,k} \setminus \{i\}} B_j = \varnothing$ $(j = \overline{1,k})$; $\left|H \setminus \bigcup_{i=1}^{k} V(B_i, \varepsilon_i)\right| > 0$. Let us choose a density point x_{k+1} from $H \setminus \bigcup_{i=1}^{k} V(B_i, \varepsilon_i)$ for which $\operatorname{dist}\left(x_{k+1}, \bigcup_{i=1}^{k} V(B_i, \varepsilon_i)\right) > 0$. Now taking a ball B_{k+1} with center at x_{k+1} and $\varepsilon_{k+1} > 0$ small enough we have

$$V(B_i, \varepsilon_i) \cap \bigcup_{j \in \overline{1,k} \setminus \{i\}} B_j = \varnothing \quad (i = \overline{1,k+1});$$

$$\left|H \setminus \bigcup_{i=1}^{k} V(B_i, \varepsilon_i)\right| > 0.$$

Thus the needed sequences $\{B_k\}$ and $\{\varepsilon_k\}$ are constructed. The lemma is proved. □

Lemma 4. *Let $X \in \mathbb{R}^n$, $|X| > 0$, $E \in \mathbb{S}(X)$, $c_1 > 0$, $0 < c_2 \leq 1$ and $(w, v) \in \mathbb{W}(X)$. Then for any $c \in (0, c_2)$, $c' \in (0, c_{w,v})$ and $\varepsilon > 0$ there is a set $H \subset \{x \in X : w(x)v(x) > 0, v(x)/w(x) \geq c'\}$, $0 < |H| < \infty$ such that for any set $A \subset H$, $|A| > 0$, we have*
1) *if $f \in E_v$ is such that $|\{f \geq c_1\} \cap A| \geq c_2|A|$, then*

$$\|f\chi_A\|_{E_v} \geq c_1(\underline{\psi}_E(c) - \varepsilon)\|\chi_A\|_{E_v};$$

2) *if* $f \in E(v)$ *is such that* $|\{f \geq c_1\} \cap A| \geq c_2|A|$, *then*

$$\|f\chi_A\|_{E(v)} \geq c_1(1-\varepsilon)(\underline{\psi}_E(c_2)-\varepsilon)\|\chi_A\|_{E(v)};$$

3) $\|\chi_A\|_{E_v} \geq (\underline{\psi}_E(c')-\varepsilon)\|\chi_A\|_{E_w}$.

Proof. Let $\delta > 0$ be such that $\varphi_E(c_2t)/\varphi_E(t) > \underline{\psi}_E(c_2)-\varepsilon$, $\varphi_E(ct)/\varphi_E(t) > \underline{\psi}_E(c)-\varepsilon$ and $\varphi_E(c't)/\varphi_E(t) > \underline{\psi}_E(c')-\varepsilon$ when $0 < t < \delta$. Clearly, there are $0 < b_1 < b_2$ with $b_1/b_2 > c/c_2$ for which the set

$$M = \left\{x \in X : w(x)v(x) > 0, \ \ b_1 < v(x) < b_2, \ \ \frac{v(x)}{w(x)} \geq c'\right\}$$

is of positive measure. Let H be a subset of M such that $0 < |H| < \infty$ and

$$(b_2/c' + b_2 + 1)|H| < \delta.$$

Suppose $A \subset H$, $|A| > 0$. Due to the choice of H we have

$$\|\chi_A\|_{E_v} = \varphi_E(|A|_v) \geq \varphi_E(c'|A|_w) \geq (\underline{\psi}_E(c')-\varepsilon)\varphi_E(|A|_w) = (\underline{\psi}_E(c')-\varepsilon)\|\chi_A\|_{E_w};$$

if $f \in E_v$ and $|\{f \geq c_1\} \cap A| \geq c_2|A|$, then

$$\begin{aligned}\|f\chi_A\|_{E_v} &\geq \|c_1\chi_{\{f\geq c_1\}\cap A}\|_{E_v} = c_1\varphi_E(|\{f \geq c_1\}\cap A|_v)\\ &\geq c_1\varphi_E(b_1|\{f \geq c_1\}\cap A|) \geq c_1\varphi_E(b_1c_2|A|) \geq c_1\varphi_E\left(\frac{b_1c_2}{b_2}\,|A|_v\right)\\ &\geq c_1(\underline{\psi}_E(c)-\varepsilon)\varphi_E(|A|_v) = c_1(\underline{\psi}_E(c)-\varepsilon)\|\chi_A\|_{E_v};\end{aligned}$$

