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APPROXIMATING COMMON FIXED POINTS OF NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. Let K be a nonempty closed convex subset of a real uniformly convex Banach space E and $S, T : K \to K$ two nonexpansive mappings such that $F(S) \cap F(T) := \{x \in K : Sx = Tx = x\} \neq \varnothing$. Suppose $\{x_n\}$ is generated iteratively by

$$x_1 \in K, \;\; x_{n+1} = (1-\alpha_n)x_n + \alpha_n S[(1-\beta_n)x_n + \beta_n T x_n],$$

$n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[0, 1]$. In this paper, we discuss the weak and strong convergence of $\{x_n\}$ to some $x^* \in F(S) \cap F(T)$.

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1. Introduction

Let K be a nonempty subset of a real normed linear space E. Let T be a self-mapping of K. Then T is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \tag{1.1}$$

for all $x, y \in K$. The set of fixed points of T is denoted by $F(T)$. For the last thirty years, weak and strong convergence theorems for nonexpansive mappings have been established by a number of authors (see, e.g., [1], [2], [4], [10], [12], [14]–[16]).

In 1995, Xu [18] introduced and studied the Mann and Ishikawa iteration schemes with errors. Since then, these schemes have been further investigated by a number of authors for approximating fixed points of nonlinear mappings. Recently, Khan and Fukhar-ud-din [7] studied the following iterative scheme with errors involving two nonexpansive mappings

$$\begin{cases} x_1 \in K, \\ x_{n+1} = \alpha_n x_n + \beta_n S y_n + \gamma_n v_n, \\ y_n = \alpha'_n x_n + \beta'_n T x_n + \gamma'_n u_n, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ are real sequences in $[0, 1]$ such that

$$\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n, \tag{1.3}$$

$$\sum_{n=1}^{\infty} \gamma_n < \infty, \quad \sum_{n=1}^{\infty} \gamma'_n < \infty, \tag{1.4}$$

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and $\{u_n\}$, $\{v_n\}$ are bounded sequences in K. They obtained the following results.

Theorem KF1 ([7, Theorem 1]). *Let E be a uniformly convex Banach space satisfying* Opial's conditon *and K a nonempty closed convex bounded subset of E. Let $S, T : K \to K$ be two nonexpansive mappings with $F := F(S) \cap F(T) \neq \varnothing$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ be real sequences in $[0,1]$ satisfying* (1.3), (1.4) *and $\alpha_n, \alpha'_n \in [\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. For arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion* (1.2). *Then $\{x_n\}$ converges* weakly *to some common fixed point of S and T.*

Theorem KF2 ([7, Theorem 2]), *Let E be a uniformly convex Banach space and K a nonempty closed bounded convex subset of E. Let $S, T : K \to K$ be two nonexpansive mappings with $F := F(S) \cap F(T) \neq \varnothing$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ be real sequences in $[0,1]$ satisfying* (1.3), (1.4) *and $\alpha_n, \alpha'_n \in [\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. For arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion* (1.2). *Suppose S and T satisfy* condition (A'), *i.e. there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that $\frac{1}{2}(\|x - Tx\| + \|x - Sx\|) \geq f(d(x,F))$ for all $x \in K$. Then $\{x_n\}$ converges* strongly *to some common fixed point of S and T.*

Remark 1.1.

(1) If $S = T$ and $\gamma_n = \gamma'_n = 0$ for all n, then the iteration scheme (1.2) reduces to the Ishikawa iteration scheme [4].

(2) If $S = T$ and $\beta'_n = \gamma'_n = 0$ for all n or if $T = I$ and $\gamma'_n = 0$ for all n , then the iteration scheme (1.2) reduces to the Mann iteration scheme with errors and if, in addition, $\gamma_n = 0$ for all n, it reduces to the well-known Mann iteration process [9].

The purpose of this paper is to study the following iteration scheme without error terms

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n S y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[0,1]$.

We note that Theorem KF1 is not new; see, for instance, the following results.

