Georgian Mathematical Journal
Volume 13 (2006), Number 3, 539–572

ON SYSTEMS OF LINEAR FUNCTIONAL DIFFERENTIAL INEQUALITIES

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Abstract. The Cauchy problem for n-dimensional systems of linear functional differential equations is studied. Some new theorems on systems of functional differential inequalities are established.

2000 Mathematics Subject Classification: 34K06, 34K10.
Key words and phrases: Functional differential system, Cauchy problem, differential inequalities, sign-constant solutions.

1. Introduction

On the interval $[a, b]$, we consider the initial value problem

$$u'(t) = \ell(u)(t) + q(t), \qquad u(a) = c, \tag{1.1}$$

where $\ell : C([a, b]; \mathbb{R}^n) \to L([a, b]; \mathbb{R}^n)$ is a linear bounded operator, $q \in L([a, b]; \mathbb{R}^n)$, and $c \in \mathbb{R}^n$. By a solution of problem (1.1) is understood an absolutely continuous vector function $u : [a, b] \to \mathbb{R}^n$ satisfying the equation in (1.1) almost everywhere on $[a, b]$ and also the initial condition $u(a) = c$. A special case of problem (1.1) is the initial value problem for the system of ordinary differential equations

$$u' = P(t)u + q(t), \qquad u(a) = c, \tag{1.2}$$

where $P : [a, b] \to \mathbb{R}^{n\times n}$ is an integrable matrix function. The following proposition on systems of ordinary differential inequalities is well-known (see, e.g., [10]).

Proposition 1.1. *Let the matrix function $P = (p_{ik})_{i,k=1}^n$ satisfy*

$$p_{ik}(t) \ge 0 \quad \text{for a. a.} \quad t \in [a, b], \ i, k = 1, \dots, n, \ i \ne k.$$

Then every absolutely continuous vector function $x : [a, b] \to \mathbb{R}^n$ such that

$$x'(t) \le P(t)x(t) + q(t) \quad \text{for a. a.} \quad t \in [a, b], \qquad x(a) \le c,$$

satisfies

$$x(t) \le u(t) \quad \text{for} \quad t \in [a, b],$$

where u is a solution of problem (1.2).

Theorems on functional differential inequalities and their systems are studied, e.g., in [3, 4, 5, 6, 11, 16, 15, 8, 7, 12, 14, 1]. In this paper, we establish new theorems on systems of linear functional differential inequalities complementing the known results. Below we will show (see Remark 3.1) that, under the

ISSN 1072-947X / $8.00 /

assumptions of the theorems obtained, the problem (1.1) has a unique solution for any q and c. Moreover, this solution is of constant sign if q and c satisfy some natural assumptions.

The paper is organized as follows. In Section 3, we prove the main results – theorems on systems of functional differential inequalities. Their corollaries for the operators with deviating arguments are given in Section 4. The results obtained are optimal, in a certain sense, which is isslustrated by the counter-examples constructed in Section 5.

2. Notation and Definitions

The following notation is used throughout the paper.

$\mathbb{N}$ is the set of all natural numbers, $\mathbb{R}$ is the set of all real numbers, $\mathbb{R}_+ = [0, +\infty[$.

$\mathbb{R}^n$ is the space of n-dimensional column vectors $x = (x_i)_{i=1}^n$ with elements $x_i \in \mathbb{R}$ $(i = 1, \dots, n)$ and the norm

$$\|x\| = \sum_{i=1}^{n} |x_i|.$$

$\mathbb{R}^n_+ = \{(x_i)_{i=1}^n \in \mathbb{R}^n : x_i \geq 0 \text{ for } i = 1, \dots, n\}$.

$\mathbb{R}^{n\times n}$ is the space of $n \times n$-matrices $X = (x_{ik})_{i,k=1}^n$ with elements $x_{ik} \in \mathbb{R}$ $(i, k = 1, \dots, n)$ and the norm

$$\|X\| = \sum_{i,k=1}^{n} |x_{ik}|.$$

If $x, y \in \mathbb{R}^n$ and $X, Y \in \mathbb{R}^{n\times n}$, then

$$\begin{aligned} x \leq y &\quad \text{if and only if} \quad x_i \leq y_i \quad \text{for} \quad i = 1, \dots, n, \\ x < y &\quad \text{if and only if} \quad x_i < y_i \quad \text{for} \quad i = 1, \dots, n, \\ X \leq Y &\quad \text{if and only if} \quad x_{ik} \leq y_{ik} \quad \text{for} \quad i, k = 1, \dots, n. \end{aligned}$$

If $x \in \mathbb{R}^n$ then $|x| = (|x_i|)_{i=1}^n$ and

$$[x]_+ = \frac{1}{2}(|x| + x), \qquad [x]_- = \frac{1}{2}(|x| - x).$$

E is the unit matrix, Θ is the zero matrix.

X^{-1} is the inverse matrix to $X \in \mathbb{R}^{n\times n}$.

$r(X)$ is the spectral radius of a matrix $X \in \mathbb{R}^{n\times n}$;

X^T is the transposed matrix to an $n \times m$-matrix X.

If $x_i \in \mathbb{R}$ $(i = 1, \dots, n)$, then

$$\operatorname{diag}(x_1, \dots, x_n) = \begin{pmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x_n \end{pmatrix}.$$

$C([a,b];\mathbb{R}^n)$ is the Banach space of continuous vector functions $u:[a,b]\to\mathbb{R}^n$ equipped with the norm

$$\|u\|_C = \max\{\|u(t)\| : t\in[a,b]\}.$$

$C_a^\Sigma([a,b];\mathbb{R}_+^n)$, where $\Sigma = (\sigma_1,\dots,\sigma_n)$ with elements $\sigma_i \in \{-1,1\}$ $(i = 1,\dots,n)$, is the set of functions $u\in C([a,b];\mathbb{R}^n)$ such that

$$u(a)=0, \qquad \operatorname{diag}(\sigma_1,\dots,\sigma_n)u(t)\in\mathbb{R}_+^n \quad \text{for} \quad t\in[a,b].$$

$\widetilde{C}([a,b];\mathbb{R}^n)$ is the set of absolutely continuous vector functions $u:[a,b]\to\mathbb{R}^n$.

$L([a,b];\mathbb{R}^n)$ is the Banach space of Lebesgue integrable vector functions $h:[a,b]\to\mathbb{R}^n$ equipped with the norm

$$\|h\|_L = \int_a^b \|h(s)\|ds.$$

In what follows, the equalities and inequalities with integrable functions are understood to hold almost everywhere.

$\mathcal{L}_{ab}^n$ is the set of linear bounded operators $\ell : C([a,b];\mathbb{R}^n)\to L([a,b];\mathbb{R}^n)$. We write $\mathcal{L}_{ab}$ instead of $\mathcal{L}_{ab}^1$.

For any $\varphi\in\mathcal{L}_{ab}^n$, the operators $\varphi_i : C([a,b];\mathbb{R}^n)\to L([a,b];\mathbb{R})$, $\varphi_{ik} : C([a,b];\mathbb{R})\to L([a,b];\mathbb{R})$ $(i,k=1,\dots,n)$, and the matrix function $P_\varphi:[a,b]\to\mathbb{R}^{n\times n}$ are defined in the following way:

- For any $v\in C([a,b];\mathbb{R}^n)$, $\varphi_i(v)$ is the i-th component of a vector function $\varphi(v)$.
- For any $z\in C([a,b];\mathbb{R})$, we put $\varphi_{ik}(z)=\varphi_i(\widehat{z})$, where

  $$\widehat{z} = (0,\dots,0,\overbrace{z}^{\text{k-th}},0,\dots,0)^T.$$

  Obviously, $\varphi_{ik}\in\mathcal{L}_{ab}$ for $i,k=1,\dots n$.
- For almost all $t\in[a,b]$, we put $P_\varphi(t) = (\varphi_{ik}(1)(t))_{i,k=1}^n$.

Remark 2.1. If $\ell\in\mathcal{L}_{ab}^n$ and $u=(u_i)_{i=1}^n\in C([a,b];\mathbb{R}^n)$, it is clear that

$$\ell(u) = (\ell_i(u))_{i=1}^n \quad \text{and} \quad \ell_i(u) = \sum_{k=1}^n \ell_{ik}(u_k) \quad \text{for} \quad i=1,\dots,n.$$

Definition 2.1. Let $\Sigma=(\sigma_1,\dots,\sigma_n)$ be a row vector with elements $\sigma_i\in\{-1,1\}$ $(i=1,\dots,n)$. An operator $\ell\in\mathcal{L}_{ab}^n$ is said to be Σ-nonnegative if the relation

$$\operatorname{diag}(\sigma_1,\dots,\sigma_n)\,\ell(u)(t)\ge 0 \quad \text{for} \quad t\in[a,b]$$

holds for every function $u\in C([a,b];\mathbb{R}^n)$ satisfying

$$\operatorname{diag}(\sigma_1,\dots,\sigma_n)\,u(t)\ge 0 \quad \text{for} \quad t\in[a,b].$$

The set of Σ-nonnegative operators is denoted by $\mathcal{P}_{ab}^{n,\Sigma}$. We say that an operator $\ell\in\mathcal{L}_{ab}^n$ is Σ-nonpositive if $-\ell\in\mathcal{P}_{ab}^{n,\Sigma}$.

Remark 2.2. It is obvious that $\mathcal{P}_{ab}^{n,\Sigma} = \mathcal{P}_{ab}^{n,-\Sigma}$. Therefore, following [6], we write $\mathcal{P}_{ab}$ instead of $\mathcal{P}_{ab}^{1,\sigma_1}$.

It is not difficult either to verify that $\ell \in \mathcal{P}_{ab}^{n,\Sigma}$ if and only if

$$\sigma_i \sigma_k \ell_{ik} \in \mathcal{P}_{ab} \quad \text{for} \quad i, k = 1, \dots, n. \tag{2.1}$$

Definition 2.2. An operator $\ell \in \mathcal{L}_{ab}^n$ is said to be an a-Volterra operator if, for every $b_0 \in]a, b]$ and $v \in C([a, b]; \mathbb{R}^n)$ satisfying

$$v(t) = 0 \quad \text{for} \quad t \in [a, b_0],$$

we have

$$\ell(v)(t) = 0 \quad \text{for} \quad t \in [a, b_0].$$

Remark 2.3. It is clear that $\ell \in \mathcal{L}_{ab}^n$ is an a-Volterra operator if and only if ℓ_{ik} $(i, k = 1, \dots, n)$ are a-Volterra operators.

3. Theorems on Systems of Differential Inequalities

In what follows, we fix a row vector $\Sigma = (\sigma_1, \dots, \sigma_n)$ with elements $\sigma_i \in \{-1, 1\}$ $(i = 1, \dots, n)$. In order to simplify the formulation of the main results we first introduce the following definition.

Definition 3.1. We say that an operator $\ell \in \mathcal{L}_{ab}^n$ belongs to the set $\mathcal{S}_{ab}^{n,\Sigma}(a)$ if every vector function $u \in \widetilde{C}([a, b]; \mathbb{R}^n)$ such that

$$\operatorname{diag}(\sigma_1, \dots, \sigma_n)\Big(u'(t) - \ell(u)(t)\Big) \geq 0 \quad \text{for} \quad t \in [a, b], \tag{3.1}$$

$$\operatorname{diag}(\sigma_1, \dots, \sigma_n)\, u(a) \geq 0 \tag{3.2}$$

satisfies

$$\sigma_i u_i(t) \geq 0 \quad \text{for} \quad t \in [a, b],\ i = 1, \dots, n. \tag{3.3}$$

Since $\mathcal{S}_{ab}^{n,\Sigma}(a) = \mathcal{S}_{ab}^{n,-\Sigma}$, following [6], we write $\mathcal{S}_{ab}(a)$ instead of $\mathcal{S}_{ab}^{1,\sigma_1}(a)$.

Remark 3.1. It follows from Definition 3.1 that if the inclusion $\ell \in \mathcal{S}_{ab}^{n,\Sigma}(a)$ is true, then the homogeneous problem

$$u'(t) = \ell(u)(t), \qquad u(a) = 0 \tag{3.4}$$

has only the trivial solution. Consequently, according to the Fredholm property of the problem (1.1) (see, e.g., [9, 11, 17] and the references therein), the problem (1.1) is uniquely solvable for any $q \in L([a, b]; \mathbb{R}^n)$ and $c \in \mathbb{R}^n$.

If, moreover, q and c are such that

$$\operatorname{diag}(\sigma_1, \dots, \sigma_n)\, q(t) \geq 0 \quad \text{for} \quad t \in [a, b], \qquad \operatorname{diag}(\sigma_1, \dots, \sigma_n)\, c \geq 0,$$

then the solution u of the problem (1.1) satisfies (3.3).

Remark 3.2. The following analogue of Proposition 1.1 for the problem (1.1) is true if $\ell \in \mathcal{S}_{ab}^{n,\Sigma}(a)$.

Let $\ell \in \mathcal{S}_{ab}^{n,\Sigma}(a)$. Then every vector function $x \in \widetilde{C}([a, b]; \mathbb{R}^n)$ such that

$$\operatorname{diag}(\sigma_1, \dots, \sigma_n)\big(x'(t) - \ell(x)(t) - q(t)\big) \leq 0 \quad \textit{for a. a.} \quad t \in [a, b],$$

$$\operatorname{diag}(\sigma_1, \dots, \sigma_n)\big(x(a) - c\big) \leq 0,$$

satisfies

$$\operatorname{diag}(\sigma_1,\dots,\sigma_n)\big(x(t)-u(t)\big)\le 0 \quad \textit{for} \quad t\in[a,b],$$

where u is a solution of the problem (1.1).

A certain "characteristic" structure of the set $\mathcal{S}_{ab}^{n,\Sigma}(a)$ is described in the following theorem the proof of which is given in Subsection 3.3.

Theorem 3.1. *Let $\ell=\ell^+-\ell^-$, where $\ell^+,\ell^-\in\mathcal{P}_{ab}^{n,\Sigma}$ are such that*

$$\ell^+\in\mathcal{S}_{ab}^{n,\Sigma}(a), \qquad -\ell^-\in\mathcal{S}_{ab}^{n,\Sigma}(a). \tag{3.5}$$

Then $\ell\in\mathcal{S}_{ab}^{n,\Sigma}(a)$.

Analogous results for first and second order scalar functional differential equations are given in [6] and [13], respectively.