and if $f \in E(v)$ and $|\{f \geq c_1\} \cap A| \geq c_2|A|$, then

$$\begin{aligned}\|f\chi_A\|_{E(v)} &= \|f\chi_A v\|_E \geq c_1\|\chi_{\{f\geq c_1\}\cap A}v\|_E \geq c_1b_1\|\chi_{\{f\geq c_1\}\cap A}\|_E\\ &= c_1b_1\varphi_E(|\{f \geq c_1\}\cap A|) \geq c_1b_1\varphi_E(c_2|A|)\\ &\geq c_1b_1(\underline{\psi}_E(c_2)-\varepsilon)\varphi_E(|A|) \geq c_1b_1(\underline{\psi}_E(c_2)-\varepsilon)\|\chi_A\|_E\\ &= \frac{c_1b_1}{b_2}\,(\underline{\psi}_E(c_2)-\varepsilon)\|\chi_A\|_{E(v)} \geq c_1(1-\varepsilon)(\underline{\psi}_E(c_2)-\varepsilon)\|\chi_A\|_{E(v)}.\end{aligned}$$

The lemma is proved. □

It is easy to verify the validity of our next assertion.

Lemma 5. *Let* $E \in \mathbb{I}(X,\mu)$, $(w,v) \in \mathbb{W}_E(X,\mu)$ *and* $c \geq 0$. *If* $H \in S_\mu$, $\chi_H \in E(w) \cap E(v)$ *and* $H \subset \{x : v(x) \geq cw(x)\}$, *then* $\|\chi_H\|_{E(v)} \geq c\|\chi_H\|_{E(w)}$.

4. Proof of Theorem 1

Assertion 1). Without loss of generality assume that T maps $\Pi(S_{|\cdot|}) \cap E_w$ into E_v. Let $c \in (0, c_2)$, $c' \in (0, c_{w,v})$ and $\varepsilon > 0$. Suppose H is the set from Lemma 4. Take numbers a_1 and a_2 with $0 < a_1 < a_2$ so that the set $\widetilde{H} = H \cap \{a_1 \leq w \leq a_2\}$ be of positive measure.

By virtue of Lemma 3 there are sequences of balls $\{B_k\}$ and numbers $\{\varepsilon_k\}$ such that $|B_k \cap \widetilde{H}| > 0$ and $\left|V(B_k,\varepsilon_k) \cap \bigcup\limits_{m\neq k} B_m\right| > 0$ $(k \in \mathbb{N})$.

It is easy to see that we can choose a sequence $A_k \in S_{|\cdot|}(B_k \cap \widetilde{H})$ $(k \in \mathbb{N})$ such that $|A_k|_w > 0$ $(k \in \mathbb{N})$, $\lim_{k\to\infty} |A_k|_w = 0$ and for any $k \in \mathbb{N}$

$$
\begin{aligned}
&|\{|T\chi_{A_k}| \geq c_1\} \cap A_k| \geq c_2|A_k|,\\
&\frac{\varphi_E(c'|A_k|_w)}{\varphi_E(|A_k|_w)} \geq \overline{\psi}_E(c) - \varepsilon,\\
&\frac{1}{\varphi_E(|A_{k+1}|_w)} \geq 2^{k+1} \max_{1\leq i\leq k} \frac{1}{\varphi_E(|A_i|_w)} \cdot \sup |T\chi_{A_i} \cdot \chi_{X\setminus V(B_i,\varepsilon_i)}|.
\end{aligned}
$$

We also have that (see Lemma 4) for any $k \in \mathbb{N}$

$$\|T\chi_{A_k} \cdot \chi_{A_k}\|_{E_v} \geq c_1(\underline{\psi}_E(c) - \varepsilon)\|\chi_{A_k}\|_{E_v}.$$