Theorem TT ([15, Theorem 3.3]). *Let E be a uniformly convex Banach space satisfying Opial's condition or whose norm is Fréchet differentiable and K a nonempty closed convex subset of E. Let $S, T : K \to K$ be two nonexpansive mappings with $F := F(S) \cap F(T) \neq \varnothing$. Let $\{\alpha_n\}$, $\{\beta_n\}$ be real sequences in $[a,b]$ for some $a, b \in \mathbf{R}$ with $0 < a \leq b < 1$. For arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion* (1.5). *Then $\{x_n\}$ converges weakly to some common fixed point of S and T.*

Theorem KKT ([8, Theorem 3.5]). *Let E be a uniformly convex Banach space satisfying Opial's condition or whose norm is Fréchet differentiable and K a nonempty closed convex subset of E. Let $S, T : K \to K$ be two nonexpansive* *mappings with* $F := F(S) \cap F(T) \neq \varnothing$*. Let* $\{\alpha_n\}$*,* $\{\beta_n\}$*,* $\{\gamma_n\}$*,* $\{\alpha'_n\}$*,* $\{\beta'_n\}$*,* $\{\gamma'_n\}$ *be real sequences in* $[0,1]$ *satisfying* (1.3)*,* (1.4) *and* $\alpha_n, \alpha'_n \in [a,b]$ *for some* $a, b \in \mathbf{R}$ *with* $0 < a \leq b < 1$*. For arbitrary* $x_1 \in K$*, define the sequence* $\{x_n\}$ *by the recursion* (1.2)*. Then* $\{x_n\}$ *converges weakly to some common fixed point of* S *and* T*.*

We remark that once a convergence theorem has been proved for an iteration scheme *without errors*, such as (1.5), it is not always difficult to establish the corresponding result for the case *with errors* such as Theorem KF1, Theorem KF2 and Theorem KKT above under the conditions (1.3) and (1.4). As pointed out by Chidume [2], if error terms satisfying (1.3) and (1.4) are introduced in either the Mann or the Ishikawa iterative scheme, the proofs of the results are basically unnecessary repetitions of the proofs when no error terms are added. Usually, we are interested, in mathematics, in simpler algorithms, unless the better rate of convergence or some other advantage is gained. This is not the case with both Theorem KF1 and Theorem KF2. We use the iteration process (1.5) for approximating the common fixed point of two nonexpansive maps (when such a common fixed point exists) and to prove some strong and weak convergence theorems for such maps. Our results improve, complement and extend some known results including Theorem KF1 and Theorem KF2. It is worth mentioning that our weak convergence result applies not only to L^p-spaces with $1 < p < \infty$ but also to other spaces which do not satisfy *Opial's condition* or have a *Fréchet differentiable norm* ([5]).

2. Preliminaries

Let E be a real Banach space. Then E is said to have the *Kadec-Klee property* if for every sequence $\{x_n\}$ in E, $x_n \to x$ weakly and $\|x_n\| \to \|x\|$ strongly together imply $\|x_n - x\| \to 0$.

The following lemmas are needed in the sequel.

Lemma 2.1 (see, e.g., [16])**.** *Let* $\{\lambda_n\}$ *and* $\{\sigma_n\}$ *be sequences of nonnegative real numbers such that* $\lambda_{n+1} \leq \lambda_n + \sigma_n$*,* $\forall\, n \geq 1$ *and* $\sum_{n=1}^{\infty} \sigma_n < \infty$*. Then* $\lim_{n\to\infty} \lambda_n$ *exists. Moreover, if there exists a subsequence* $\{\lambda_{n_j}\}$ *of* $\{\lambda_n\}$ *such that* $\lambda_{n_j} \to 0$ *as* $j \to \infty$*, then* $\lambda_n \to 0$ *as* $n \to \infty$*.*