Remark 3.3. The assumption (3.5) in Theorem 3.1 can be replaced neither by the assumption

$$(1-\varepsilon)\ell^+\in\mathcal{S}_{ab}^{n,\Sigma}(a), \qquad -\ell^-\in\mathcal{S}_{ab}^{n,\Sigma}(a)$$

nor by the assumption

$$\ell^+\in\mathcal{S}_{ab}^{n,\Sigma}(a), \qquad -(1-\varepsilon)\ell^-\in\mathcal{S}_{ab}^{n,\Sigma}(a),$$

no matter how small $\varepsilon>0$ is (see Examples 5.1 and 5.2).

In order to use Theorem 3.1 we should find some conditions sufficient for the inclusion $\ell\in\mathcal{S}_{ab}^{n,\Sigma}(a)$ in both cases where the operator ℓ is Σ-nonnegative and Σ-nonpositive. The conditions indicated are given in Subsections 3.1 and 3.2.

Below we will show (see Theorem 3.3) that if $\ell\in\mathcal{S}_{ab}^{n,\Sigma}(a)$ is a Σ-nonpositive operator then neccessarily $\ell_{ik}\equiv 0$ for $i,k=1,\dots,n$, $i\neq k$. Consequently, using Theorem 3.1 and the results of Subsections 3.1 and 3.2, we can derive several efficient conditions sufficient for the inclusion $\ell\in\mathcal{S}_{ab}^{n,\Sigma}(a)$ if the operator ℓ satisfies

$$\ell_{ii}=\ell_{ii}^+-\ell_{ii}^- \quad \text{with} \quad \ell_{ii}^+,\ell_{ii}^-\in\mathcal{P}_{ab} \quad (i=1,\dots,n)$$

and

$$\sigma_i\sigma_k\ell_{ik}\in\mathcal{P}_{ab} \quad (i,k=1,\dots,n,\ i\neq k).$$

3.1. **The Case $\ell\in\mathcal{P}_{ab}^{n,\Sigma}$.** In this part, we give some efficient conditions for the inclusion $\ell\in\mathcal{S}_{ab}^{n,\Sigma}(a)$ in the case where the operator ℓ is Σ-nonnegative. The proofs of the statements formulated below can be found in Subsection 3.3. We first introduce the following notation.

Notation 3.1. Let $\ell\in\mathcal{L}_{ab}^n$ where $n\ge 2$. For any $k\in\{1,\dots,n\}$, we define the operator $\ell^k=(\ell_i^k)_{i=1}^{n-1}\in\mathcal{L}_{ab}^{n-1}$ by setting

$$\ell_i^k(v)(t) \overset{\text{def}}{=} \begin{cases} \displaystyle\sum_{j=1}^{k-1} \ell_{ij}(v_j)(t) + \sum_{j=k}^{n-1} \ell_{ij+1}(v_j)(t) \\ \qquad\qquad \text{for} \quad t \in [a,b], \ i < k, \\ \displaystyle\sum_{j=1}^{k-1} \ell_{i+1j}(v_j)(t) + \sum_{j=k}^{n-1} \ell_{i+1j+1}(v_j)(t) \\ \qquad\qquad \text{for} \quad t \in [a,b], \ k \le i \le n-1. \end{cases} \tag{3.6}$$

Proposition 3.1. *Let $\ell \in \mathcal{P}_{ab}^{n,\Sigma}$, where $n \ge 2$, and let $k \in \{1,\dots,n\}$ be such that*

$$\ell_{kj} \equiv 0 \quad \text{for} \quad j = 1,\dots,n, \ j \ne k. \tag{3.7}$$

Then $\ell \in \mathcal{S}_{ab}^{n,\Sigma}(a)$ if and only if $\ell_{kk} \in \mathcal{S}_{ab}(a)$ and $\ell^k \in \mathcal{S}_{ab}^{n-1,\Sigma^k}(a)$, where $\Sigma^k = (\sigma_1,\dots,\sigma_{k-1},\sigma_{k+1},\dots,\sigma_n)$.

Let us note that the efficient conditions sufficient for the inclusion $\ell \in \mathcal{S}_{ab}(a)$ are given in [6].

Theorem 3.2. *Let $\ell \in \mathcal{P}_{ab}^{n,\Sigma}$. Then $\ell \in \mathcal{S}_{ab}^{n,\Sigma}(a)$ if and only if there exists a function $\gamma \in \widetilde{C}([a,b];\mathbb{R}^n)$ satisfying*

$$\operatorname{diag}(\sigma_1,\dots,\sigma_n)\,\gamma(t) > 0 \quad \text{for} \quad t \in [a,b], \tag{3.8}$$

$$\operatorname{diag}(\sigma_1,\dots,\sigma_n)\Big(\gamma'(t) - \ell(\gamma)(t)\Big) \ge 0 \quad \text{for} \quad t \in [a,b]. \tag{3.9}$$

By a suitable choice of the function γ in Theorem 3.2 we obtain the following corollary.

Corollary 3.1. *Let $\ell \in \mathcal{P}_{ab}^{n,\Sigma}$ and let there exist numbers $m,k \in \mathbb{N}$ and $\alpha \in [0,1[$ such that $m > k$ and*

$$\operatorname{diag}(\sigma_1,\dots,\sigma_n)\Big(\rho^m(t) - \alpha\rho^k(t)\Big) \le 0 \quad \text{for} \quad t \in [a,b], \tag{3.10}$$

where $\rho^1 \in \mathbb{R}^n$ is such that

$$\operatorname{diag}(\sigma_1,\dots,\sigma_n)\,\rho^1 > 0, \tag{3.11}$$

$$\rho^{i+1}(t) = \varphi(\rho^i)(t) \quad \text{for} \quad t \in [a,b], \ i \in \mathbb{N}, \tag{3.12}$$

and

$$\varphi(v)(t) = \int_a^t \ell(v)(s)ds \quad \text{for} \quad t \in [a,b], \ v \in C([a,b];\mathbb{R}). \tag{3.13}$$

Then $\ell \in \mathcal{S}_{ab}^{n,\Sigma}(a)$.

Remark 3.4. The assumption $\alpha \in [0,1[$ in Corollary 3.1 cannot be replaced by the assumption $\alpha \in [0,1]$ (see Example 5.3).

From the last corollary we get

Corollary 3.2. *Let $\ell \in \mathcal{P}_{ab}^{n,\Sigma}$ and there exist numbers $\delta_i > 0$ $(i = 1, \dots, n)$ such that*

$$\max\left\{\frac{1}{\delta_i}\sum_{k=1}^{n}\delta_k\int_a^b|\ell_{ik}(1)(s)|ds : i = 1, \dots, n\right\} < 1. \tag{3.14}$$

Then $\ell \in \mathcal{S}_{ab}^{n,\Sigma}(a)$.

Remark 3.5. Example 5.3 shows that, in general, the strict inequality (3.14) in Corollary 3.2 cannot be replaced by the nonstrict one. However, in the case where the equality

$$\max\left\{\frac{1}{\delta_i}\sum_{k=1}^{n}\delta_k\int_a^b|\ell_{ik}(1)(s)|ds : i = 1, \dots, n\right\} = 1 \tag{3.15}$$

is satisfied with some $\delta_i > 0$ $(i = 1, \dots, n)$, the inclusion $\ell \in \mathcal{S}_{ab}^{n,\Sigma}(a)$ is still true under additional assumptions. Some of such additional conditions are presented in the next proposition.

Proposition 3.2. *Let $\ell \in \mathcal{P}_{ab}^{n,\Sigma}$ and let there exist numbers $\delta_i > 0$ $(i = 1, \dots, n)$ and a set $J \subset \{1, \dots, n\}$ such that $J \neq \varnothing$,*

$$\frac{1}{\delta_i}\sum_{k=1}^{n}\delta_k\int_a^b|\ell_{ik}(1)(s)|ds < 1 \quad \textit{for} \quad i \in J, \tag{3.16}$$

$$\frac{1}{\delta_i}\sum_{k=1}^{n}\delta_k\int_a^b|\ell_{ik}(1)(s)|ds = 1 \quad \textit{for} \quad i \in \{1, \dots, n\} \setminus J, \tag{3.17}$$

and

$$\sum_{\substack{k=1\\k\neq i}}^{n}|\ell_{ik}(1)| \not\equiv 0 \quad \textit{for} \quad i \in \{1, \dots, n\} \setminus J. \tag{3.18}$$

Then $\ell \in \mathcal{S}_{ab}^{n,\Sigma}(a)$.

The last proposition cannot be applied in the case where

$$\frac{1}{\delta_i}\sum_{k=1}^{n}\delta_k\int_a^b|\ell_{ik}(1)(s)|ds = 1 \quad \text{for} \quad i = 1, \dots, n$$

with some $\delta_i > 0$ $(i = 1, \dots, n)$. Nevertheless, the following more general statement can be used in the case indicated.

Proposition 3.3. *Let $\ell \in \mathcal{P}_{ab}^{n,\Sigma}$ and let there exist numbers $\delta_i > 0$ $(i = 1, \dots, n)$ such that* (3.15) *is fulfilled. Then $\ell \in \mathcal{S}_{ab}^{n,\Sigma}(a)$ if and only if the homogeneous problem* (3.4) *has only the trivial solution.*

The next corollary of Theorem 3.2 contains another type of conditions sufficient for the inclusion $\ell \in \mathcal{S}_{ab}^{n,\Sigma}(a)$.

Corollary 3.3. *Let $\ell \in \mathcal{P}_{ab}^{n,\Sigma}$ and let $Y : [a,b] \to \mathbb{R}^{n\times n}$ be a fundamental matrix of the system*

$$x' = \widetilde{P}(t)x, \tag{3.19}$$

where the matrix function $\widetilde{P} = (\widetilde{p}_{ik})_{i,k=1}^n : [a,b] \to \mathbb{R}^{n\times n}$ is defined by

$$\widetilde{p}_{ii} \equiv 0 \quad \text{for} \quad i = 1,\dots,n,$$

$$\widetilde{p}_{ik}(t) = |\ell_{ik}(1)(t)| \exp\left(\int_a^t \Big[\ell_{kk}(1)(s) - \ell_{ii}(1)(s)\Big] ds\right) \tag{3.20}$$
$$\text{for} \quad t \in [a,b],\ i,k = 1,\dots,n,\ i \neq k.$$

Let, moreover, there exist an operator $\overline{\ell} \in \mathcal{P}_{ab}^{n,\Sigma}$ such that the inequality

$$\operatorname{diag}(\sigma_1,\dots,\sigma_n)\Big(\ell\big(\varphi(v)\big)(t) - P_\ell(t)\varphi(v)(t)\Big) \leq$$
$$\leq \operatorname{diag}(\sigma_1,\dots,\sigma_n)\overline{\ell}(v)(t) \quad \text{for} \quad t \in [a,b] \tag{3.21}$$

holds on the set $C_a^\Sigma([a,b];\mathbb{R}_+^n)$ and

$$BY(b)\int_a^b Y^{-1}(s)\widetilde{q}(s)ds < (1,\dots,1)^T, \tag{3.22}$$

where φ is given by (3.13), *$\widetilde{q} = (\widetilde{q}_i)_{i=1}^n \in L([a,b];\mathbb{R}^n)$ is defined by*

$$\widetilde{q}_i(t) = \sigma_i \overline{\ell}_i\big(\Sigma^T\big)(t)\ \mathrm{e}^{-\int_a^t \ell_{ii}(1)(s)ds} \quad \text{for} \quad t \in [a,b],\ i = 1,\dots,n, \tag{3.23}$$

and

$$B = \operatorname{diag}\left(\mathrm{e}^{\int_a^b \ell_{11}(1)(s)ds},\dots,\mathrm{e}^{\int_a^b \ell_{nn}(1)(s)ds}\right). \tag{3.24}$$

Then $\ell \in \mathcal{S}_{ab}^{n,\Sigma}(a)$.

In Corollaries 3.4 and 3.5, the efficient conditions are given, under which the fundamental matrix Y of the system (3.19) satisfies (3.22).

Corollary 3.4. *Let $\ell \in \mathcal{P}_{ab}^{n,\Sigma}$ and let there exist an operator $\overline{\ell} \in \mathcal{P}_{ab}^{n,\Sigma}$ such that the inequality* (3.21) *holds on the set $C_a^\Sigma([a,b];\mathbb{R}_+^n)$ and*

$$\mathrm{e}^{\max\left\{\int_a^b \ell_{ii}(1)(s)ds : i=1,\dots,n\right\}} \int_a^b h(s)\,\mathrm{e}^{\int_s^b p(\xi)d\xi} ds < 1, \tag{3.25}$$

where φ is given by (3.13),

$$h(t) \overset{\text{def}}{=} \max\Big\{\widetilde{q}_i(t) : i = 1,\dots,n\Big\} \quad \text{for} \quad t \in [a,b], \tag{3.26}$$

$$p(t) \overset{\text{def}}{=} \max\left\{\sum_{k=1}^n \widetilde{p}_{ik}(t) : i = 1,\dots,n\right\} \quad \text{for} \quad t \in [a,b], \tag{3.27}$$

and $\widetilde{p}_{ik}$, $\widetilde{q}_i$ $(i,k=1,\dots,n)$ *are defined by* (3.20) *and* (3.23), *respectively. Then* $\ell\in\mathcal{S}^{n,\Sigma}_{ab}(a)$.

Remark 3.6. The strict inequality (3.25) in Corollary 3.4 cannot be replaced by the nonstrict one (see Example 5.3).

For two-dimensional systems of differential inequalities, we get from Corollary 3.3

Corollary 3.5. *Let* $n=2$, $\ell\in\mathcal{P}^{2,\Sigma}_{ab}$, *and let there exist an operator* $\overline{\ell}\in\mathcal{P}^{2,\Sigma}_{ab}$ *such that the inequality* (3.21) *holds on the set* $C^{\Sigma}_a([a,b];\mathbb{R}^2_+)$. *Let, moreover,*

$$\max\left\{\lambda_1\,\mathrm{e}^{\int_a^b\ell_{11}(1)(s)ds},\ \lambda_2\,\mathrm{e}^{\int_a^b\ell_{22}(1)(s)ds}\right\}<1, \tag{3.28}$$

where

$$\lambda_i=\int_a^b\cosh\left(\int_s^b p(\xi)d\xi\right)\widetilde{q}_i(s)ds+\int_a^b\sinh\left(\int_s^b p(\xi)d\xi\right)\widetilde{q}_{3-i}(s)ds \quad \text{for} \quad i=1,2, \tag{3.29}$$

$$p(t)\overset{\text{def}}{=}\max\left\{\widetilde{p}_{12}(t),\widetilde{p}_{21}(t)\right\}\quad \text{for}\quad t\in[a,b], \tag{3.30}$$

and $\widetilde{p}_{ik}$, $\widetilde{q}_i$ $(i,k=1,2)$ *are defined by* (3.20) *and* (3.23), *respectively. Then* $\ell\in\mathcal{S}^{2,\Sigma}_{ab}(a)$.