Denote $f_k = \chi_{A_k}/\|\chi_{A_k}\|_{E_w}$, $\alpha_k = 1/\|\chi_{A_k}\|_{E_w}$ and $\beta_k = \max_{1\leq i\leq k} \alpha_i \sup |T\chi_{A_i} \cdot \chi_{X\setminus V(B_i,\varepsilon_i)}|$ $(k \in \mathbb{N})$. Taking into account the construction of $\{A_k\}$ for any $k > m$ we can write

$$
\begin{aligned}
\|Tf_k - Tf_m\|_{E_v} &\geq \|(Tf_k - Tf_m)\chi_{A_k}\|_{E_v} \geq \|Tf_k \cdot \chi_{A_k}\|_{E_w} - \|Tf_m \cdot \chi_{A_k}\|_{E_w}\\
&\geq c_1(\underline{\psi}_E(c) - \varepsilon)\alpha_k\|\chi_{A_k}\|_{E_v} - \beta_m\|\chi_{A_k}\|_{E_v}\\
&\geq c_1(\underline{\psi}_E(c) - \varepsilon)\Big(1 - \frac{1}{c_1 2^k}\Big)\alpha_k\varphi_E(c'|A_k|_w)\\
&\geq \Big(1 - \frac{1}{c_1 2^k}\Big)c_1\ (\underline{\psi}_E(c) - \varepsilon)(\overline{\psi}_E(c') - \varepsilon).
\end{aligned}
$$

This completes the proof of assertion 1).

The other assertions are direct consequences of Lemmas 1–5. The theorem is proved.

5. Some Applications of Theorem 1

In this section we give examples of some classes of operators to which Theorem 1 is applicable. In particular, the properties of operators are established, which, by virtue of Theorem 1, clearly imply the corresponding conclusions on their non-compactness.

5.1. **Hardy–Littlewood maximal operators.** A mapping $\mathbf{B}$ defined on $\mathbb{R}^n$ is said to be a *differentiation basis* in $\mathbb{R}^n$ (see, e.g., [4]) if for every $x \in \mathbb{R}^n$, $\mathbf{B}(x)$ is a family of open bounded sets containing the point x such that there exists a sequence $\{R_k\} \subset \mathbf{B}(x)$ with $\operatorname{diam} R_k \to 0$ $(k \to \infty)$.

A Hardy-Littlewood maximal operator $M_{\mathbf{B},G}$ corresponding to a differentiation basis $\mathbf{B}$ and an open non-empty set $G \subset \mathbb{R}^n$ is defined as follows: $f \in L_{loc}(G)$ and $x \in G$

$$M_{\mathbf{B},G}f(x) = \sup_{R\in\mathbf{B}(x)} \frac{1}{|R|} \int_{R\cap G} |f|.$$

Denote $M_{\mathbf{B}} = M_{\mathbf{B},\mathbb{R}^n}$.

A basis $\mathbf{B}$ is said to *differentiate the integral of the function* f if for almost every $x \in \mathbb{R}^n$ the integral mean $\frac{1}{|R|} \int_R f$ tends to $f(x)$ when $R \in \mathbf{B}(x)$, $\operatorname{diam} R \to 0$.

A basis $\mathbf{B}$ is called:

a density basis if $\mathbf{B}$ differentiates the integral of the characteristic function of every measurable set,

convex if for every $x \in \mathbb{R}^n$ the collection $\mathbf{B}(x)$ consists of convex sets,

translation invariant if $\mathbf{B}(x) = \{x + R : R \in \mathbf{B}(0)\}$ for any $x \in \mathbb{R}^n$,

Busemann–Feller if for any $R \in \bigcup_{x \in \mathbb{R}^n} \mathbf{B}(x)$ we have $R \in \mathbf{B}(y)$ for any $y \in R$.

We say that a basis $\mathbf{B}$ *contains* a basis $\mathbf{B}'$ if $\mathbf{B}(x) \supset \mathbf{B}'(x)$ for every $x \in \mathbb{R}^n$.

Note that: 1) *any translation invariant convex basis contains a density basis* (see [4, Ch. I, §3]); 2) *if* $\mathbf{B}$ *is either translation invariant or Busemann–Feller, then* $M_{\mathbf{B}} f \in \Delta(\mathbb{R}^n)$ *for any* $f \in L_{loc}(\mathbb{R}^n)$ (as is easy to verify).

We call *a strip* in $\mathbb{R}^n$ an open set bounded by two different parallel hyperplanes. The *strip width* will be called the distance between the hyperplanes that bound the strip.