Lemma 2.2 (see, e.g., [5])**.** *Let* E *be a real reflexive Banach space such that its dual* E^* *has the Kadec-Klee property. Let* $\{x_n\}$ *be a bounded sequence in* E *and* $x^*, y^* \in \omega_w(x_n)$*; here* $\omega_w(x_n)$ *denotes the weak* w*-limit set of* $\{x_n\}$*. Suppose* $\lim_{n\to\infty} \|tx_n + (1-t)x^* - y^*\|$ *exists for all* $t \in [0,1]$*. Then* $x^* = y^*$*.*

Lemma 2.3 (see, e.g., [8])**.** *Let* K *be a nonempty closed convex subset of a Banach space* E*. Let* $S, T : K \to K$ *be two nonexpansive mappings with* $x^* \in F := F(S) \cap F(T)$*. Suppose that* $\{x_n\}$ *is defined by* (1.2) *and that for every* n*, a mapping* $T_n : K \to K$ *is defined by*

$$T_n x = \alpha_n x + \beta_n S[\alpha'_n x + \beta'_n Tx + \gamma'_n x] + \gamma_n x$$

for $x \in K$. If there are $\alpha_n, \alpha_n' \in [a,b]$ for some $a, b \in \mathbf{R}$ with $0 < a \leq b < 1$, then $\{T_n T_{n-1} ... T_1 x - x_{n+1}\}$ converges strongly to 0 as $n \to \infty$.

Lemma 2.4 (see, e.g., [13]). *Let E be a uniformly convex Banach space and $\{\alpha_n\}$ a sequence in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. Suppose $\{x_n\}$ and $\{y_n\}$ are sequences in E such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$, and $\limsup_{n\to\infty} \|\alpha_n x + (1-\alpha_n) y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.5 (see, e.g., [17]). *Let $p > 1$ and $R > 1$ be two fixed numbers and E a Banach space. Then E is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that $\|\lambda x + (1-\lambda) y\|^p \leq \lambda \|x\|^p + (1-\lambda)\|y\|^p - W_p(\lambda) g(\|x - y\|)$ for all $x, y \in B_R(0) = \{x \in E : \|x\| \leq R\}$, and $\lambda \in [0,1]$, where $W_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$.*

3. Main Results

Lemma 3.1. *Let E be a real normed space and K a nonempty closed convex subset of E. Let $S, T : K \to K$ be two nonexpansive mappings with $x^* \in F := F(S) \cap F(T)$. Let $\{\alpha_n\}$, $\{\beta_n\}$ be real sequences in $[0,1]$. For arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion (1.5). Then $\lim_{n\to\infty} \|x_n - x^*\|$ exists.*

Proof. Notice that

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|(1-\alpha_n)x_n + \alpha_n S y_n - x^*\| \\
&\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n \|S y_n - x^*\| \\
&\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n \|(1-\beta_n)x_n + \beta_n T x_n - x^*\| \\
&\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n [(1-\beta_n)\|x_n - x^*\| + \beta_n \|x_n - x^*\|] \\
&= \|x_n - x^*\|.
\end{aligned}$$

Hence $\lim_{n\to\infty} \|x_n - x^*\|$ exists and so $\{x_n\}$ is bounded. This completes the proof. □

Lemma 3.2. *Let E be a real uniformly convex Banach space and K a nonempty closed convex subset of E. Let $S, T : K \to K$ be two nonexpansive mappings with $F := F(S) \cap F(T) \neq \varnothing$. Let $\{\alpha_n\}$, $\{\beta_n\}$ be real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. For arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion (1.5). Then*

$$\lim_{n\to\infty} \|x_n - T x_n\| = 0 = \lim_{n\to\infty} \|x_n - S x_n\|.$$

Proof. Let $x^* \in F$ and $y_n = (1-\beta_n)x_n + \beta_n T x_n$. Then, by Lemma 3.1, $\lim_{n\to\infty} \|x_n - x^*\|$ exists and so $\{x_n\}$ is bounded. Therefore, there exists $R > 0$ such that $x_n - x^*,\ y_n - x^* \in B_R(0)$ for all $n \geq 1$. Set $r = \lim_{n\to\infty} \|x_n - x^*\|$. If $r = 0$, then by the continuity of S and T the conclusion follows. Now suppose $r > 0$. We follow [14] (see also [15], [8] or [7]). Using Lemma 2.5, we obtain