Remark 3.7. Example 5.3 shows that the strict inequality (3.28) in Corollary 3.5 cannot be replaced by the nonstrict one.

The next proposition follows from Corollary 3.3.

Proposition 3.4. *Let* $\ell\in\mathcal{P}^{n,\Sigma}_{ab}$ *be an a-Volterra operator. Then* ℓ *belongs to the set* $\mathcal{S}^{n,\Sigma}_{ab}(a)$.

3.2. **The Case** $-\ell\in\mathcal{P}^{n,\Sigma}_{ab}$**.** In the case where ℓ is a Σ-nonpositive operator, we have a sufficient and necessary condition for the inclusion $\ell\in\mathcal{S}^{n,\Sigma}_{ab}(a)$.

Theorem 3.3. *Let* $-\ell\in\mathcal{P}^{n,\Sigma}_{ab}$. *Then* $\ell\in\mathcal{S}^{n,\Sigma}_{ab}(a)$ *if and only if the following two conditions are satisfied:*

(a) $\ell_{ii}\in\mathcal{S}_{ab}(a)$ *for every* $i\in\{1,\dots,n\}$;
(b) $\ell_{ik}\equiv 0$ *for every* $i,k\in\{1,\dots,n\}$, $i\neq k$.

The proof of Theorem 3.3 is given in Subsection 3.3. Using the results obtained in [6], we can immediately formulate the following corollary of Theorem 3.3.

Corollary 3.6. *Let the operator* $\ell\in\mathcal{L}^n_{ab}$ *be defined by*

$$\ell(v)(t)\overset{\text{def}}{=}\Big(\ell_{11}(v_1)(t),\dots,\ell_{nn}(v_n)(t)\Big)^T\quad \text{for}\quad t\in[a,b],$$

where $-\ell_{ii}\in\mathcal{P}_{ab}$ *and* ℓ_{ii} *are a-Volterra operators* $(i=1,\dots,n)$. *Let, moreover, for every* $i\in\{1,\dots,n\}$, *at least one of the following conditions be satisfied:*

(a) *there exists an absolutely continuous function* $\gamma : [a,b] \to \mathbb{R}_+$ *such that*

$$\gamma(t) > 0 \quad \textit{for} \quad t \in [a,b[\,,$$
$$\gamma'(t) \le \ell_{ii}(\gamma)(t) \quad \textit{for} \quad t \in [a,b];$$

(b)
$$\int_a^b |\ell_{ii}(1)(s)| ds \le 1;$$

(c)
$$\int_a^b \left| \widetilde{\ell}_{ii}(1)(s) \right| \mathrm{e}^{\int_a^s |\ell_{ii}(1)(\xi)| d\xi} ds \le 1, \ \textit{where}$$

$$\widetilde{\ell}_{ii}(z)(t) = \ell_{ii}\left(\theta_i(z)\right)(t) - \ell_{ii}(1)(t)\theta_i(z)(t) \quad \textit{for} \quad t \in [a,b],\ z \in C([a,b];\mathbb{R}),$$

$$\theta_i(z)(t) = \int_a^t \ell_{ii}(z_i)(s) ds \quad \textit{for} \quad t \in [a,b],\ z \in C([a,b];\mathbb{R}),$$

$$z_i(t) = z(t) e^{\int_a^t |\ell_{ii}(1)(s)| ds} \quad \textit{for} \quad t \in [a,b],\ z \in C([a,b];\mathbb{R}).$$

Then $\ell \in \mathcal{S}_{ab}^{n,\Sigma}(a)$.

Note that, according to Theorem 2 in [2], the assumption on the operators ℓ_{ii} to be a-Volterra ones is necessary in Corollary 3.6.

3.3. **Proofs.** In this part, we prove all the statements formulated above.

Proof of Theorem 3.1. Let $u \in \widetilde{C}([a,b];\mathbb{R}^n)$ be a function satisfying the conditions (3.1) and (3.2). We will show that (3.3) is true. Put

$$\widetilde{u}_i(t) = \sigma_i[\sigma_i u_i(t)]_- \quad \text{for} \quad t \in [a,b],\ i = 1,\dots,n. \tag{3.31}$$

According to the inclusion $-\ell^- \in \mathcal{S}_{ab}^{n,\Sigma}(a)$ and Remark 3.1, the problem

$$w'(t) = -\ell^-(w)(t) - \ell^+\big(\widetilde{u}\big)(t), \tag{3.32}$$
$$w(a) = 0 \tag{3.33}$$

has a unique solution w and

$$\sigma_i w_i(t) \le 0 \quad \text{for} \quad t \in [a,b],\ i = 1,\dots,n. \tag{3.34}$$

We first note that, for any $t \in [a,b]$ and $i = 1,\dots,n$,

$$\sigma_i\Big(u_i(t) + \widetilde{u}_i(t)\Big) = \sigma_i u_i(t) + [\sigma_i u_i(t)]_- = [\sigma_i u_i(t)]_+ \ge 0. \tag{3.35}$$

Therefore, in view of (3.1), (3.32), (3.35), and the assumption $\ell^+ \in \mathcal{P}_{ab}^{n,\Sigma}$, we get

$$\sigma_i\Big[u_i'(t) - w_i'(t) + \ell_i^-(u-w)(t)\Big] \ge$$
$$\ge \sigma_i \ell_i^+\big(u + \widetilde{u}\big)(t) \ge 0 \quad \text{for} \quad t \in [a,b],\ i = 1,\dots,n,$$

i.e.,

$$\operatorname{diag}(\sigma_1,\dots,\sigma_n)\Big[u'(t)-w'(t)+\ell^-(u-w)(t)\Big]\geq 0 \quad \text{for} \quad t\in[a,b]. \tag{3.36}$$

On the other hand, (3.2) and (3.33) yield

$$\operatorname{diag}(\sigma_1,\dots,\sigma_n)\Big(u(a)-w(a)\Big)\geq 0. \tag{3.37}$$

Since $-\ell^- \in \mathcal{S}_{ab}^{n,\Sigma}(a)$, the inequalities (3.36) and (3.37) result in

$$\sigma_i u_i(t)\geq \sigma_i w_i(t) \quad \text{for} \quad t\in[a,b],\ i=1,\dots,n. \tag{3.38}$$

Using (3.34), from (3.38) we get

$$-[\sigma_i u_i(t)]_- \geq \sigma_i w_i(t) \quad \text{for} \quad t\in[a,b],\ i=1,\dots,n,$$

i.e.,

$$-\operatorname{diag}(\sigma_1,\dots,\sigma_n)\,w(t)\geq \operatorname{diag}(\sigma_1,\dots,\sigma_n)\,\widetilde{u}(t) \quad \text{for} \quad t\in[a,b]. \tag{3.39}$$

Finally, by virtue of (3.34), (3.39), and the assumptions $\ell^+,\ell^-\in\mathcal{P}_{ab}^{n,\Sigma}$, it follows from (3.32) that

$$\begin{aligned}\operatorname{diag}(\sigma_1,\dots,\sigma_n)\Big(w'(t)-\ell^+(w)(t)\Big)&\geq \operatorname{diag}(\sigma_1,\dots,\sigma_n)\Big(w'(t)+\ell^+\big(\widetilde{u}\big)(t)\Big)\\ &=-\operatorname{diag}(\sigma_1,\dots,\sigma_n)\,\ell^-(w)(t)\geq 0 \quad \text{for} \quad t\in[a,b].\end{aligned}$$

Hence, the inclusion $\ell^+\in\mathcal{S}_{ab}^{n,\Sigma}(a)$, on account of (3.33), implies

$$\sigma_i w_i(t)\geq 0 \quad \text{for} \quad t\in[a,b],\ i=1,\dots,n,$$

which, together with (3.38), guarantees the condition (3.3). □

Proof of Proposition 3.1. First suppose that

$$\ell_{kk}\in\mathcal{S}_{ab}(a), \qquad \ell^k\in\mathcal{S}_{ab}^{n-1,\Sigma^k}(a).$$

Let $u\in\widetilde{C}([a,b];\mathbb{R}^n)$ be a function satisfying the conditions (3.1) and (3.2). We will show that (3.3) is true. In view of (3.7), it follows from (3.1) and (3.2) that

$$\sigma_k u_k'(t)\geq \ell_{kk}(\sigma_k u_k)(t) \quad \text{for} \quad t\in[a,b], \qquad \sigma_k u_k(a)\geq 0, \tag{3.40}$$

and thus the assumption $\ell_{kk}\in\mathcal{S}_{ab}(a)$ implies

$$\sigma_k u_k(t)\geq 0 \quad \text{for} \quad t\in[a,b]. \tag{3.41}$$

According to the assumption $\ell\in\mathcal{P}_{ab}^{n,\Sigma}$ and Remark 2.2, we have $\sigma_i\sigma_j\ell_{ij}\in\mathcal{P}_{ab}$ for $i,j=1,\dots,n$. Therefore, using (3.41) in (3.1), we get

$$\begin{aligned}\sigma_i u_i'(t)\geq \sigma_i\sum_{j=1}^n \ell_{ij}(u_j)(t)&=\sigma_i\sum_{\substack{j=1\\ j\neq k}}^n \ell_{ij}(u_j)(t)+\sigma_i\sigma_k\ell_{ik}(\sigma_k u_k)(t)\\ &\geq \sigma_i\sum_{\substack{j=1\\ j\neq k}}^n \ell_{ij}(u_j)(t) \quad \text{for} \quad t\in[a,b],\ i=1,\dots,n,\ i\neq k.\end{aligned}$$

Hence, the assumption $\ell^k \in \mathcal{S}_{ab}^{n-1,\Sigma^k}(a)$ yields

$$\sigma_i u_i(t) \geq 0 \quad \text{for} \quad t \in [a,b],\ i = 1,\dots,n,\ i \neq k,$$

which, together with (3.41), guarantees the condition (3.3). Consequently, $\ell \in \mathcal{S}_{ab}^{n,\Sigma}(a)$.

Now suppose that $\ell \in \mathcal{S}_{ab}^{n,\Sigma}(a)$. First we will show that $\ell^k \in \mathcal{S}_{ab}^{n-1,\Sigma^k}(a)$. Indeed, let a function $v = (v_j)_{j=1}^{n-1} \in \widetilde{C}([a,b];\mathbb{R}^{n-1})$ be such that

$$\operatorname{diag}(\sigma_1,\dots,\sigma_{k-1},\sigma_{k+1},\dots,\sigma_n)\Big(v'(t) - \ell^k(v)(t)\Big) \geq 0 \quad \text{for} \quad t \in [a,b],$$
$$\operatorname{diag}(\sigma_1,\dots,\sigma_{k-1},\sigma_{k+1},\dots,\sigma_n)\, v(a) \geq 0.$$

Put

$$u_i(t) = \begin{cases} v_i(t) & \text{for} \quad t \in [a,b],\ i < k, \\ 0 & \text{for} \quad t \in [a,b],\ i = k, \\ v_{i-1}(t) & \text{for} \quad t \in [a,b],\ k < i \leq n. \end{cases}$$

It is clear that the function $u = (u_i)_{i=1}^n$ satisfies (3.1) and (3.2). Therefore, the assumption $\ell \in \mathcal{S}_{ab}^{n,\Sigma}(a)$ yields (3.3). Consequently,

$$\operatorname{diag}(\sigma_1,\dots,\sigma_{k-1},\sigma_{k+1},\dots,\sigma_n)\, v(t) \geq 0 \quad \text{for} \quad t \in [a,b],$$

and thus $\ell^k \in \mathcal{S}_{ab}^{n-1,\Sigma^k}(a)$.

It remains to show that $\ell_{kk} \in \mathcal{S}_{ab}(a)$. Let $z \in \widetilde{C}([a,b];\mathbb{R})$ be a function satisfying

$$z'(t) \geq \ell_{kk}(z)(t) \quad \text{for} \quad t \in [a,b], \qquad z(a) \geq 0. \tag{3.42}$$

We will show that

$$z(t) \geq 0 \quad \text{for} \quad t \in [a,b]. \tag{3.43}$$

Let the function $h = (h_j)_{j=1}^{n-1} \in L([a,b];\mathbb{R}^{n-1})$ be defined by

$$h_j(t) = \begin{cases} \ell_{jk}(\sigma_k z)(t) & \text{for} \quad t \in [a,b],\ j < k, \\ \ell_{j+1k}(\sigma_k z)(t) & \text{for} \quad t \in [a,b],\ k \leq j \leq n-1. \end{cases}$$

According to Remark 3.1 and the above-proved inclusion $\ell^k \in \mathcal{S}_{ab}^{n-1,\Sigma^k}(a)$, the problem

$$v'(t) = \ell^k(v)(t) + h(t), \qquad v(a) = 0$$

has a unique solution $v = (v_j)_{j=1}^{n-1}$. Put

$$u_i(t) = \begin{cases} v_i(t) & \text{for} \quad t \in [a,b],\ i < k, \\ \sigma_k z(t) & \text{for} \quad t \in [a,b],\ i = k, \\ v_{i-1}(t) & \text{for} \quad t \in [a,b],\ k < i \leq n. \end{cases}$$

It is clear that the function $u = (u_i)_{i=1}^n$ satisfies (3.1) and (3.2). Therefore, the assumption $\ell \in \mathcal{S}_{ab}^{n,\Sigma}(a)$ yields (3.3). Consequently, (3.43) is satisfied and thus $\ell_{kk} \in \mathcal{S}_{ab}(a)$. □

Proof of Theorem 3.2. First suppose that there exists $\gamma \in \widetilde{C}([a,b];\mathbb{R}^n)$ satisfying the conditions (3.8) and (3.9). Let $u \in \widetilde{C}([a,b];\mathbb{R}^n)$ be such that the conditions (3.1) and (3.2) hold. We will show that (3.3) is true. Put

$$A = \Big\{\lambda \in \mathbb{R}_+ : \sigma_i\Big(\lambda\gamma_i(t) + u_i(t)\Big) \geq 0 \quad \text{for} \quad t \in [a,b],\ i = 1,\dots,n\Big\}. \tag{3.44}$$

Since the function γ satisfies (3.8), we have $A \neq \varnothing$. Let

$$\lambda_0 = \inf A. \tag{3.45}$$

Now we put

$$w(t) = \lambda_0\gamma(t) + u(t) \quad \text{for} \quad t \in [a,b]. \tag{3.46}$$

It is clear that $\lambda_0 \geq 0$, $w \in \widetilde{C}([a,b];\mathbb{R}^n)$, and

$$\operatorname{diag}(\sigma_1,\dots,\sigma_n)\, w(t) \geq 0 \quad \text{for} \quad t \in [a,b]. \tag{3.47}$$

Therefore, by virtue of the assumption $\ell \in \mathcal{P}_{ab}^{n,\Sigma}$, we get

$$\operatorname{diag}(\sigma_1,\dots,\sigma_n)\, w'(t) \geq \operatorname{diag}(\sigma_1,\dots,\sigma_n)\, \ell(w)(t) \geq 0 \quad \text{for} \quad t \in [a,b]. \tag{3.48}$$

Assume that

$$\lambda_0 > 0. \tag{3.49}$$

Then it follows from (3.2), (3.8), and (3.49) that

$$\operatorname{diag}(\sigma_1,\dots,\sigma_n)\, w(a) > 0.$$

Hence, using (3.48), we get

$$\operatorname{diag}(\sigma_1,\dots,\sigma_n)\, w(t) > 0 \quad \text{for} \quad t \in [a,b]. \tag{3.50}$$

Consequently, there exists $\varepsilon \in\,]0,\lambda_0]$ such that

$$\sigma_i w_i(t) \geq \varepsilon\sigma_i\gamma_i(t) \quad \text{for} \quad t \in [a,b],\ i = 1,\dots,n,$$

i.e.,

$$\sigma_i\Big[(\lambda_0 - \varepsilon)\gamma_i(t) + u_i(t)\Big] \geq 0 \quad \text{for} \quad t \in [a,b],\ i = 1,\dots,n.$$

Hence, by virtue of (3.44), we get $\lambda_0 - \varepsilon \in A$, which contradicts (3.45).