Theorem 2. *Let* $\mathbf{B}$ *be a differentiation basis in* $\mathbb{R}^n$ *and* $M_{\mathbf{B}}$ *map* $\Pi(S^*_{|\cdot|})$ *into* $\Delta(\mathbb{R}^n)$. *Then the following assertions are true:*

1) *If* $\mathbf{B}$ *contains a density basis, then for any measurable set* $H \subset \mathbb{R}^n$ *we have that* $M_{\mathbf{B}} \chi_H(x) = 1$ *for a.e.* $x \in H$, *and* $M_{\mathbf{B}} \chi_H(x) \le 1$ *for any* $x \in \mathbb{R}^n$. *Consequently,* $M_{\mathbf{B},G} \in \mathbb{T}_1(1,1) \cap \mathbb{T}_4$ *for any open non-empty set* $G \subset \mathbb{R}^n$;

2) *If* $\mathbf{B}$ *is a convex basis, then for any strip* H

$$M_{\mathbf{B}} \chi_H(x) \le \frac{2^n (\text{width of } H)}{\operatorname{dist}(x, H)} \quad \textit{when} \quad \operatorname{dist}(x, H) \ge (\text{width of } H).$$

Consequently, $M_{\mathbf{B},G} \in \mathbb{T}_3$ *for any open non-empty set* $G \subset \mathbb{R}^n$.

The first assertion of Theorem 2 is obvious, while the second one is proved in [7] (see also [8]).

For the conclusions on the non-compactness of Hardy–Littlewood operators derived on the basis of Theorem 2 and assertions 1) and 2) of Theorem 1, the reader is referred to [8] and [9]. Previously, a similar result was obtained in [2] when $\mathbf{B}$ is a basis of balls and $E = L^p$ $(p > 1)$ (note that the results of the same type for fractional maximal operators and for some singular integral operators were established in [2] and [6], respectively.)

Remark 1. The conclusions on the non-compactness of a Hardy–Littlewood operator M corresponding to the basis of balls can be extended to the non-atomic homogeneous type measure spaces. Here we formulate only a relevant particular result that reads as follows: for any $p \ge 1$ and $(w, v) \in \mathbb{W}(X)$, $\alpha(M, L^p_w, L^p_v) \ge c^{1/p}_{w,v}$. For the proof it is sufficient to use Lemma 1 and the fact that for homogeneous type spaces we have an analog of Bezicovitch' covering lemma and the Lebesgue theorem on the differentiation of integrals (see, e.g., [11]).

Furthermore, note that Lemmas 1 and 2 allow us to extend the conclusions of Theorem 1 in a more or less complete form to the cases of more general measure spaces, but we will not dwell on this topic here.

5.2. **Fefferman–Stein maximal operator.** For a differentiation basis $\mathbf{B}$, under $F_{\mathbf{B}}$ we mean the *Fefferman–Stein maximal operator corresponding to* $\mathbf{B}$, i.e., for $f \in L_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$

$$F_{\mathbf{B}}f(x) = \sup_{R \in \mathbf{B}(x)} \frac{1}{|R|} \int\limits_R |f - f_R|,$$

where $f_R = \frac{1}{|R|} \int\limits_R |f|$.

We call a basis $\mathbf{B}$ *homothety invariant* if for any $x \in \mathbb{R}^n$, $R \in \mathbf{B}(x)$ and a homothety H with center at x we have $H(R) \in \mathbf{B}(x)$.

It is easy to verify that *if* $\mathbf{B}$ *is either translation invariant or Busemann–Feller, then* $F_{\mathbf{B}}f \in \Delta(\mathbb{R}^n)$ *for any* $f \in L_{loc}(\mathbb{R}^n)$.

Theorem 3. *Let* $\mathbf{B}$ *be a differentiation basis in* $\mathbb{R}^n$*, and* $F_{\mathbf{B}}$ *map* $\Pi(S^*_{|\cdot|})$ *into* $\Delta(\mathbb{R}^n)$*. Then the following assertions are true:*

1) *If* $\mathbf{B}$ *contains a density homothety invariant Busemann–Feller basis, then* $F_{\mathbf{B}} \in \mathbb{T}_1(c, 1)$ *for any* $c \in (0, 1/2)$*;*
2) *If* $\mathbf{B}$ *contains a convex translation invariant basis, then* $F_{\mathbf{B}} \in \mathbb{T}_2(1/4^{n+1}n^{2n}, 1/4)$*;*
3) *If* $\mathbf{B}$ *is a convex basis, then* $F_{\mathbf{B}} \in \mathbb{T}_3$.