$$\|y_n - x^*\|^2 = \|(1-\beta_n)x_n + \beta_n T x_n - x^*\|^2$$

$$
\begin{aligned}
&= \|\beta_n(Tx_n - x^*) + (1-\beta_n)(x_n - x^*)\|^2 \\
&\le \beta_n\|Tx_n - x^*\|^2 + (1-\beta_n)\|x_n - x^*\|^2 \\
&\quad - W_2(\beta_n)g(\|Tx_n - x_n\|) \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|x_n - x^*\|^2 \\
&= \|x_n - x^*\|^2,
\end{aligned}
$$

so

$$\limsup_{n\to\infty} \|y_n - x^*\| \le r,$$

and

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(1-\alpha_n)x_n + \alpha_n S y_n - x^*\|^2 \\
&\le \alpha_n\|y_n - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 \\
&\quad - W_2(\alpha_n)g(\|Sy_n - x_n\|) \\
&\le \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 \\
&\quad - W_2(\alpha_n)g(\|Sy_n - x_n\|) \\
&\le \|x_n - x^*\|^2 - W_2(\alpha_n)g(\|Sy_n - x_n\|). \qquad (3.1)
\end{aligned}
$$

Since $W_2(\alpha_n) \ge 2\epsilon^3$, we have from (3.1) that

$$2\epsilon^3 \sum_{n=1}^{\infty} g(\|Sy_n - x_n\|) \le \|x_1 - x^*\|^2 < \infty$$

Thus we have $\lim_{n\to\infty} g(\|Sy_n - x_n\|) = 0$. Since g is strictly increasing and continuous at 0, we have

$$\lim_{n\to\infty} \|Sy_n - x_n\| = 0.$$

Since S is nonexpansive, we have

$$\|x_n - x^*\| \le \|x_n - Sy_n\| + \|y_n - x^*\|,$$

which on taking $\liminf_{n\to\infty}$ yields

$$r \le \liminf_{n\to\infty} \|y_n - x^*\|.$$

Consequently,

$$\lim_{n\to\infty} \|\beta_n(Tx_n - x^*) + (1-\beta_n)(x_n - x^*)\| = \lim_{n\to\infty} \|y_n - x^*\| = r.$$

Since

$$\limsup_{n\to\infty} \|Tx_n - x^*\| \le r,$$

by Lemma 2.4 we have $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. Also,

$$
\begin{aligned}
\|Sx_n - x_n\| &\le \|Sx_n - Sy_n\| + \|Sy_n - x_n\| \\
&\le \|x_n - y_n\| + \|Sy_n - x_n\| \\
&\le \|Tx_n - x_n\| + \|Sy_n - x_n\|.
\end{aligned}
$$

This implies that $\lim_{n\to\infty} \|Sx_n - x_n\| = 0$. This completes the proof. □

The following result was proved by Shahzad in [14] (using Lemma 2.2), which contains Theorem TT for the case when E is a uniformly convex space whose norm is Fréchet differentiable.

Theorem 3.3. *Let E be a real uniformly convex Banach space such that its dual E^* has the Kadec–Klee property and K a nonempty closed convex subset of E. Let $S, T : K \to K$ be two nonexpansive mappings with $F := F(S) \cap F(T) \neq \varnothing$. Let $\{\alpha_n\}$, $\{\beta_n\}$ be real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. For arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion* (1.5)*. Then $\{x_n\}$ converges weakly to some common fixed point of S and T.*

As we have remarked above, once a result has been proved for (1.5), it is not difficult to prove it for the iteration process (1.2). For example, combining Theorem 3.3 and Lemma 2.3, we can obtain the following result which can be applied to the spaces not covered by Theorem KF1 and Theorem KKT. For details, see [8], [5], and [14].