The contradiction obtained proves that $\lambda_0 = 0$. Consequently, (3.46) and (3.47) yield (3.3) and thus $\ell \in \mathcal{S}_{ab}^{n,\Sigma}(a)$.

Now suppose that $\ell \in \mathcal{S}_{ab}^{n,\Sigma}(a)$. Then, according to Remark 3.1, the problem

$$\gamma'(t) = \ell(\gamma)(t), \qquad \gamma(a) = \Sigma^T \tag{3.51}$$

has a unique solution γ and

$$\operatorname{diag}(\sigma_1,\dots,\sigma_n)\, \gamma(t) \geq 0 \quad \text{for} \quad t \in [a,b]. \tag{3.52}$$

By virtue of the assumption $\ell \in \mathcal{P}_{ab}^{n,\Sigma}$, the equation in (3.51) implies

$$\operatorname{diag}(\sigma_1,\dots,\sigma_n)\, \gamma'(t) = \operatorname{diag}(\sigma_1,\dots,\sigma_n)\, \ell(\gamma)(t) \geq 0 \quad \text{for} \quad t \in [a,b]. \tag{3.53}$$

Therefore, in view of the initial condition $\gamma(a) = \Sigma^T$, we get

$$\sigma_i\gamma_i(t) \geq \sigma_i\gamma(a) = 1 \quad \text{for} \quad t \in [a,b].$$

Consequently, $\gamma \in \widetilde{C}([a,b];\mathbb{R}^n)$ satisfies (3.8) and (3.9). $\square$

Proof of Corollary 3.1. Put

$$\gamma(t)=(1-\alpha)\sum_{j=1}^{k}\rho^j(t)+\sum_{j=k+1}^{m}\rho^j(t) \quad \text{for} \quad t\in[a,b].$$

Obviously, $\gamma\in\widetilde{C}([a,b];\mathbb{R}^n)$. According to (3.11)–(3.13) and the assumption $\ell\in\mathcal{P}_{ab}^{n,\Sigma}$, it is not difficult to verify that

$$\operatorname{diag}(\sigma_1,\dots,\sigma_n)\,\gamma(t)\geq(1-\alpha)\operatorname{diag}(\sigma_1,\dots,\sigma_n)\,\rho^1>0 \quad \text{for} \quad t\in[a,b],$$

i.e., the condition (3.8) is satisfied. Moreover,

$$\gamma'(t)=(1-\alpha)\sum_{j=1}^{k-1}\ell\big(\rho^j\big)(t)+\sum_{j=k}^{m-1}\ell\big(\rho^j\big)(t) \\ =\ell(\gamma)(t)+\ell\big(\alpha\rho^k-\rho^m\big)(t) \quad \text{for} \quad t\in[a,b].$$

Therefore, in view of (3.10) and the assumption $\ell\in\mathcal{P}_{ab}^{n,\Sigma}$, the condition (3.9) is true. Hence, by Theorem 3.2, we get $\ell\in\mathcal{S}_{ab}^{n,\Sigma}(a)$. □

Proof of Corollary 3.2. The validity of the corollary follows immediately from Corollary 3.1 with $k=1$, $m=2$, $\rho^1=(\sigma_1\delta_1,\dots,\sigma_n\delta_n)^T$, and

$$\alpha=\max\left\{\frac{1}{\delta_i}\sum_{k=1}^{n}\delta_k\int_a^b|\ell_{ik}(1)(s)|ds : i=1,\dots,n\right\}.$$ □

Proof of Proposition 3.2. Put $\rho^1=(\sigma_1\delta_1,\dots,\sigma_n\delta_n)^T$. At first we will show that

$$\sigma_i\int_a^b\ell_i\big(\varphi\big(\rho^1\big)\big)(s)ds<\delta_i \quad \text{for} \quad t\in[a,b],\ i=1,\dots,n, \tag{3.54}$$

where φ is given by (3.13). Indeed, according to the assumption $\ell\in\mathcal{P}_{ab}^{n,\Sigma}$ and Remark 2.2, the relations (2.1) are true. Therefore, we get

$$\sigma_i\int_a^b\ell_i\big(\varphi\big(\rho^1\big)\big)(s)ds=\sum_{k=1}^{n}\sigma_i\sigma_k\int_a^b\ell_{ik}\big(\sigma_k\varphi_k\big(\rho^1\big)\big)(s)ds \\ \leq\sum_{k=1}^{n}\sigma_k\int_a^b\ell_k\big(\rho^1\big)(\eta)d\eta\int_a^b|\ell_{ik}(1)(s)|ds \\ =\sum_{k=1}^{n}\int_a^b|\ell_{ik}(1)(s)|ds\sum_{j=1}^{n}\int_a^b\sigma_k\ell_{kj}(\sigma_j\delta_j)(\eta)d\eta \\ =\sum_{k=1}^{n}\int_a^b|\ell_{ik}(1)(s)|ds\sum_{j=1}^{n}\int_a^b\delta_j|\ell_{kj}(1)(\eta)|d\eta \quad \text{for} \quad i=1,\dots,n. \tag{3.55}$$

A) Let $i \in J$. Then, by virtue of (3.16) and (3.17), we obtain

$$\sum_{k=1}^{n} \int_a^b |\ell_{ik}(1)(s)|ds \sum_{j=1}^{n} \int_a^b \delta_j |\ell_{kj}(1)(\eta)|d\eta \;\leq\; \sum_{k=1}^{n} \delta_k \int_a^b |\ell_{ik}(1)(s)|ds \;<\; \delta_i\,.$$

B) Let $i \in \{1,\dots,n\} \setminus J$. Then, according to (3.18), there exists $k_0 \in J$ such that

$$\int_a^b |\ell_{ik_0}(1)(s)|ds > 0.$$

Using (3.16) and (3.17), we get

$$\sum_{k=1}^{n} \int_a^b |\ell_{ik}(1)(s)|ds \sum_{j=1}^{n} \int_a^b \delta_j |\ell_{kj}(1)(\eta)|d\eta$$

$$< \delta_{k_0} \int_a^b |\ell_{ik_0}(1)(s)|ds + \sum_{\substack{k=1\\k\neq k_0}}^{n} \delta_k \int_a^b |\ell_{ik}(1)(s)|ds = \sum_{k=1}^{n} \delta_k \int_a^b |\ell_{ik}(1)(s)|ds = \delta_i\,.$$

Consequently, (3.55) yields (3.54). Put

$$\alpha = \max\left\{\frac{\sigma_i}{\delta_i} \int_a^b \ell_i\big(\varphi(\rho^1)\big)(s)ds : i = 1,\dots,n\right\}. \tag{3.56}$$

Obviously, $\alpha \in [0,1[$ and

$$\operatorname{diag}(\sigma_1,\dots,\sigma_n)\,\rho^3(b) \leq \alpha(\delta_1,\dots,\delta_n)^T = \operatorname{diag}(\sigma_1,\dots,\sigma_n)\,\alpha\rho^1,$$

where ρ^3 is defined by (3.12). Consequently, the assumptions of Corollary 3.1 are satisfied with $k = 1$, $m = 3$, and α given by (3.56). □

Proof of Proposition 3.3. First suppose that the homogeneous problem (3.4) has only the trivial solution. We will show that $\ell \in \mathcal{S}_{ab}^{n,\Sigma}(a)$. According to the Fredholm property of the problem (1.1) (see, e.g., [9, 11, 17]), the problem (3.51) has a unique solution γ. Put

$$\gamma^* = \max\left\{\frac{1}{\delta_i}\max\Big\{-\sigma_i\gamma_i(t) : t \in [a,b]\Big\} : i = 1,\dots,n\right\}. \tag{3.57}$$

Then there exist $t_0 \in [a,b]$ and $i_0 \in \{1,\dots,n\}$ such that

$$-\sigma_{i_0}\gamma_{i_0}(t_0) = \gamma^*\delta_{i_0}\,. \tag{3.58}$$

It is clear that

$$-\sigma_i\gamma_i(t) \leq \gamma^*\delta_i \quad \text{for} \quad t \in [a,b],\ i = 1,\dots,n. \tag{3.59}$$

Note also that, by virtue of the assumption $\ell \in \mathcal{P}_{ab}^{n,\Sigma}$ and Remark 2.2, the relations (2.1) are true.

Assume that

$$\gamma^* \geq 0. \tag{3.60}$$

Then (3.51) yields

$$\sigma_i - \gamma_i(t_0) = -\int_a^{t_0} \ell_i(\gamma)(s)ds \quad \text{for} \quad i = 1, \dots, n.$$

Therefore, in view of (2.1), (3.15), and (3.58)-(3.60), we get

$$\begin{aligned} 1 + \gamma^* \delta_{i_0} &= -\sigma_{i_0} \int_a^{t_0} \ell_{i_0}(\gamma)(s)ds = -\sigma_{i_0} \int_a^{t_0} \sum_{k=1}^n \ell_{i_0 k}(\gamma_k)(s)ds \\ &= \sum_{k=1}^n \int_a^{t_0} \sigma_{i_0}\sigma_k \ell_{i_0 k}(-\sigma_k\gamma_k)(s)ds \leq \sum_{k=1}^n \gamma^*\delta_k \int_a^{t_0} \sigma_{i_0}\sigma_k \ell_{i_0 k}(1)(s)ds \\ &= \gamma^* \sum_{k=1}^n \delta_k \int_a^{t_0} |\ell_{i_0 k}(1)(s)|ds \leq \gamma^* \delta_{i_0}\,, \end{aligned}$$

which is impossible.

The contradiction obtained proves that $\gamma^* < 0$. Hence, (3.59) implies

$$\operatorname{diag}(\sigma_1, \dots, \sigma_n)\,\gamma(t) \geq -\gamma^*(\delta_1, \dots, \delta_n)^T > 0 \quad \text{for} \quad t \in [a,b],$$

and thus Theorem 3.2 guarantees $\ell \in \mathcal{S}_{ab}^{n,\Sigma}(a)$.

The converse implication is obvious (see Remark 3.1). □

To prove Corollary 3.3 we need the following lemma.

Lemma 3.1. *Let $\ell \in \mathcal{P}_{ab}^{n,\Sigma}$. Then $\ell \in \mathcal{S}_{ab}^{n,\Sigma}(a)$ if and only if there is no nontrivial function $v \in \widetilde{C}([a,b];\mathbb{R}^n)$ satisfying*

$$\operatorname{diag}(\sigma_1, \dots, \sigma_n)\,v(t) \geq 0 \quad \text{for} \quad t \in [a,b], \tag{3.61}$$

$$\operatorname{diag}(\sigma_1, \dots, \sigma_n)\Big(v'(t) - \ell(v)(t)\Big) \leq 0 \quad \text{for} \quad t \in [a,b], \tag{3.62}$$

$$v(a) = 0. \tag{3.63}$$

Proof. If $\ell \in \mathcal{S}_{ab}^{n,\Sigma}(a)$, then it is clear that every function $v \in \widetilde{C}([a,b];\mathbb{R}^n)$ satisfying (3.61)–(3.63) is trivial.

Conversely, let there be no nontrivial function $v \in \widetilde{C}([a,b];\mathbb{R}^n)$ fulfilling (3.61)–(3.63) and let $u \in \widetilde{C}([a,b];\mathbb{R}^n)$ satisfy (3.1) and (3.2). We will show that (3.3) is true. Indeed, put

$$v_i(t) = \sigma_i[\sigma_i u_i(t)]_- \quad \text{for} \quad t \in [a,b],\ i = 1, \dots, n. \tag{3.64}$$

According to (3.2), it is clear that $v = (v_i)_{i=1}^n$ satisfies (3.61) and (3.63). Moreover, by virtue of the assumption $\ell \in \mathcal{P}_{ab}^{n,\Sigma}$, (2.1) holds. Therefore, using (3.1), we get

$$\begin{aligned}\sigma_i v_i'(t) &= \frac{1}{2}\,\sigma_i u_i'(t)\Big(\operatorname{sgn}\big(\sigma_i u_i(t)\big)-1\Big) \le \frac{1}{2}\,\sigma_i \ell_i(u)(t)\Big(\operatorname{sgn}\big(\sigma_i u_i(t)\big)-1\Big)\\ &= \frac{1}{2}\left(\sum_{k=1}^{n}\sigma_i\sigma_k\ell_{ik}(\sigma_k u_k)(t)\operatorname{sgn}\big(\sigma_i u_i(t)\big)-\sigma_i\ell_i(u)(t)\right)\\ &\le \frac{1}{2}\left(\sum_{k=1}^{n}\sigma_i\sigma_k\ell_{ik}(|\sigma_k u_k|)(t)-\sigma_i\sum_{k=1}^{n}\sigma_k\ell_{ik}(\sigma_k u_k)(t)\right)\\ &= \sigma_i\sum_{k=1}^{n}\sigma_k\ell_{ik}([\sigma_k u_k]_-)(t)=\sigma_i\sum_{k=1}^{n}\ell_{ik}(v_k)(t)\\ &= \sigma_i\ell_i(v)(t) \quad \text{for} \quad t\in[a,b], \quad i=1,\dots,n.\end{aligned}$$

Hence, v satisfies also the condition (3.62) and thus $v\equiv 0$. However, this means that the condition (3.3) is true. Consequently, $\ell\in\mathcal{S}_{ab}^{n,\Sigma}(a)$. □

Proof of Corollary 3.3. According to Lemma 3.1, to prove the corollary it is sufficient to show that there is no nontrivial function $v\in\widetilde{C}([a,b];\mathbb{R}^n)$ satisfying (3.61)–(3.63).

