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THE DIRICHLET PROBLEM FOR A SECOND ORDER ELLIPTIC EQUATION DEGENERATING ON THE WHOLE BOUNDARY WHEN THE BOUNDARY CONDITION IS SATISFIED IN THE PRESENCE OF SOME WEIGHT

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Abstract. The first boundary value problem with weight is investigated for a general-type second order elliptic type equation degenerating on the whole boundary.

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Let us consider the elliptic equation

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + au_x + bu_y + cu = 0 \tag{1}$$

in the circle $D = \{(x, y) : x^2 + y^2 < 1\}$ and assume that in a sufficiently small neighborhood $D_1 \subset D$ of the boundary $\Gamma = \partial D$ the representation

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = \frac{\partial^2 u}{\partial \ell_1^2} + H^p \frac{\partial^2 u}{\partial \ell_2^2} \tag{2}$$

holds, where $D_1 = \{(x, y) : 1 - \varepsilon < x^2 + y^2 < 1\}$, $\ell_1 = (-y, x)$, $\ell_2 = (x, y)$, $H = 1 - \sqrt{x^2 + y^2}$, $p = const > 0$, $0 < \varepsilon < 1$.

In what follows it is assumed that the coefficients $a, b \in C^{2,\alpha}(\overline{D})$, $c \in C^{0,\alpha}(\overline{D})$, $c \le 0$, $(x, y) \in \overline{D}$, $0 < \alpha < 1$.

It is not difficult to show that equation (1) with assumption (2) on the boundary $\Gamma = \partial D$ parabolically degenerates and the characteristic direction coincides with the tangent direction, since

$$(AH_x^2 + 2BH_xH_y + CH_y^2)|_{D_1} = H^p G, \quad G = 1.$$

As is known [1], for $p = 1$, $IG^{-1}|_{\partial D} > 1$, $I = L(H) - cH$, the Dirichlet problem is not correctly posed. In that case we can consider the following problem.

Problem. Find a regular solution $u \in C^2(D) \cap C(\overline{D})$ of equation (1) in the domain D and the boundary condition is satisfied with weight, i.e.,

$$\lim_{H \to 0} \tau(x, y)u(x, y) = f(x, y), \tag{3}$$

where the weight function $\tau(x, y) = \frac{1}{W(x,y)}$, $W \in C^2(\overline{D})$ is some function uniformly tending to ∞ as $H \to 0$, $f(x, y)$ is a given continuous function. In such a formulation the problem was for the first time proposed by A.V. Bitsadze [2]. The work [3] by S.A. Tersenev dealing with the same topic should also be noted.

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Equation (1) with (2) taken into account can be rewritten in terms of polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$ as

$$\frac{\partial^2 u}{\partial\varphi^2} + r^2(1-r)^p \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\,(b\cos\varphi - a\sin\varphi)\,\frac{\partial u}{\partial\varphi} + (H^p r + a\cos\varphi + b\sin\varphi)\,\frac{\partial u}{\partial r} + cu = 0. \quad (4)$$

Since

$$H = 1 - r, \quad \frac{\partial u}{\partial r} = -\frac{\partial u}{\partial H}\,, \quad \frac{\partial^2 u}{\partial r^2} = \frac{\partial^2 u}{\partial H^2}\,,$$

equation (4) takes the form

$$L_1(u) = \frac{\partial^2 u}{\partial\varphi^2} + (1-H)^2 H^p \frac{\partial^2 u}{\partial H^2} + a_1(H,\varphi)\,\frac{\partial u}{\partial\varphi} + b_1(H,\varphi)\,\frac{\partial u}{\partial H} + c(H,\varphi)u = 0, \quad (5)$$

where

$$a_1(H,\varphi) = \frac{b\cos\varphi - a\sin\varphi}{1-H}\,,$$
$$b_1(H,\varphi) = -\left[H^p(1-H) + a\cos\varphi + b\sin\varphi\right].$$

Let us introduce a new unknown function $v(\varphi, H)$ assuming that

$$u = Wv. \quad (6)$$

Then equation (5) reduces to the form

$$\frac{\partial^2 v}{\partial\varphi^2} + (1-H)^2 H^p \frac{\partial^2 v}{\partial H^2} + \left[a_1 + \frac{2W_\varphi}{W}\right]\frac{\partial v}{\partial\varphi} + \left[b_1 + 2(1-H)^2 H^p \frac{W_H}{H}\right]\frac{\partial v}{\partial H} + \frac{L_1(W)}{W} = 0. \quad (7)$$

Choose a function $W(\varphi, H)$ such that for $p = 1$

$$\lim_{H\to 0}\left[b_1(H,\varphi) + 2(1-H)^2 H^p \frac{W_H}{W}\right] < 1, \quad (8)$$

$$W(\varphi,H) > 0, \quad L_1(W) < 0 \ \text{ in } \ D. \quad (9)$$

Let us now construct the function $W(\varphi, H)$ satisfying conditions (8) and (9).