Theorem 3.4. *Let E be a real uniformly convex Banach space such that its dual E^* has the Kadec–Klee property and K a nonempty closed convex subset of E. Let $S, T : K \to K$ be two nonexpansive mappings with $F := F(S) \cap F(T) \neq \varnothing$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ be real sequences in* $[0,1]$ *satisfying* (1.3)*,* (1.4) *and $\alpha_n, \alpha'_n \in [\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. For arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion* (1.2)*. Then $\{x_n\}$ converges weakly to some common fixed point of S and T.*

The mappings $S, T : K \to K$ with $F := F(S) \cap F(T) \neq \varnothing$ are said to satisfy *condition (B)* if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that for all $x \in K$

$$\max\{\|x - Tx\|, \|x - Sx\|\} \geq f(d(x, F)).$$

When $S = I$, the identity map or $S = T$, condition (B) reduces to condition (I) of Senter and Dotson [12]. Our condition (B) also contains condition (A') of Khan and Fakhar-ud-din [7] (see, for the definition of condition (A'), the statement of Theorem KF2). We further note that when $S = I$, condition (A') of Khan and Fakhar-ud-din [7] does not reduce to condition (I) of Senter and Dotson [12]. A mapping $T : K \to K$ is called (1) *demicompact* if any bounded sequence $\{x_n\}$ in K such that $\{x_n - Tx_n\}$ converges has a convergent subsequence; (2) *semi-compact* (or *hemicompact*) if any bounded sequence $\{x_n\}$ in K satisfying $\|x_n - Tx_n\| \to 0$ as $n \to \infty$ has a convergent subsequence. Every demicompact mapping is semi-compact but the converse is not true in general. It is known [12] that if $T : K \to K$ is nonexpansive and demicompact, then T satisfies condition (I).

Theorem 3.5. *Let E be a real uniformly convex Banach space and K a nonempty closed convex subset of E. Let $S, T : K \to K$ be two nonexpansive mappings with $F := F(S) \cap F(T) \neq \varnothing$. Let $\{\alpha_n\}$, $\{\beta_n\}$ be real sequences* *in* $[\epsilon, 1-\epsilon]$ *for some* $\epsilon \in (0,1)$. *For arbitrary* $x_1 \in K$, *define the sequence* $\{x_n\}$ *by the recursion* (1.5). *Suppose* S *and* T *satisfy condition (B). Then* $\{x_n\}$ *converges strongly to some common fixed point of* S *and* T.

Proof. Let $x^* \in F$. Then, by Lemma 3.1, $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Also

$$\|x_{n+1} - x^*\| \le \|x_n - x^*\|$$

for all $n \ge 1$. This implies that $d(x_{n+1}, F) \le d(x_n, F)$ and so, by Lemma 2.1, $\lim_{n\to\infty} d(x_n, F)$ exists. Also by Lemma 3.2, $\lim_{n\to\infty} \|x_n - Sx_n\| = 0 = \lim_{n\to\infty} \|x_n - Tx_n\|$. Since S and T satisfy condition (B), it follows that

$$\lim_{n\to\infty} f(d(x_n, F)) = 0.$$

This implies that $\lim_{n\to\infty} d(x_n, F) = 0$. So we can find a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence $\{x_j^*\} \subset F$ satisfying $\|x_{n_j} - x_j^*\| \le 2^{-j}$. Put $n_{j+1} = n_j + k$ for some $k \ge 1$. Then

$$\|x_{n_{j+1}} - x_j^*\| \le \|x_{n_j+k-1} - x_j^*\| \le \|x_{n_j} - x_j^*\| \le \frac{1}{2^j}$$

and so we have $\|x_{j+1}^* - x_j^*\| \le \frac{3}{2^{j+1}}$. Thus $\{x_j^*\}$ is a Cauchy sequence and so there exists $y^* \in K$ such that $x_j^* \to y^*$ as $j \to \infty$. Since F is closed, $y^* \in F$. As a result, we have $x_{n_j} \to y^*$ as $j \to \infty$. Since $\lim_{n\to\infty} \|x_n - y^*\|$ exists by Lemma 3.1, the conclusion follows. □

Combining Theorem 3.5 and Lemma 2.3, we obtain the following result, which contains Theorem KF2 as a special case. Unlike Khan and Fakhar-ud-din [7], we do not impose the boundedness condition on K.