Lemma 6. *Let* R *be an open bounded convex set in* $\mathbb{R}^n$*. Then there is a rectangle* I *such that* $I \subset R \subset n^2 I$.

Proof. As is known (see, e.g., [4, Ch. VI, §2]), there is an open ellipsoid E with $E \subset R \subset nE$. Let J be a minimal rectangle among the rectangles containing E and having edges parallel to the principal axis of E. Take $I = \frac{1}{n} J$. It is easy to see that $I \subset E \subset nI$. Consequently, $I \subset R \subset n^2 I$. The lemma is proved. ☐

Proof of Theorem 3. *Assertion* 1). Let $\mathbf{B}'$ be a density and Busemann–Feller basis contained in $\mathbf{B}$. Suppose $H \in S_{|\cdot|}$ and B is a ball with $|B \cap H| > 0$. Let $x \in B \cap H$ be such that

$$\lim_{\operatorname{diam} R \to 0,\ R \in \mathbf{B}'(x)} \frac{|R \cap B \cap H|}{|R|} = 1.$$

Take arbitrary $R \in \mathbf{B}'(x)$. Let $R' \subset R$ be such that $|R'| = \frac{1}{2}|R|$. For $t > 0$ denote $R_t = f_t(R)$, $R'_t = f_t(R')$, $A_t = R'_t \cap B \cap H$, where f_t is the homothety with center at x and coefficient t. It is easy to verify that

$$\lim_{t \to 0+} \frac{1}{|R_t|} \int\limits_{R_t} \left| \chi_{A_t} - \frac{|A_t \cap R_t|}{|R_t|} \right| = \frac{1}{2},$$

wherefrom, due to the Busemann–Feller property of $\mathbf{B}'$, for any $c \in (0, 1/2)$ we can find $t_0 > 0$ such that $\{F_{\mathbf{B}}\chi_{A_t} \geq c\} \supset A_t$ if $0 < t < t_0$, which readily implies the validity of assertion 1).

Assertion 2). Without loss of generality assume that $\mathbf{B}$ is translation invariant.

Since almost every point of a measurable set is a density point, there is a ball $B(x_0, r)$ for which

$$|B(x_0,r)\cap H| > \left(1-\frac{1}{2\cdot 4^n n^{2n}}\right)|B(x_0,r)|.$$

Take $\delta > 0$ such that

$$|B(x_0,r)\cap H| - |B(x_0,r)\setminus B(x_0,r-\delta)| > \left(1-\frac{1}{2\cdot 4^n n^{2n}}\right)|B(x_0,r)|.$$

Assume $R \in \mathbf{B}(0)$ and $\operatorname{diam} R < \delta/n^2$. Let I be a rectangle (see Lemma 6) with $I \subset R \subset n^2 I$. Let us divide $\mathbb{R}^n$ into nonoverlapping rectangles obtained by translations of $n^2 I$, and denote their family by Γ. Due to the choice of δ there is a rectangle $J \in \Gamma$ with

$$|J\cap H| > \left(1-\frac{1}{2\cdot 4^n n^{2n}}\right)|J|.$$

Taking into account that $\mathbf{B}$ is translation invariant we can find $x \in J$ such that $x + R \in \mathbf{B}(x)$ and $\frac{1}{n^2}J \subset x + R \subset J$. Denote $J_1 = \frac{1}{n^2}J$ and $J_2 = \frac{1}{4n^2}J$. Let J_2^* be a rectangle obtained by translation of J_2 and such that $J_2^* \subset J$ and x is a vertex of J_2^*. We have

$$|J_2^*\cap H| \geq |J_2^*| - |J\setminus H| > \frac{|J|}{4^n n^{2n}} - \frac{|J|}{2\cdot 4^n n^{2n}} = \frac{|J_2^*|}{2}\,.$$

Similarly, we obtain $|J_2 \cap H| > |J_2|/2$.