If $IG^{-1}|_{\partial D} = b_1(\varphi, 0) \neq 0$, then solutions of equation (5) in the neighborhood of the parabolic degeneration circumference behave mainly like solutions of the equation

$$(1-H)^2 H^p \frac{\partial^2 u}{\partial H^2} + b_1(\varphi, H)\,\frac{\partial u}{\partial H} = 0$$

or

$$H^p \frac{\partial^2 u}{\partial H^2} + b^*(\varphi, H)\,\frac{\partial u}{\partial H} = 0, \quad (10)$$

where

$$b^*(\varphi, H) = -\frac{H^p(1-H) + a\cos\varphi + b\sin\varphi}{(1-H)^2}, \quad b^*(\varphi, 0) = b_1(\varphi, 0).$$

Let $p = 1$ and $(1 - IG^{-1})|_{\partial D} < 0$ or, which is the same, $b_0 = b^*(\varphi, 0) > 1$, $0 \le \varphi \le 2\pi$.

Consider the function

$$\omega(\varphi, H) = \int\limits_H^1 \exp\left[\int\limits_t^1 b^*(\varphi, 0)\tau^{-1}\, d\tau\right] dt = \frac{H^{1-b_0} - 1}{b_0 - 1}, \quad b_0 \ne 1.$$

It is easy to see that $\omega(\varphi, H)$ satisfies equation (10). Also note that for $H > 0$ the function $\omega(\varphi, H)$ is continuous together with its derivatives of all orders and, for $H \to 0$, by virtue of the assumption that $b_0 > 1$ it has singularities at the points of the circumference.

It is obvious that for sufficiently small H the following estimate is true:

$$\frac{\omega_H}{H} = (1 - b_0)H^{-1} + O\left(H^{b_0-2}\right), \quad \omega_H = \frac{\partial\omega}{\partial H}. \tag{11}$$

Since $b_0 = b_1$ and $H = 0$, estimate (11) immediately implies that

$$\begin{aligned}
&\lim_{H\to 0}\left[b_1 + 2(1-H)^2 H\,\frac{\omega_H}{\omega}\right] \\
&\quad= \lim_{H\to 0}\left\{b_0 + 2(1-H)^2 H\left[\frac{-H^{b_0}(1-b_0)}{1 - H^{1-b_0}} + \frac{1-b_0}{H} - \frac{1-b_0}{H}\right]\right\} \\
&\quad= \lim_{H\to 0}\left\{2 - b_0 - 2(1-H)^2(1-b_0)\left[\frac{H^{b_0-1}}{H^{b_0-1} - 1}\right]\right\} = 2 - b_0 < 1.
\end{aligned}$$

Let us now consider the function

$$\begin{aligned}
W(\varphi, H) = {} & \left\{\omega(\varphi, H)\left[1 + \left(\ln\frac{1}{H}\right)^{-\alpha}\right] - (\varphi - \delta)^n + K\right\}\Phi_1(H) \\
& + \left[-(\varphi - \delta_1)^{n_1} + K_1\right]\Phi_2(H),
\end{aligned}$$

where $0 < \alpha < 1$; n, n_1, K and K_1 are positive numbers; δ and δ_1 are chosen so that $\varphi - \delta > 1$ and $\varphi - \delta_1 > 1$,

$$\Phi_1(x) = \frac{\Psi_{0,\lambda}(x)}{\Psi_{0,\lambda}(x) + \Psi_{1,1-\varepsilon}(x)} \in C^\infty[0, 1],$$

$$\Phi_2(x) = \frac{\Psi_{1,1-\varepsilon}(x)}{\Psi_{0,\lambda}(x) + \Psi_{1,1-\varepsilon}(x)} \in C^\infty[0, 1].$$

Here the functions $\Psi_{a,\varepsilon}(x)$ are defined by the equality [4]

$$\Psi_{a,\varepsilon}(x) = \begin{cases} e^{-\frac{1}{\varepsilon^2 - (x-a)^2}}, & |x - a| < \varepsilon, \\ 0, & |x - a| \ge \varepsilon. \end{cases}$$

Let us show that $W(\varphi, H)$ satisfies conditions (8) and (9). We consider three cases.