At first we prove that

$$\widetilde{\ell}\in\mathcal{S}_{ab}^{n,\Sigma}(a), \tag{3.65}$$

where

$$\widetilde{\ell}(w)(t)\stackrel{\text{def}}{=}P_\ell(t)w(t)+\overline{\ell}(w)(t) \quad \text{for} \quad t\in[a,b]. \tag{3.66}$$

According to the inequality (3.22), there exists a vector $\varepsilon=(\varepsilon_i)_{i=1}^n>0$ such that

$$B\left(Y(b)Y^{-1}(a)\varepsilon+Y(b)\int_a^b Y^{-1}(s)\widetilde{q}(s)ds\right)\le(1,\dots,1)^T. \tag{3.67}$$

Put

$$z(t)=Y(t)Y^{-1}(a)\varepsilon+Y(t)\int_a^t Y^{-1}(s)\widetilde{q}(s)ds \quad \text{for} \quad t\in[a,b]. \tag{3.68}$$

It is clear that z is a solution of the Cauchy problem

$$z'=\widetilde{P}(t)z+\widetilde{q}(t), \qquad z(a)=\varepsilon, \tag{3.69}$$

because Y is a fundamental matrix of the system (3.19). In view of (3.20) and (3.23), we get

$$\widetilde{P}(t)\ge\Theta, \qquad \widetilde{q}(t)\ge 0 \quad \text{for} \quad t\in[a,b], \tag{3.70}$$

and thus

$$0<z(t)\le z(b) \quad \text{for} \quad t\in[a,b]. \tag{3.71}$$

Put

$$\gamma_i(t)=\sigma_i z_i(t)e^{\int_a^t \ell_{ii}(1)(s)ds} \quad \text{for} \quad t\in[a,b],\ i=1,\dots,n. \tag{3.72}$$

It is not difficult to verify that, on account of (3.20), (3.23), (3.69), and (3.72), $\gamma = (\gamma_i)_{i=1}^n \in \widetilde{C}([a,b];\mathbb{R}^n)$ is a solution of the system

$$\gamma' = P_\ell(t)\gamma + \overline{\ell}\big(\Sigma^T\big)(t). \tag{3.73}$$

Moreover, (3.67) and (3.71) imply

$$0 < \operatorname{diag}(\sigma_1,\dots,\sigma_n)\,\gamma(t) \le Bz(b) \le (1,\dots,1)^T \quad \text{for} \quad t\in[a,b],$$

i.e., the condition (3.8) is true and

$$\operatorname{diag}(\sigma_1,\dots,\sigma_n)\Big(\Sigma^T - \gamma(t)\Big) \ge 0 \quad \text{for} \quad t\in[a,b]. \tag{3.74}$$

Therefore, in view of (3.74) and the assumption $\overline{\ell} \in \mathcal{P}_{ab}^{n,\Sigma}$, from (3.73) we get

$$\begin{aligned}\operatorname{diag}(\sigma_1,\dots,\sigma_n)\big(\gamma'(t)-\widetilde{\ell}(\gamma)(t)\big) &= \operatorname{diag}(\sigma_1,\dots,\sigma_n)\big(\gamma'(t)-P_\ell(t)\gamma(t)-\overline{\ell}(\gamma)(t)\big)\\ &\ge \operatorname{diag}(\sigma_1,\dots,\sigma_n)\big(\gamma'(t)-P_\ell(t)\gamma(t)-\overline{\ell}\big(\Sigma^T\big)(t)\big) = 0 \quad \text{for} \quad t\in[a,b].\end{aligned}$$

Consequently, using Theorem 3.2, we show that the inclusion (3.65) is true.

Let $v \in \widetilde{C}([a,b];\mathbb{R}^n)$ satisfy (3.61)–(3.63). We will show that $v \equiv 0$. Put

$$u(t) = \varphi(v)(t) \quad \text{for} \quad t\in[a,b], \tag{3.75}$$

where φ is defined by (3.13). Obviously, (3.61)–(3.63) and (3.75) yield

$$u'(t) = \ell(v)(t) \quad \text{for} \quad t\in[a,b], \tag{3.76}$$

$$u(a) = 0, \tag{3.77}$$

and

$$\begin{aligned}0 \le \operatorname{diag}(\sigma_1,\dots,\sigma_n)\,v(t) &\le \operatorname{diag}(\sigma_1,\dots,\sigma_n)\int_a^t \ell(v)(s)ds\\ &= \operatorname{diag}(\sigma_1,\dots,\sigma_n)\,\varphi(v)(t) = \operatorname{diag}(\sigma_1,\dots,\sigma_n)\,u(t) \quad \text{for} \quad t\in[a,b].\end{aligned} \tag{3.78}$$

On the other hand, since $\ell \in \mathcal{P}_{ab}^{n,\Sigma}$ we have

$$\begin{aligned}\operatorname{diag}(\sigma_1,\dots,\sigma_n)\,u'(t) &= \operatorname{diag}(\sigma_1,\dots,\sigma_n)\,\ell(v)(t)\\ &\le \operatorname{diag}(\sigma_1,\dots,\sigma_n)\,\ell(u)(t) = \operatorname{diag}(\sigma_1,\dots,\sigma_n)\,P_\ell(t)u(t)\\ &\quad + \operatorname{diag}(\sigma_1,\dots,\sigma_n)\Big(\ell\big(\varphi(v)\big) - P_\ell(t)\varphi(v)(t)\Big) \quad \text{for} \quad t\in[a,b].\end{aligned}$$

By virtue of (3.21), (3.77), and (3.78), the last relations yield

$$\operatorname{diag}(\sigma_1,\dots,\sigma_n)\,u'(t) \le \operatorname{diag}(\sigma_1,\dots,\sigma_n)\Big(P_\ell(t)u(t) + \overline{\ell}(v)(t)\Big) \quad \text{for} \quad t\in[a,b].$$

Hence, in view of (3.78) and the assumption $\overline{\ell} \in \mathcal{P}_{ab}^{n,\Sigma}$, we get

$$\begin{aligned}\operatorname{diag}(\sigma_1,\dots,\sigma_n)\,u'(t) &\le \operatorname{diag}(\sigma_1,\dots,\sigma_n)\Big(P_\ell(t)u(t) + \overline{\ell}(u)(t)\Big)\\ &= \operatorname{diag}(\sigma_1,\dots,\sigma_n)\,\widetilde{\ell}(u)(t) \quad \text{for} \quad t\in[a,b].\end{aligned}$$

Consequently, the inclusion (3.65) yields

$$\operatorname{diag}(\sigma_1,\dots,\sigma_n)\,u(t)\le 0 \quad \text{for} \quad t\in[a,b],$$

and thus, it follows from (3.78) that $v\equiv 0$. □

Proof of Corollary 3.4. We will show that the assumptions of Corollary 3.3 are satisfied. Clearly, it is sufficient to show that the condition (3.25) yields the inequality (3.22). Since Y is a fundamental matrix of the system (3.19), the condition (3.22) is satisfied if and only if the solution $x=(x_i)_{i=1}^n$ of the Cauchy problem

$$x'=\widetilde{P}(t)x+\widetilde{q}(t), \qquad x(a)=0 \tag{3.79}$$

satisfies

$$x_i(b)e^{\int_a^b \ell_{ii}(1)(s)ds}<1 \quad \text{for} \quad i=1,\dots,n. \tag{3.80}$$

Put

$$v_i(t)=\int_a^t h(s)e^{\int_s^t p(\xi)d\xi}ds \quad \text{for} \quad t\in[a,b],\ i=1,\dots,n. \tag{3.81}$$

It is clear that

$$v_i(t)\ge 0 \quad \text{for} \quad t\in[a,b],\ i=1,\dots,n,$$

because $\widetilde{P}$ and $\widetilde{q}$ satisfy (3.70). Therefore, from (3.81) we get

$$\begin{aligned} v_i'(t)=p(t)v_i(t)+h(t)&\ge\sum_{k=1}^n \widetilde{p}_{ik}(t)v_i(t)+\widetilde{q}_i(t)\\ &=\sum_{k=1}^n \widetilde{p}_{ik}(t)v_k(t)+\widetilde{q}_i(t) \quad \text{for} \quad t\in[a,b],\ i=1,\dots,n. \end{aligned}$$

But this means that $v=(v_i)_{i=1}^n$ is a solution of the system of differential inequalities

$$v'(t)\ge\widetilde{P}(t)v(t)+\widetilde{q}(t) \quad \text{for} \quad t\in[a,b].$$

According to Proposition 1.1, we get

$$x(t)\le v(t) \quad \text{for} \quad t\in[a,b],$$

where x is a solution of the problem (3.79). Consequently, by virtue of (3.25) and (3.81), the solution x of the problem (3.79) satisfies (3.80), i.e., the inequality (3.22) holds. Hence, the assumptions of Corollary 3.3 are satisfied. □

To prove Corollary 3.5 we need the following lemma.

Lemma 3.2. *Let $p\in L([a,b];\mathbb{R})$, $\widetilde{q}\in L([a,b];\mathbb{R}^2)$, and let $v=(v_1,v_2)^T$ be a solution of the problem*

$$v'=A(t)v+\widetilde{q}, \qquad v(a)=0, \tag{3.82}$$

where

$$A(t)=\begin{pmatrix}0 & p(t)\\ p(t) & 0\end{pmatrix} \quad \textit{for} \quad t\in[a,b]. \tag{3.83}$$

Then

$$v_i(t) = \int_a^t \cosh\Big(\int_s^t p(\xi)d\xi\Big)\widetilde{q}_i(s)ds + \int_a^t \sinh\Big(\int_s^t p(\xi)d\xi\Big)\widetilde{q}_{3-i}(s)ds$$
$$\text{for} \quad t \in [a,b], \ i = 1,2. \quad (3.84)$$

Proof. It is easy to see that, for an abitrary $s \in [a,b]$, we have

$$A(t)\Big(\int_s^t A(s)ds\Big) = \Big(\int_s^t A(s)ds\Big)A(t) \quad \text{for} \quad t \in [a,b],$$

and thus, the solution v of the problem (3.82) has the form

$$v(t) = \int_a^t e^{\int_s^t A(\xi)d\xi}\,\widetilde{q}(s)ds \quad \text{for} \quad t \in [a,b]. \tag{3.85}$$

It can be verified by direct calculation that

$$e^{\int_s^t A(\xi)d\xi} = \Big(\sum_{k=0}^{+\infty}\frac{1}{(2k)!}\Big(\overline{p}(s,t)\Big)^{2k}\Big)E + \Big(\sum_{k=0}^{+\infty}\frac{1}{(2k+1)!}\Big(\overline{p}(s,t)\Big)^{2k}\Big)\int_s^t A(\xi)d\xi \quad \text{for} \quad a \le s \le t \le b,$$

where

$$\overline{p}(s,t) = \int_s^t p(\xi)d\xi \quad \text{for} \quad a \le s \le t \le b.$$

Whence we get

$$e^{\int_s^t A(\xi)d\xi} = \begin{pmatrix} \cosh\overline{p}(s,t) & \sinh\overline{p}(s,t) \\ \sinh\overline{p}(s,t) & \cosh\overline{p}(s,t)\end{pmatrix} \quad \text{for} \quad a \le s \le t \le b. \tag{3.86}$$

Therefore, (3.85) and (3.86) result in (3.84). □

Proof of Corollary 3.5. We will show that the assumptions of Corollary 3.3 with $n = 2$ are satisfied. Clearly, it is sufficient to show that the condition (3.28) yields the inequality (3.22). Since Y is a fundamental matrix of the system (3.19), the condition (3.22) is fulfilled if and only if the solution $x = (x_1, x_2)^T$ of the problem (3.79) satisfies (3.80).

Let $x = (x_1, x_2)^T$ be a solution of the problem (3.79). The functions x_1 and x_2 are nonnegative because (3.70) is true. Therefore, (3.79) yields

$$x'(t) \le A(t)x(t) + \widetilde{q}(t) \quad \text{for} \quad t \in [a,b], \tag{3.87}$$

where the matrix function A is given by (3.83). According to Proposition 1.1, we get

$$x(t) \le v(t) \quad \text{for} \quad t \in [a,b],$$

where v is a solution of the problem (3.82). Consequently, by virtue of (3.28) and Lemma 3.2, the solution $x = (x_1, x_2)^T$ of the problem (3.79) satisfies (3.80), i.e., the inequality (3.22) holds. Hence, the assumptions of Corollary 3.3 with $n = 2$ are satisfied. □

Proof of Proposition 3.4. According to Remark 2.3, the operators ℓ_{ik} $(i, k = 1, \dots, n)$ are a-Volterra ones. Consequently, it is not difficult to verify that the assumptions of Corollary 3.3 are satisfied for $\overline{\ell} \equiv 0$. □

Proof of Theorem 3.3. First suppose that the conditions (a) and (b) are satisfied. Let $u \in \widetilde{C}([a, b]; \mathbb{R}^n)$ be such that the conditions (3.1) and (3.2) hold. According to (b), we have

$$\sigma_i u_i'(t) \geq \ell_{ii}(\sigma_i u_i)(t) \quad \text{for} \quad t \in [a, b], \quad \sigma_i u_i(a) \geq 0 \qquad (i = 1, \dots, n),$$

and thus, by virtue of (a), the condition (3.3) is satisfied. Consequently, $\ell \in \mathcal{S}_{ab}^{n,\Sigma}(a)$.