Theorem 3.6. *Let* E *be a real uniformly convex Banach space and* K *a nonempty closed convex subset of* E. *Let* $S, T : K \to K$ *be two nonexpansive mappings with* $F := F(S) \cap F(T) \neq \varnothing$. *Let* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ *be real sequences in* $[0,1]$ *satisfying* (1.3), (1.4) *and* $\alpha_n, \alpha'_n \in [\epsilon, 1-\epsilon]$ *for some* $\epsilon \in (0,1)$. *For arbitrary* $x_1 \in K$, *define the sequence* $\{x_n\}$ *by the recursion* (1.2). *Suppose* S *and* T *satisfy condition (B). Then* $\{x_n\}$ *converges strongly to some common fixed point of* S *and* T.

Finally we prove the following strong convergence theorem.

Theorem 3.7. *Let* E *be a real uniformly convex Banach space and* K *a nonempty closed convex subset of* E. *Let* $S, T : K \to K$ *be two nonexpansive mappings with* $F := F(S) \cap F(T) \neq \varnothing$. *Let* $\{\alpha_n\}$, $\{\beta_n\}$ *be real sequences in* $[\epsilon, 1-\epsilon]$ *for some* $\epsilon \in (0,1)$. *For arbitrary* $x_1 \in K$, *define the sequence* $\{x_n\}$ *by the recursion* (1.5). *Suppose one of* S *and* T *is semi-compact. Then* $\{x_n\}$ *converges strongly to some common fixed point of* S *and* T.

Proof. We may assume that T is semi-compact. By Lemma 3.1, $\{x_n\}$ is bounded and by Lemma 3.2, we have $\lim_{n\to\infty} \|x_n - Tx_n\| = 0 = \lim_{n\to\infty} \|x_n - Sx_n\|$. So there exists a subsequence $\{x_m\}$ of $\{x_n\}$ such that $x_m \to x^* \in K$ as $j \to \infty$. Now

Lemma 3.2 guarantees that $\lim_{m\to\infty} \|x_m - Tx_m\| = 0 = \lim_{m\to\infty} \|x_m - Sx_m\|$ and so $\|x^* - Tx^*\| = 0 = \|x^* - Sx^*\|$. This implies that $x^* \in F$. Since $\lim_{n\to\infty} d(x_n, F) = 0$, it follows, as in the proof of Theorem 3.5, that $\{x_n\}$ converges strongly to some common fixed point of S and T. This completes the proof. □

The following proposition was noted in [3] (see [3] for definitions).

Proposition 3.8. *Let E be a uniformly convex Banach space and K be a nonempty closed bounded convex subset of E. Suppose $T : K \to K$. Then T is semi-compact if T satisfies any of the following conditions:*
(1) *T is either set-condensing or ball-condensing (or compact);*
(2) *T is a generalized contraction;*
(3) *T is uniformly strictly contractive;*
(4) *T is strictly semicontractive;*
(5) *T is of strictly semicontractive type;*
(6) *T is of strongly semicontractive type.*

Remarks.

(1) B. E. Rhoades in MR:2004m:47143 and C. E. Chidume in MR2003m:47133 pointed out that generalizing a convergence theorem for an iteration scheme without errors to the corresponding one with errors contributes nothing to the theory. Moreover, Rhoades in [11] has mentioned the practical impossibility of carrying out such an iteration scheme.
(2) Let E be a reflexive Banach space. Then the dual E^* of E has the Kadec-Klee property if and only if E is asymptotically smooth [6].
(3) It is possible to replace the semi-compactness assumption in Theorem 3.7 by any one of the contractive assumptions (1)–(6) of Proposition 3.8.

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