Let $H^* = (J_2 \cup J_2^*) \cap H$. For $y \in J_2^* \cap H$ we have

$$\frac{1}{|y+R|}\int\limits_{y+R}\chi_{_{H^*}} \leq \frac{2|J_2|}{|J_1|} \leq \frac{1}{2}\,.$$

Consequently, taking into account the obvious inclusions $y+R \supset y-x+J_1 \supset J_2$ and the translation invariance of $\mathbf{B}$ we write

$$F_{\mathbf{B}}\chi_{_{H^*}}(y) \geq \frac{1}{|y+R|}\int\limits_{y+R}|\chi_{_{H^*}} - 1/2| \geq \frac{1}{|J|}\int\limits_{J_2\cap H^*}\frac{1}{2} \geq \frac{1}{4}\,\frac{|J_2|}{|J|} = \frac{1}{4^{n+1}n^{2n}}\,.$$

Thus

$$\left\{F_{\mathbf{B}}\chi_{_{H^*}} > 1/4^{n+1}n^{2n}\right\} \supset J_2^*\cap H^*.$$

Therefore

$$\left|\left\{F_{\mathbf{B}}\chi_{_{H^*}} > 1/4^{n+1}n^{2n}\right\}\cap H^*\right| \geq \frac{1}{4}\,|H^*|.$$

Assertion 2) is proved.

Assertion 3) is a corollary of Theorem 2 and an obvious estimation $F_{\mathbf{B}}f \leq 2M_{\mathbf{B}}f$ $(f \in L_{loc}(\mathbb{R}^n))$. The theorem if proved. □

5.3. **Majorant of partial sums.**

Theorem 4. *Let $X \subset \mathbb{R}^n$, $|X| > 0$ and $\{\varphi_k\}_{k\in\mathbb{N}}$ be a complete orthonormal system in $L^2(X)$. If Ω is a collection of subsets of $\mathbb{N}$ such that for any $k \in \mathbb{N}$ there is $N \in \Omega$ with $\overline{1,k} \subset N$, and*

$$Tf(x) = \sup_{N\in\Omega} |S_N f(x)| \quad (f \in L^2(X), \ \ x \in X),$$

where $S_N f(x) = \sum\limits_{k\in N} (f, \varphi_k)\varphi_k(x)$, then $T \in \mathbb{T}_1(1,1)$.

Proof. Let us consider a sequence $\{N_k\} \subset \Omega$ such that $\overline{1,k} \subset N_k$ $(k \in \mathbb{N})$. Due to the Parseval equality for arbitrary $f \in L^2(X)$ we have $\lim\limits_{k\to\infty} \|S_{N_k} f - f\|_{L^2(X)} = 0$. By virtue of the Riesz theorem on the convergence in measure there is a subsequence $\{S_{N_{k(i)}} f\}$ that converges to f a.e., wherefrom $Tf(x) \geq |f(x)|$ a.e. The theorem is proved. □

Theorem 5. *Let $\{\varphi_k\}_{k\in\mathbb{N}}$ be a trigonometric system on $[-\pi, \pi]$. If Ω is a collection of sets of the form $\mathop{\times}\limits_{i=1}^{n} \overline{-k_i, k_i}$ such that for any $k \in \mathbb{N}$ there is $N \in \Omega$ with $\overline{-k,k}^n \subset N$, and*

$$Tf(x) = \sup_{N\in\Omega} |S_N f(x)| \quad (f \in L[-\pi,\pi]^n, \ \ x \in [-\pi,\pi]^n),$$

where $S_N f(x) = \sum\limits_{k\in N} (f, \varphi_{k_1}\cdots\varphi_{k_n})\varphi_{k_1}(x_1)\cdots\varphi_{k_n}(x_n)$, then $T \in \mathbb{T}_1(1,1) \cap \mathbb{T}_3$.

Proof. The conclusion $T \in \mathbb{T}_1(1,1)$ follows from Theorem 4. The Dirichlet kernels of rectangular sums with respect to $\{\varphi_{k_1}, \dots, \varphi_{k_n}\}$ are uniformly bounded outside a cross-like neighborhood of 0, i.e., on $\{x \in [-\pi,\pi]^n : |x_i| > \varepsilon \ (i \in \overline{1,n})\}$ for any $\varepsilon > 0$ (see, e.g., [12, Ch. II, §5 and Ch. XVII, §2]), wherefrom it readily follows that $T \in \mathbb{T}_3$. The theorem is proved. □