Now suppose that $\ell \in \mathcal{S}_{ab}^{n,\Sigma}(a)$. According to the assumption $-\ell \in \mathcal{P}_{ab}^{n,\Sigma}$ and Remark 2.2, we have

$$-\sigma_i \sigma_k \ell_{ik} \in \mathcal{P}_{ab}. \tag{3.88}$$

Condition (a). We will show that $\ell_{11} \in \mathcal{S}_{ab}(a)$ (the proof for $i \neq 1$ is analogous). Since $\mathcal{S}_{ab}^{n,\Sigma}(a) = \mathcal{S}_{ab}^{n,-\Sigma}(a)$, without loss of generality we can assume that $\sigma_1 = 1$. Let $u_1 \in \widetilde{C}([a, b]; \mathbb{R})$ satisfy

$$u_1'(t) \geq \ell_{11}(u_1)(t) \quad \text{for} \quad t \in [a, b], \qquad u_1(a) \geq 0. \tag{3.89}$$

Put

$$u_i(t) = \sigma_i \int_a^t |\ell_{i1}(u_1)(s)| ds \quad \text{for} \quad t \in [a, b], \ i = 2, \dots, n. \tag{3.90}$$

Then it is clear that

$$\sigma_i u_i'(t) = |\ell_{i1}(u_1)(t)| \geq \sigma_i \ell_{i1}(u_1)(t) \quad \text{for} \quad t \in [a, b], \ i = 2, \dots, n. \tag{3.91}$$

Moreover, (3.90) guarantees

$$\sigma_k u_k(t) \geq 0 \quad \text{for} \quad t \in [a, b], \ k = 2, \dots, n,$$

and thus (3.88) implies

$$\sigma_i \ell_{ik}(u_k)(t) = \sigma_i \sigma_k \ell_{ik}(\sigma_k u_k)(t) \leq 0 \\ \text{for} \quad t \in [a, b], \ i = 1, \dots, n, \ k = 2, \dots, n. \tag{3.92}$$

From (3.89) and (3.91), in view of (3.92), we get

$$\sigma_i u_i'(t) \geq \sigma_i \ell_{i1}(u_1)(t) \geq \sigma_i \sum_{k=1}^n \ell_{ik}(u_k)(t) = \sigma_i \ell_i(u)(t) \ \text{ for } \ t \in [a, b], \ i{=}1, \dots, n,$$

where $u = (u_i)_{i=1}^n$. Consequently, u satisfies (3.1) and (3.2) which, together with the assumption $\ell \in \mathcal{S}_{ab}^{n,\Sigma}(a)$, guarantees the condition (3.3). Hence,

$$u_1(t) = \sigma_1 u_1(t) \geq 0 \quad \text{for} \quad t \in [a, b]$$

and thus $\ell_{11} \in \mathcal{S}_{ab}(a)$.

Condition (b). We will show that $\ell_{21} \equiv 0$ (the other cases can be proved analogously). We have proved above that $\ell_{11} \in \mathcal{S}_{ab}(a)$ and thus, by virtue of (3.88) and Theorem 2 in [2], ℓ_{11} is an a-Volterra operator. Define operators $\widetilde{\ell}, \overline{\ell} \in \mathcal{L}_{ab}^n$ by setting

$$\widetilde{\ell}_1(v)(t) \stackrel{\text{def}}{=} \sum_{k=2}^{n} \ell_{1k}(v_k)(t) \quad \text{for} \quad t \in [a,b], \qquad \widetilde{\ell}_i \equiv \ell_i \quad \text{for} \quad i = 2, \dots, n,$$

$$\overline{\ell}_1(v)(t) \stackrel{\text{def}}{=} -\ell_{11}(v_1)(t) \quad \text{for} \quad t \in [a,b], \qquad \overline{\ell}_i \equiv 0 \quad \text{for} \quad i = 2, \dots, n.$$

It is clear that $\overline{\ell} \in \mathcal{P}_{ab}^{n,\Sigma}$ and $\widetilde{\ell} = \overline{\ell} + \ell$. Since ℓ_{11} is an a-Volterra operator, the operator $\overline{\ell}$ is also an a-Volterra one (see Remark 2.3). Hence, using Proposition 3.4 we get $\overline{\ell} \in \mathcal{S}_{ab}^{n,\Sigma}(a)$. By Theorem 3.1, we obtain that $\widetilde{\ell} \in \mathcal{S}_{ab}^{n,\Sigma}(a)$. Consequently, the problem

$$u'(t) = \widetilde{\ell}(u)(t), \tag{3.93}$$

$$u_1(a) = \sigma_1, \quad u_i(a) = 0 \quad (i = 2, \dots, n) \tag{3.94}$$

has a unique solution $u = (u_i)_{i=1}^n$ and the condition (3.3) is true (see Remark 3.1). Therefore, in view of the assumption $-\ell \in \mathcal{P}_{ab}^{n,\Sigma}$, (3.93) implies

$$\sigma_i u_i'(t) = \sigma_i \widetilde{\ell}_i(u)(t) = \sigma_i \ell_i(u)(t) \le 0 \quad \text{for} \quad t \in [a,b], \ i = 2, \dots, n,$$

which, together with (3.3) and (3.94), yields $u_i \equiv 0$ for $i = 2, \dots, n$. Hence, from (3.93) we get $u_1'(t) = 0$ for $t \in [a,b]$, i.e., $u_1 \equiv \sigma_1$. Finally, (3.93) implies

$$0 = u_2'(t) = \widetilde{\ell}_2(u)(t) = \sigma_1 \ell_{21}(1)(t) \quad \text{for} \quad t \in [a,b],$$

i.e., $\ell_{21}(1) \equiv 0$. However, this means $\ell_{21} \equiv 0$ because the operator ℓ_{21} is monotone. □

4. Corollaries for Operators With Deviating Arguments

In this section, some corollaries of the main results will be established for the operators with deviating arguments, i.e., for the cases where $\ell = (\ell_i)_{i=1}^n \in \mathcal{L}_{ab}^n$ is defined by

$$\ell_i(v)(t) \stackrel{\text{def}}{=} \sum_{k=1}^{n} p_{ik}(t) v_k\big(\tau_{ik}(t)\big) \quad \text{for} \quad t \in [a,b], \ i = 1, \dots, n, \tag{4.1}$$

$$\ell_i(v)(t) \stackrel{\text{def}}{=} -g_i(t) v_i\big(\mu_i(t)\big) \quad \text{for} \quad t \in [a,b], \ i = 1, \dots, n, \tag{4.2}$$

and

$$\ell_i(v)(t) \stackrel{\text{def}}{=} \sum_{k=1}^{n} p_{ik}(t) v_k\big(\tau_{ik}(t)\big) - g_i(t) v_i\big(\mu_i(t)\big) \ \text{for} \ t \in [a,b], \ i = 1, \dots, n. \tag{4.3}$$

Here, $p_{ik} \in L([a,b], \mathbb{R})$, $g_i \in L([a,b], \mathbb{R}_+)$, and $\tau_{ik}, \mu_i : [a,b] \to [a,b]$ are measurable functions $(i, k = 1, \dots, n)$.

As above, we have fixed a row vector $\Sigma = (\sigma_1, \dots, \sigma_n)$ with elements $\sigma_i \in \{-1, 1\}$ $(i = 1, \dots, n)$. Throughout this section, the following notation will be used:

$$\tau^* = \max\Big\{ \operatorname{ess\,sup} \big\{ \tau_{ik}(t) : t \in [a, b] \big\} : i, k = 1, \dots, n \Big\}.$$

All the statements are formulated in Subsection 4.1, their proofs are given in Subsection 4.2.

4.1. **Formulation of the Results.** From Corollary 3.1 we get

Theorem 4.1. *Let*

$$\sigma_i \sigma_k p_{ik}(t) \geq 0 \quad for \quad t \in [a, b], \ i, k = 1, \dots, n. \tag{4.4}$$

Let, moreover, there exist $\alpha \in [0, 1[$ and numbers $\delta_i > 0$ $(i = 1, \dots, n)$ such that

$$\sum_{j=1}^{n} \int_a^t |p_{ij}(s)| \bigg(\sum_{k=1}^{n} \delta_k \int_a^{\tau_{ij}(s)} |p_{jk}(\xi)| d\xi \bigg) ds \leq \alpha \sum_{k=1}^{n} \delta_k \int_a^t |p_{ik}(s)| ds \quad for \quad t \in [a, b], \ i = 1, \dots, n. \tag{4.5}$$

Then the operator $\ell = (\ell_i)_{i=1}^n$ defined by (4.1) *belongs to the set $\mathcal{S}_{ab}^{n,\Sigma}(a)$.*

Remark 4.1. Example 5.3 shows that the assumption $\alpha \in [0, 1[$ in Theorem 4.1 cannot be replaced by the assumption $\alpha \in [0, 1]$.

The following corollary follows immediately from Theorem 4.1.

Corollary 4.1. *Let the condition* (4.4) *hold and let there exist numbers $\delta_i > 0$ $(i = 1, \dots, n)$ such that*

$$\max\bigg\{ \frac{1}{\delta_i} \sum_{k=1}^{n} \delta_k \int_a^{\tau^*} |p_{ik}(s)| ds : i = 1, \dots, n \bigg\} < 1. \tag{4.6}$$

Then the operator $\ell = (\ell_i)_{i=1}^n$ defined by (4.1) *belongs to the set $\mathcal{S}_{ab}^{n,\Sigma}(a)$.*

Remark 4.2. Example 5.3 shows that, in general, the strict inequality (4.6) in Corollary 4.1 cannot be replaced by the nonstrict one. However, in the case where the equality

$$\max\bigg\{ \frac{1}{\delta_i} \sum_{k=1}^{n} \delta_k \int_a^{\tau^*} |p_{ik}(s)| ds : i = 1, \dots, n \bigg\} = 1 \tag{4.7}$$

is satisfied with some $\delta_i > 0$ $(i = 1, \dots, n)$, the operator $\ell = (\ell_i)_{i=1}^n$ defined by (4.1) still belongs to the set $\mathcal{S}_{ab}^{n,\Sigma}(a)$ under additional assumptions. Some such additional conditions are presented in the next proposition.

Proposition 4.1. *Let the condition* (4.4) *hold and let there exist numbers* $\delta_i > 0$ $(i = 1, \dots, n)$ *such that the equality* (4.7) *is satisfied. If, moreover,* $r(A) < 1$, *where the matrix* $A = (a_{ik})_{i,k=1}^n$ *is defined by*

$$a_{ik} = \frac{\delta_k}{\delta_i} \sum_{j=1}^{n} \int_a^{\tau^*} |p_{ij}(s)| \left(\int_a^{\tau_{ij}(s)} |p_{jk}(\xi)| d\xi \right) ds \quad (i, k = 1, \dots, n), \tag{4.8}$$

then the operator $\ell = (\ell_i)_{i=1}^n$ *defined by* (4.1) *belongs to the set* $\mathcal{S}_{ab}^{n,\Sigma}(a)$.

Remark 4.3. The assumption $r(A) < 1$ in the last proposition cannot be replaced by the assumption $r(A) \le 1$ (see Example 5.3).

The following theorem can be regarded as a complement of Corollary 4.1.

Theorem 4.2. *Let the condition* (4.4) *hold and let the inequality*

$$\max \left\{ \frac{1}{\delta_i} \sum_{k=1}^{n} \delta_k \int_a^{\tau^*} |p_{ik}(s)| ds : i = 1, \dots, n \right\} \ge 1$$

is satisfied for every $\delta_i > 0$ $(i = 1, \dots, n)$. *Let, moreover,*

$$\operatorname{ess\,sup} \left\{ \int_t^{\tau_{ik}(t)} p(s) ds : t \in [a, b] \right\} < \eta^* \quad for \quad i, k = 1, \dots, n, \tag{4.9}$$

where

$$\eta^* = \sup \left\{ \frac{1}{x} \ln \left(x + \frac{x}{\exp\left(x \int_a^{\tau^*} p(s) ds \right) - 1} \right) : x > 0 \right\} \tag{4.10}$$

and

$$p(t) \stackrel{\text{def}}{=} \max \left\{ \sum_{k=1}^{n} |p_{ik}(t)| : i = 1, \dots, n \right\} \quad for \quad t \in [a, b]. \tag{4.11}$$

Then the operator $\ell = (\ell_i)_{i=1}^n$ *defined by* (4.1) *belongs to the set* $\mathcal{S}_{ab}^{n,\Sigma}(a)$.

The next theorem follows from Corollary 3.3.

Theorem 4.3. *Let the condition* (4.4) *hold and let* $Y : [a, b] \to \mathbb{R}^{n \times n}$ *be a fundamental matrix of the system* (3.19), *where the matrix function* $\widetilde{P} = (\widetilde{p}_{ik})_{i,k=1}^n : [a, b] \to \mathbb{R}^{n \times n}$ *is defined by*

$$\widetilde{p}_{ii} \equiv 0 \quad for \quad i = 1, \dots, n,$$

$$\widetilde{p}_{ik}(t) = |p_{ik}(t)| \exp\left(\int_a^t \Big[p_{kk}(s) - p_{ii}(s) \Big] ds \right) \tag{4.12}$$

$$for \quad t \in [a, b], \ i, k = 1, \dots, n, \ i \ne k.$$

Let, moreover, the inequality (3.22) *be satisfied, where* $\widetilde{q} = (\widetilde{q}_i)_{i=1}^n \in L([a,b];\mathbb{R}^n)$ *is defined by*

$$\widetilde{q}_i(t) = \sum_{k=1}^n |p_{ik}(t)|\, \omega_{ik}(t) \Bigg(\sum_{j=1}^n \int_t^{\tau_{ik}(t)} |p_{kj}(s)| ds \Bigg) \, \mathrm{e}^{-\int_a^t p_{ii}(\eta)d\eta} \quad \text{for} \quad t \in [a,b],\ i = 1,\dots,n, \tag{4.13}$$

$$\omega_{ik}(t) = \frac{1}{2}\Big(1 + \operatorname{sgn}\big(\tau_{ik}(t) - t\big)\Big) \quad \text{for} \quad t \in [a,b],\ i,k = 1,\dots,n, \tag{4.14}$$

and

$$B = \operatorname{diag}\Bigg(\mathrm{e}^{\int_a^b p_{11}(s)ds}, \dots, \mathrm{e}^{\int_a^b p_{nn}(s)ds} \Bigg). \tag{4.15}$$

Then the operator $\ell = (\ell_i)_{i=1}^n$ *defined by* (4.1) *belongs to the set* $\mathcal{S}_{ab}^{n,\Sigma}(a)$.

In the following two corollaries, the efficient conditions are presented under which the fundamental matrix Y of the system (3.19) satisfies (3.22) in the case where the matrix function $\widetilde{P}$ is given by (4.12).

Corollary 4.2. *Let the condition* (4.4) *hold and let*

$$\mathrm{e}^{\max\left\{ \int_a^b p_{ii}(s)ds : i=1,\dots,n \right\}} \int_a^b h(s)\, \mathrm{e}^{\int_s^b p(\xi)d\xi} ds < 1, \tag{4.16}$$

where h *and* p *are given by* (3.26) *and* (3.27)*, respectively, and* $\widetilde{p}_{ik}$, $\widetilde{q}_i$ $(i,k = 1,\dots,n)$ *are defined by* (4.12)*–*(4.14)*. Then the operator* $\ell = (\ell_i)_{i=1}^n$ *given by* (4.1) *belongs to the set* $\mathcal{S}_{ab}^{n,\Sigma}(a)$.

Remark 4.4. The strict inequality (4.16) in the last corollary cannot be replaced by the nonstrict one (Example 5.3).

For two-dimensional systems of differential inequalities with argument deviations, we get from Corollary 3.5

Corollary 4.3. *Let* $n = 2$*, the condition* (4.4) *hold, and let*

$$\max\Bigg\{ \lambda_1\, \mathrm{e}^{\int_a^b p_{11}(s)ds}, \lambda_2\, \mathrm{e}^{\int_a^b p_{22}(s)ds} \Bigg\} < 1, \tag{4.17}$$

where λ_1, λ_2 *are given by* (3.29), (3.30) *and* $\widetilde{p}_{12}, \widetilde{p}_{21}, \widetilde{q}_1, \widetilde{q}_2$ *are defined by* (4.12)*–*(4.14)*. Then the operator* $\ell = (\ell_1, \ell_2)^T$ *defined by* (4.1) *belongs to the set* $\mathcal{S}_{ab}^{2,\Sigma}(a)$.

Remark 4.5. Example 5.3 shows that the strict inequality (4.17) in Corollary 4.3 cannot be replaced by the nonstrict one.

Theorem 3.2 also yields the following proposition which is a particular case of Theorem 3.2 (a) in [15].

Proposition 4.2. *Let* p_{ik} $(i,k=1,\dots,n)$ *be the essentially bounded functions satisfying the condition* (4.4)*. If, moreover,*

$$\max\left\{\frac{1}{\delta_i}\ \text{ess sup}\left\{\sum_{k=1}^{n}\delta_k|p_{ik}(t)|\big(\tau_{ik}(t)-a\big) : t\in[a,b]\right\} : i=1,\dots,n\right\}<1,$$

then the operator $\ell=(\ell_i)_{i=1}^n$ *defined by* (4.1) *belongs to the set* $\mathcal{S}_{ab}^{n,\Sigma}(a)$*.*

From Corollary 3.6 and the results of [6] we obtain

Theorem 4.4. *Let*

$$g_i(t)\big(\mu_i(t)-t\big)\le 0 \quad \textit{for} \quad t\in[a,b],\ i=1,\dots,n$$

and let, for every $i\in\{1,\dots,n\}$*, at least one of the following conditions be fulfilled:*

(a) $$\int_a^b g_i(s)ds\le 1;$$

(b) $$\int_a^b g_i(s)\int_{\mu_i(s)}^{s} g_i(\xi)\exp\left(\int_{\mu_i(\xi)}^{s} g_i(\eta)d\eta\right)d\xi ds\le 1;$$

(c) $g_i\not\equiv 0$ *and*

$$\text{ess sup}\left\{\int_{\mu_i(t)}^{t} g_i(s)ds : t\in[a,b]\right\}<\eta_i^*,$$

where

$$\eta_i^*=\sup\left\{\frac{1}{x}\ln\left(x+\frac{x}{\exp\left(x\int_a^b g_i(s)ds\right)-1}\right) : x>0\right\}.$$

Then the operator $\ell=(\ell_i)_{i=1}^n$ *defined by* (4.2) *belongs to the set* $\mathcal{S}_{ab}^{n,\Sigma}(a)$*.*

The next theorem follows from Theorem 3.1 and the statements formulated in this section.

Theorem 4.5. *Let the functions* p_{ik},τ_{ik} $(i,k=1,\dots,n)$ *satisfy the assumption of one of Theorems* 4.1–4.2 *and Corollaries* 4.2 *and* 4.3*, whereas the functions* g_i,μ_i $(i=1,\dots,n)$ *satisfy the assumptions of Theorem* 4.4*. Then the operator* $\ell=(\ell_i)_{i=1}^n$ *defined by* (4.3) *belongs to the set* $\mathcal{S}_{ab}^{n,\Sigma}(a)$*.*

4.2. **Proofs.**

Proof of Theorem 4.1. Let the operator $\ell=(\ell_i)_{i=1}^n\in\mathcal{L}_{ab}^n$ be defined by (4.1). It is clear that, in view of (4.4), $\ell\in\mathcal{P}_{ab}^{n,\Sigma}$. Moreover, according to (4.5), we have

$$\text{diag}(\sigma_1,\dots,\sigma_n)\Big(\rho^3(t)-\alpha\rho^2(t)\Big)\le 0 \quad \text{for} \quad t\in[a,b],$$

where ρ^2, ρ^3 are given by (3.12) and

$$\rho^1 = (\sigma_1\delta_1, \dots, \sigma_n\delta_n)^T.$$

Therefore, the assumptions of Corollary 3.1 are satisfied with $k=2$ and $m=3$. □

Proof of Corollary 4.1. The validity of the corollary follows immediately from Theorem 4.1 with

$$\alpha = \max\left\{\frac{1}{\delta_i}\sum_{k=1}^{n}\delta_k\int_a^{\tau^*}|p_{ik}(s)|ds : i=1,\dots,n\right\}.$$ □

Proof of Proposition 4.1. Let the operator $\ell = (\ell_i)_{i=1}^n \in \mathcal{L}_{ab}^n$ be defined by (4.1). It is clear that, in view of (4.4), $\ell \in \mathcal{P}_{ab}^{n,\Sigma}$. Let, moreover, the operator $\ell^* = (\ell_i^*)_{i=1}^n \in \mathcal{L}_{a\tau^*}^n$ be defined by

$$\ell_i^*(v)(t) = \sum_{k=1}^{n} p_{ik}(t)v_k\big(\tau_{ik}(t)\big) \quad \text{for} \quad t\in[a,\tau^*],\ i=1,\dots,n. \tag{4.18}$$

In other words, ℓ^* is the restriction of ℓ to the space $C([a,\tau^*];\mathbb{R}^n)$. Since

$$\tau_{ik}(t) \le \tau^* \quad \text{for} \quad t\in[a,b],\ i,k=1,\dots,n$$

and the condition (4.4) holds, it is clear that $\ell \in \mathcal{S}_{ab}^{n,\Sigma}(a)$ if and only if $\ell^* \in \mathcal{S}_{a\tau^*}^{n,\Sigma}(a)$. However, according to (4.7) and Proposition 3.3, $\ell^* \in \mathcal{S}_{a\tau^*}^{n,\Sigma}(a)$ if and only if the homogeneous problem

$$u_i'(t) = \sum_{k=1}^{n} p_{ik}(t)u_k\big(\tau_{ik}(t)\big) \qquad \big(t\in[a,\tau^*],\ i=1,\dots,n\big), \tag{4.19}$$

$$u_i(a) = 0 \qquad \big(i=1,\dots,n\big) \tag{4.20}$$

has only the trivial solution.

Let $u = (u_i)_{i=1}^n$ be a solution of the problem (4.19), (4.20). We will show that $u \equiv 0$. Put

$$u_i^* = \frac{1}{\delta_i}\max\Big\{\,|u_i(t)| : t\in[a,\tau^*]\Big\} \quad \text{for} \quad i=1,\dots,n \tag{4.21}$$

and choose $t_i \in [a,\tau^*]$ $(i=1,\dots,n)$ such that

$$|u_i(t_i)| = \delta_i u_i^* \quad \text{for} \quad i=1,\dots,n. \tag{4.22}$$

The integration of (4.19) from a to t, in view of (4.20) and (4.21), results in

$$|u_i(t)| \le \sum_{k=1}^{n}\int_a^t \big|p_{ik}(s)u_k\big(\tau_{ik}(s)\big)\big|\,ds$$

$$\le \sum_{k=1}^{n}\delta_k u_k^*\int_a^t |p_{ik}(s)|\,ds \quad \text{for} \quad t\in[a,\tau^*],\ i=1,\dots,n. \tag{4.23}$$

By virtue of (4.22), (4.23) yields

$$\delta_i u_i^* \le \sum_{j=1}^n \int_a^{t_i} |p_{ij}(s)| \left(\sum_{k=1}^n \delta_k u_k^* \int_a^{\tau_{ij}(s)} |p_{jk}(\xi)|\, d\xi \right) ds \quad \text{for} \quad i = 1, \dots, n.$$

Hence we get

$$u_i^* \le \sum_{k=1}^n \left[\frac{\delta_k}{\delta_i} \sum_{j=1}^n \int_a^{\tau^*} |p_{ij}(s)| \Big(\int_a^{\tau_{ij}(s)} |p_{jk}(\xi)| d\xi \Big) ds \right] u_k^* \quad \text{for} \quad i = 1, \dots, n,$$

i.e.,

$$u^* \le A u^*, \tag{4.24}$$

where $u^* = (u_1^*, \dots, u_n^*)^T$. However, (4.24) implies $u^* \le 0$ because $A \ge \Theta$ and $r(A) < 1$. Consequently, (4.21) yields $u \equiv 0$. □

Proof of Theorem 4.2. Let the operator $\ell = (\ell_i)_{i=1}^n \in \mathcal{L}_{ab}^n$ be defined by (4.1). It is clear that, in view of (4.4), $\ell \in \mathcal{P}_{ab}^{n,\Sigma}$. According to (4.9) and (4.10), there exist $x_0 > 0$ and $\varepsilon \in [0, 1[$ such that

$$\int_t^{\tau_{ik}(t)} p(s) ds \le \frac{1}{x_0} \ln \left(x_0 + \frac{\varepsilon x_0}{e^{x_0 \int_a^{\tau^*} p(s) ds} - \varepsilon} \right) \quad \text{for} \quad t \in [a, b],\ i, k = 1, \dots, n.$$

Hence we get

$$e^{x_0 \int_a^{\tau_{ik}(t)} p(s) ds} - \varepsilon \le x_0\, e^{x_0 \int_a^t p(s) ds} \quad \text{for} \quad t \in [a, b],\ i, k = 1, \dots, n. \tag{4.25}$$

Put

$$\gamma_i(t) = \sigma_i \Big(e^{x_0 \int_a^t p(s) ds} - \varepsilon \Big) \quad \text{for} \quad t \in [a, b],\ i = 1, \dots, n. \tag{4.26}$$

Obviously, $\gamma = (\gamma_i)_{i=1}^n \in \widetilde{C}([a, b]; \mathbb{R}^n)$ and the inequality (3.8) is satisfied. Moreover, by virtue of (4.11), (4.25), and (4.26), we get

$$\begin{aligned} \sigma_i \gamma_i'(t) &= x_0 p(t)\, e^{x_0 \int_a^t p(s) ds} \ge \sum_{k=1}^n |p_{ik}(t)|\, x_0\, e^{x_0 \int_a^t p(s) ds} \\ &\ge \sum_{k=1}^n |p_{ik}(t)| \left(e^{x_0 \int_a^{\tau_{ik}(t)} p(s) ds} - \varepsilon \right) = \sigma_i \sum_{k=1}^n p_{ik}(t)\, \sigma_k \left(e^{x_0 \int_a^{\tau_{ik}(t)} p(s) ds} - \varepsilon \right) \\ &= \sigma_i \sum_{k=1}^n p_{ik}(t) \gamma_k\big(\tau_{ik}(t)\big) = \sigma_i \ell_i(\gamma)(t) \quad \text{for} \quad t \in [a, b],\ i = 1, \dots, n. \end{aligned}$$

Therefore γ satisfies also the condition (3.9) and thus, using Theorem 3.2, we get $\ell \in \mathcal{S}_{ab}^{n,\Sigma}(a)$. □

Proof of Theorem 4.3. Let the operator $\ell = (\ell_i)_{i=1}^n \in \mathcal{L}_{ab}^n$ be defined by (4.1). It is clear that, in view of (4.4), $\ell \in \mathcal{P}_{ab}^{n,\Sigma}$. Let, moreover, the operator $\overline{\ell} = (\overline{\ell}_i)_{i=1}^n \in \mathcal{L}_{ab}^n$ be defined by setting

$$\overline{\ell}_i(v)(t) \overset{\text{def}}{=} \sigma_i \sum_{k=1}^n |p_{ik}(t)|\, \omega_{ik}(t) \Bigg(\sum_{j=1}^n \int_t^{\tau_{ik}(t)} |p_{kj}(s)|\, \sigma_j v_j\big(\tau_{kj}(s)\big) ds \Bigg) \quad \text{for} \quad t \in [a,b],\ i = 1, \dots, n, \tag{4.27}$$

where ω_{ik} $(i,k = 1,\dots,n)$ are given by (4.14). Obviously, $\overline{\ell} \in \mathcal{P}_{ab}^{n,\Sigma}$ and

$$\begin{aligned} \sigma_i \Bigg(\ell_i\big(\varphi(v)\big)(t) - \sum_{k=1}^n p_{ik}(t) \varphi_k(v)(t) \Bigg) &= \sum_{k=1}^n |p_{ik}(t)| \Bigg(\sum_{j=1}^n \int_t^{\tau_{ik}(t)} |p_{kj}(s)|\, \sigma_j v_j\big(\tau_{kj}(s)\big) ds \Bigg) \\ &\le \sigma_i \overline{\ell}_i(v)(t) \quad \text{for} \quad t \in [a,b],\ i = 1,\dots,n,\ v \in C_a^{\Sigma}([a,b];\mathbb{R}_+^n), \end{aligned} \tag{4.28}$$

where φ is given by (3.13). Consequently, the inequality (3.21) holds on the set $C_a^{\Sigma}([a,b];\mathbb{R}_+^n)$. Therefore, the assumptions of Corollary 3.3 are satisfied. □

Proof of Corollary 4.2. Let the operator $\ell = (\ell_i)_{i=1}^n \in \mathcal{L}_{ab}^n$ be defined by (4.1). It is clear that, in view of (4.4), $\ell \in \mathcal{P}_{ab}^{n,\Sigma}$. Analogously to the proof of Theorem 4.3 one can show that the inequality (3.21) holds on the set $C_a^{\Sigma}([a,b];\mathbb{R}_+^n)$, where φ is given by (3.13) and $\overline{\ell} = (\overline{\ell}_i)_{i=1}^n \in \mathcal{L}_{ab}^n$ is defined by (4.27). Moreover, in view of (4.16), the inequality (3.25) is fulfilled. Therefore, by Corollary 3.4 we get $\ell \in \mathcal{S}_{ab}^{n,\Sigma}(a)$. □