Theorem 6. *Let $\{\varphi_k\}_{k\in\mathbb{N}}$ be a bounded Vilenkin system on $[0,1]$. If Ω is a collection of sets of the form $\mathop{\times}\limits_{i=1}^{n} \overline{1, k_i}$ such that for any $k \in \mathbb{N}$ there is $N \in \Omega$ with $\overline{1,k}^n \subset N$, and*

$$Tf(x) = \sup_{N\in\Omega} |S_N f(x)| \quad (f \in L[0,1]^n, \ \ x \in [0,1]^n),$$

where $S_N f(x) = \sum\limits_{k\in N} (f, \varphi_{k_1}\cdots\varphi_{k_n})\varphi_{k_1}(x_1)\cdots\varphi_{k_n}(x_n)$, then $T \in \mathbb{T}_1(1,1) \cap \mathbb{T}_3$.

Proof. The conclusion $T \in \mathbb{T}_1(1,1)$ follows from Theorem 4. Taking into account that

1) Dirichlet kernels of rectangular sums with respect to $\{\varphi_{k_1}, \dots, \varphi_{k_n}\}$ are uniformly bounded outside any cross-like neighborhood of 0, i.e., on $\{x \in [0,1]^n : x_i > \varepsilon \ (i \in \overline{1,n})\}$ for any $\varepsilon > 0$ (see, e.g., [1, Ch. IV, §3]);

2) $\alpha \ominus \beta \geq |\alpha - \beta|$ $(\alpha, \beta \in [0,1])$, where $\ominus$ is the operation of subtraction associated with the system $\{\varphi_k\}$ (see, e.g., [3, Ch. I, §1]),

it is easy to verify that $T \in \mathbb{T}_3$. The theorem is proved. □

Remark 2. One can prove that Theorems 4–6 remain true if the partial sums in their formulations are respectively replaced by: 1) means with respect to any regular summation method; 2) (C, α) ($\alpha \in (0, 1]^n$) or Abel–Poisson means; and 3) $(C, 1)$ means.

Acknowledgement

The author wishes to thank Professor Vakhtang Kokilashvili for his helpful remarks.

References

1. G. N. Agaev, N. Ya. Vilenkin, G. M. Dzhafarli, and A. I. Rubinshteĭn, Multiplicative systems of functions and harmonic analysis on zero-dimensional groups. (Russian) *Èlm, Baku,* 1981.
2. D. E. Edmunds and A. Meskhi, On a measure of non-compactness for maximal operators. *Math. Nachr.* **254/255**(2003), 97–106.
3. B. I. Golubov, A. V. Efimov, and V. A. Skvortsov, Walsh series and transforms. (Russian) *Nauka, Moscow,* 1987.
4. M. de Guzmán, Differentiation of integrals in $\mathbb{R}^n$. *Lecture Notes in Mathematics,* Vol. 481, *Springer-Verlag, Berlin–New York,* 1975.
5. S. G. Kreĭn, Yu. I. Petunin, and E. M. Semenov, Interpolation of linear operators. (Russian) *Nauka, Moscow,* 1978.
6. A. Meskhi, On a measure of non-compactness for singular integrals. *J. Funct. Spaces Appl.* **1**(2003), No. 1, 35–43.
7. G. Oniani, On the integrability of strong maximal functions corresponding to different frames. *Georgian Math. J.* **6**(1999), No. 2, 149–168.
8. G. Oniani, On the non-compactness of maximal operators. Real Anal. Exchange **28**, no. 2, 439–446 (2002/2003).
9. G. Oniani, On the measure of non-compactness of maximal operators. *J. Funct. Spaces Appl.* **2**(2004), No. 2, 217–225.
10. V. A. Rokhlin, On the fundamental ideas of measure theory. (Russian) *Mat. Sbornik (N.S.)* **25(67)**(1949) , 107–150.
11. J.-O. Strömberg and A. Torchinsky, Weighted Hardy spaces. *Lecture Notes in Mathematics,* Vol. 1381. *Springer-Verlag, Berlin,* 1989.
12. A. Zygmund, Trigonometric series, Vols. I, II. 2nd ed. *Cambridge University Press, New York,* 1959.

(Received 11.04.2006; revised 12.06.2004)

Author's address:

A. Tsereteli Kutaisi State University
59, Tamar Mepe St., Kutaisi 4600
Georgia
E-mail: tava73@mail.ru