Proof of Corollary 4.3. Let the operator $\ell = (\ell_1, \ell_2)^T \in \mathcal{L}_{ab}^2$ be defined by (4.1). It is clear that, in view of (4.4), $\ell \in \mathcal{P}_{ab}^{2,\Sigma}$. Analogously to the proof of Theorem 4.3 one can show that the inequality (3.21) holds on the set $C_a^{\Sigma}([a,b];\mathbb{R}_+^2)$, where φ is given by (3.13) and $\overline{\ell} = (\overline{\ell}_1, \overline{\ell}_2)^T \in \mathcal{L}_{ab}^2$ is defined by (4.27). On the other hand, the inequality (4.17) yields (3.28) and thus, using Corollary 3.5, we get $\ell \in \mathcal{S}_{ab}^{2,\Sigma}(a)$. □

Proof of Proposition 4.2. Let the operator $\ell = (\ell_i)_{i=1}^n \in \mathcal{L}_{ab}^n$ be defined by (4.1). It is clear that, in view of (4.4), $\ell \in \mathcal{P}_{ab}^{n,\Sigma}$. According to the assumptions of the proposition, there exists $\varepsilon > 0$ such that

$$\sum_{k=1}^n |p_{ik}(t)| \Big[\delta_k\big(\tau_{ik}(t) - a\big) + \varepsilon \Big] \le \delta_i \quad \text{for} \quad t \in [a,b],\ i = 1,\dots,n. \tag{4.29}$$

Put

$$\gamma_i(t) = \sigma_i \Big[\delta_i(t-a) + \varepsilon \Big] \quad \text{for} \quad t \in [a,b],\ i = 1,\dots,n.$$

Obviously, $\gamma = (\gamma_i)_{i=1}^n \in \widetilde{C}([a,b];\mathbb{R}^n)$ and the inequality (3.8) is satisfied. Moreover, by virtue of (4.29), we get

$$\sigma_i\gamma_i'(t) = \delta_i \geq \sum_{k=1}^{n} |p_{ik}(t)| \Big[\delta_k\big(\tau_{ik}(t) - a\big) + \varepsilon\Big]$$
$$= \sigma_i \sum_{k=1}^{n} p_{ik}(t)\,\sigma_k \Big[\delta_k\big(\tau_{ik}(t) - a\big) + \varepsilon\Big] = \sigma_i \sum_{k=1}^{n} p_{ik}(t)\gamma_k\big(\tau_{ik}(t)\big)$$
$$= \sigma_i\ell_i(\gamma)(t) \quad \text{for} \quad t \in [a,b],\ i = 1,\dots,n. \tag{4.30}$$

Therefore, the condition (3.9) is satisfied and thus, using Theorem 3.2, we get $\ell \in \mathcal{S}_{ab}^{n,\Sigma}(a)$. □

Proof of Theorem 4.4. Let the operator $\ell = (\ell_i)_{i=1}^n \in \mathcal{L}_{ab}^n$ be defined by (4.2). It is clear that $-\ell \in \mathcal{P}_{ab}^{n,\Sigma}$ and $\ell_{ik} \equiv 0$ for $i,k = 1,\dots,n$, $i \neq k$. Moreover, by virtue of the results of [6], each of the conditions (a)–(c) in the theorem guarantees the inclusion $\ell_{ii} \in \mathcal{S}_{ab}(a)$. Therefore, the validity of the theorem follows from Theorem 3.3. □

Proof of Theorem 4.5. The validity of the theorem follows immediately from Theorem 3.1 and Theorems 4.1–4.4, Corollaries 4.2 and 4.3. □

5. Counter-Examples

Example 5.1. Let $\varepsilon \in]0,1[$ and let $p_{ik}, g_i \in L([a,b];\mathbb{R}_+)$ $(i,k = 1,2)$ be such that

$$\int_a^b p_{i1}(s)ds + \int_a^b p_{i2}(s)ds = 1 + \varepsilon, \qquad \int_a^b g_i(s)ds < 1 \quad \text{for} \quad i = 1,2, \tag{5.1}$$

$$\int_a^b p_{21}(s)ds > \varepsilon. \tag{5.2}$$

Let $\ell = \ell^+ - \ell^-$, where $\ell^+ = (\ell_1^+, \ell_2^+)^T$, $\ell^- = (\ell_1^-, \ell_2^-)^T \in \mathcal{P}_{ab}^{2,(1,1)}$ are defined by

$$\ell_i^+(v)(t) \stackrel{\text{def}}{=} \sum_{k=1}^{2} p_{ik}(t)v_k(b) \quad \text{for} \quad t \in [a,b],\ i = 1,2, \tag{5.3}$$

$$\ell_i^-(v)(t) \stackrel{\text{def}}{=} g_i(t)v_i(a) \quad \text{for} \quad t \in [a,b],\ i = 1,2. \tag{5.4}$$

According to (5.1) and Corollary 4.1 with $\delta_1 = \delta_2 = 1$, we find

$$(1-\varepsilon)\ell^+ \in \mathcal{S}_{ab}^{2,(1,1)}(a).$$

Moreover, in view of (5.1) and Theorem 4.4 (a), we get

$$-\ell^- \in \mathcal{S}_{ab}^{2,(1,1)}(a).$$

We first show that the homogeneous problem (3.4) has only the trivial solution. Indeed, let $\widehat{u} = (\widehat{u}_1, \widehat{u}_2)^T$ be a solution of the problem (3.4). Then

$$\widehat{u}_i(b) = \widehat{u}_1(b) \int_a^b p_{i1}(s)ds + \widehat{u}_2(b) \int_a^b p_{i2}(s)ds \quad \text{for} \quad i = 1, 2. \tag{5.5}$$

By virtue of (5.1) and (5.2), the last relations yield $\widehat{u}_1(b) = \widehat{u}_2(b) = 0$. Therefore, from (3.4) we get $\widehat{u} \equiv 0$. According to the Fredholm property of the problem (1.1) (see, e.g., [9, 11, 17]), the problem (1.1) with $q \equiv 0$ and $c = (1, 0)^T$ has a unique solution $u = (u_1, u_2)^T$. Obviously, u satisfies (3.1), (3.2) with $n = 2$ and $\sigma_1 = \sigma_2 = 1$.

On the other hand, it is easy to verify that

$$u_i(b) - u_i(a) = u_1(b) \int_a^b p_{i1}(s)ds + u_2(b) \int_a^b p_{i2}(s)ds \\ - u_i(a) \int_a^b g_i(s)ds \quad \text{for} \quad i = 1, 2. \tag{5.6}$$

Using (5.1), from (5.6) we get

$$-\varepsilon\left(\int_a^b p_{12}(s)ds + \int_a^b p_{21}(s)ds - \varepsilon \right) u_1(b) = \left(1 - \int_a^b g_1(s)ds\right)\left(\int_a^b p_{21}(s)ds - \varepsilon \right),$$

i.e., $u_1(b) < 0$. Consequently, $\ell \notin \mathcal{S}_{ab}^{2,(1,1)}(a)$.

This example shows that the assumption (3.5) in Theorem 3.1 cannot be replaced by the assumption

$$(1 - \varepsilon)\ell^+ \in \mathcal{S}_{ab}^{n,\Sigma}(a), \qquad -\ell^- \in \mathcal{S}_{ab}^{n,\Sigma}(a)$$

no matter how small $\varepsilon > 0$ is.

Example 5.2. Let $\varepsilon \in]0, 1[$ and let $p_{ij}, g_i \in L([a, b]; \mathbb{R}_+)$ $(i, j = 1, 2)$ be such that

$$\int_a^b p_{i1}(s)ds + \int_a^b p_{i2}(s)ds < 1, \qquad \int_a^b g_i(s)ds = 1 + \varepsilon \quad \text{for} \quad i = 1, 2. \tag{5.7}$$

Let $\ell = \ell^+ - \ell^-$, where $\ell^+ = (\ell_1^+, \ell_2^+)^T, \ell^- = (\ell_1^-, \ell_2^-)^T \in \mathcal{P}_{ab}^{2,(1,1)}$ are defined by (5.3) and (5.4), respectively. According to (5.7) and Corollary 4.1 with $\delta_1 = \delta_2 = 1$, we find

$$\ell^+ \in \mathcal{S}_{ab}^{2,(1,1)}(a).$$

Moreover, in view of (5.7) and Theorem 4.4 (a), we get

$$-(1 - \varepsilon)\ell^- \in \mathcal{S}_{ab}^{2,(1,1)}(a).$$

We first show that the homogeneous problem (3.4) has only the trivial solution. Indeed, let $\widehat{u} = (\widehat{u}_1, \widehat{u}_2)^T$ be a solution of the problem (3.4). Then (5.5) is true. By virtue of (5.7), the relations (5.5) yield $\widehat{u}_1(b) = \widehat{u}_2(b) = 0$. Therefore, from (3.4) we get $\widehat{u} \equiv 0$. According to the Fredholm property of the problem (1.1) (see, e.g., [9, 11, 17]), the problem (1.1) with $q \equiv 0$ and $c = (1,0)^T$ has a unique solution $u = (u_1, u_2)^T$. Obviously, u satisfies (3.1), (3.2) with $n = 2$ and $\sigma_1 = \sigma_2 = 1$.

On the other hand, it is easy to verify that the relations (5.6) are satisfied. Using (5.7), from (5.6) we get

$$\Bigg[\bigg(1 - \int_a^b p_{11}(s)ds\bigg)\bigg(1 - \int_a^b p_{22}(s)ds\bigg) - \\ - \int_a^b p_{12}(s)ds \int_a^b p_{21}(s)ds\Bigg] u_1(b) = -\varepsilon\bigg(1 - \int_a^b p_{22}(s)ds\bigg),$$

i.e., $u_1(b) < 0$. Consequently, $\ell \notin \mathcal{S}_{ab}^{2,(1,1)}(a)$.

This example shows that the assumption (3.5) in Theorem 3.1 cannot be replaced by the assumption

$$\ell^+ \in \mathcal{S}_{ab}^{n,\Sigma}(a), \qquad -(1-\varepsilon)\ell^- \in \mathcal{S}_{ab}^{n,\Sigma}(a),$$

no matter how small $\varepsilon > 0$ is.

Example 5.3. Let $\tau_{ik} \equiv b$ for $i, k = 1, 2$. Choose $p_{ik} \in L([a,b], \mathbb{R}_+)$ $(i, k = 1, 2)$ such that

$$p_{11} \equiv p_{22}, \quad p_{12} \equiv p_{21}, \qquad \text{and} \qquad \int_a^b p_{11}(s)ds + \int_a^b p_{12}(s)ds = 1.$$

Let the operator $\ell = (\ell_1, \ell_2)^T \in \mathcal{L}_{ab}^2$ be defined by (4.1) with $n = 2$. Obviously, $\ell \in \mathcal{P}_{ab}^{2,(1,1)}$ and, for any $m > k$, the condition (3.10) with $\sigma_1 = \sigma_2 = 1$ and $\alpha = 1$ is satisfied, where ρ^m $(m = 2, 3, \dots)$ are defined by (3.12) and $\rho^1 = (1,1)^T$. Moreover, the condition (3.15) is true with $\delta_1 = \delta_2 = 1$ and the inequality (3.21) with $\sigma_1 = \sigma_2 = 1$ holds on the set $C_a^{(1,1)}([a,b]; \mathbb{R}_+^2)$, where φ is given by (3.13) and $\overline{\ell} = \big(\overline{\ell}_1, \overline{\ell}_2\big)^T \in \mathcal{P}_{ab}^{2,(1,1)}$ is defined by

$$\overline{\ell}_i(v)(t) \overset{\text{def}}{=} \sum_{j=1}^{2} p_{ij}(t)\bigg(\int_t^b \sum_{k=1}^{2} p_{jk}(s) v_k(b) ds\bigg) \quad \text{for} \quad t \in [a,b], \; i = 1, 2.$$

Since

$$\int_a^b \sum_{j=1}^{2} p_{ij}(s)\bigg(\int_s^b \sum_{k=1}^{2} p_{jk}(\xi)d\xi\bigg) \mathrm{e}^{\int_s^b \sum_{\nu=1}^{2} p_{i\nu}(\eta)d\eta} ds = 1 \quad \text{for} \quad i = 1, 2,$$

the conditions

$$e^{\max\left\{\int_a^b \ell_{11}(1)(s)ds, \int_a^b \ell_{22}(1)(s)ds\right\}} \int_a^b h(s)\, e^{\int_s^b p(\xi)d\xi} ds = 1$$

and

$$\max\left\{\lambda_1\, e^{\int_a^b \ell_{11}(1)(s)ds}, \lambda_2\, e^{\int_a^b \ell_{22}(1)(s)ds}\right\} = 1,$$

are fulfilled, where h, p and λ_1, λ_2 are given by (3.26), (3.27) with $n = 2$, and (3.29), respectively (note that $\widetilde{p}_{12} \equiv \widetilde{p}_{12} \equiv p_{12} \equiv p_{21}$ and $\widetilde{q}_i$ is given by (3.23) with $\sigma_i = 1$ for $i = 1, 2$).

On the other hand, the function $u = (u_1, u_2)^T$, where

$$u_i(t) = \int_a^t p_{i1}(s)ds + \int_a^t p_{i2}(s)ds \quad \text{for} \quad t \in [a, b],\ i = 1, 2,$$

is a nontrivial solution of the problem (3.4). Therefore, by virtue of Remark 3.1, we get $\ell \notin \mathcal{S}_{ab}^{2,(1,1)}(a)$.

This example shows that the assumption $\alpha \in [0, 1[$ in Corollary 3.1 and Theorem 4.1 cannot be replaced by the assumption $\alpha \in [0, 1]$ and the strict inequalities (3.14), (3.25), and (3.28) in Corollaries 3.2, 3.4, and 3.5, respectively, cannot be replaced by the nonstrict ones.

Moreover, this example shows that the assumption $r(A) < 1$ in Proposition 4.1 cannot be replaced by the assumption $r(A) \leq 1$ and the strict inequalities (4.6), (4.16), and (4.17) in Corollaries 4.1, 4.2, and 4.3, respectively, cannot be replaced by the nonstrict ones.

Acknowledgement

The research was supported by the Grant Agency of the Czech Republic, Grant No. 201/04/P183, and by the Academy of Sciences of the Czech Republic, Institutional Research Plan No. AV0Z10190503.

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(Received 14.04.2006)

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