Case 1. $0 < H \le \varepsilon$. Then $\Phi_1(H) = 1$ and $\Phi_2(H) = 0$. By a direct calculation we find that

$$\begin{aligned}
&\lim_{H\to 0}\left[b_0 + 2(1-H)^2 H\,\frac{W_H}{W}\right] \\
&\quad= \lim_{H\to 0}\left\{b_0 + 2(1-H)^2 H\left[\frac{\omega_H[1+(\ln\frac{1}{H})^{-\alpha}] + \frac{\alpha\omega}{H}[\ln\frac{1}{H}]^{-\alpha-1}}{\omega[1+(\ln\frac{1}{H})^{-\alpha}] - (\varphi-\delta)^n + K}\right.\right. \\
&\qquad\left.\left. + \frac{1-b_0}{H} - \frac{1-b_0}{H}\right]\right\} \\
&\quad= \lim_{H\to 0}\left\{b_0 + 2(1-H)^2(1-b_0) + 2(1-H)^2 H\right. \\
&\qquad\left.\times\left[\frac{-H^{b_0}(1-b_0)[1+(\ln\frac{1}{H})^{-\alpha}] + \frac{\alpha}{H}(\ln\frac{1}{H})^{-\alpha-1}(1-H^{1-b_0})}{(1-H^{1-b_0})[1+(\ln\frac{1}{H})^{-\alpha}] - (1-b_0)[(\varphi-\delta)^n - K]} - \frac{1-b_0}{H}\right]\right\} \\
&\quad= 2 - b_0 < 1,
\end{aligned}$$

i.e., condition (8) is fulfilled.

We will show that W satisfies condition (9) too.

It is easy to see that

$$\frac{\omega_{\varphi\varphi}}{\omega} = O\left(|\log H|^2\right), \quad \frac{\omega_\varphi}{\omega} = O\left(|\log H|\right), \quad b - b_0 = O(H). \tag{12}$$

Using (12) we obtain

$$\begin{aligned}
L_1(W) = \frac{\alpha(1-H)^2\omega}{H}\left[\log\frac{1}{H}\right]^{-\alpha-2}&\left[\alpha + 1 + \left[\frac{b_0}{(1-H)^2} - 1 + 2H\,\frac{\omega_H}{\omega}\right]\log\frac{1}{H}\right. \\
&\left. + O(\sqrt{H})\right] - n(\varphi-\delta)^{n-2}\left[n - 1 + (\varphi-\delta)a\right] + cW.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\sigma_1 &= \frac{b_0}{(1-H)^2} - 1 + 2H\,\frac{\omega_H}{\omega} = \frac{b_0}{(1-H)^2} - 1 + 2H\,\frac{-H^{b_0}(1-b_0)}{1-H^{1-b_0}} \\
&= \frac{b_0}{(1-H)^2} - 1 + 2\,\frac{b_0-1}{H^{b_0-1}-1}\,.
\end{aligned}$$

Therefore

$$\lim_{H\to 0}\sigma_1 = b_0 - 1 - 2(b_0-1) = -(b_0-1) < 0, \tag{13}$$

from which it follows that for small H

$$\sigma_1 = \frac{b_0}{(1-H)^2} - 1 + 2H\,\frac{\omega_H}{\omega} \le \frac{2}{\log H}\,. \tag{14}$$

By virtue of (14) for small H we have

$$\alpha + 1 + \left[\frac{b_0}{(1-H)^2} - 1 + 2H\,\frac{\omega_H}{\omega}\right]\log\frac{1}{H}$$

$$\leq \alpha+1+\frac{2}{\log H}\,\log\frac{1}{H}=\alpha+1-2=\alpha-1<0. \tag{15}$$

It is not difficult to see that

$$\begin{aligned}\lim_{H\to 0}\frac{\alpha\omega[\log\frac{1}{H}]^{-\alpha-2}(1-H)^2}{H}&=\lim_{H\to 0}\frac{\alpha(1-H)^2(1-H^{1-b_0})[\log\frac{1}{H}]^{-\alpha-2}}{(1-b_0)H}\\&=\lim_{H\to 0}\frac{\alpha(1-H)^2(H^{b_0-1}-1)}{(1-b_0)H^{b_0}(\log\frac{1}{H})^{\alpha+2}}=\infty.\end{aligned} \tag{16}$$

From (12)–(15) it follows that for small H

$$\alpha+1+\left[\frac{b_0}{(1-H)^2}-1+2H\,\frac{\omega_H}{\omega}\right]\,\log\frac{1}{H}+O(\sqrt{H})\leq\alpha-1+O(\sqrt{H})<0. \tag{17}$$

By virtue of (16), (17) there exist positive numbers n_0 and K_0 such that for $n\geq n_0$ and $K\geq K_0$ the inequalities $W>0$ and $L_1(W)<0$ are fulfilled simultaneously.

Case 2. $\lambda\leq H<1$. Then $\Phi_1(H)=0$, $\Phi_2(H)=1$ and

$$W(\varphi,H)=-(\varphi-\delta_1)^{n_1}+K_1.$$

It is easy to verify that

$$L_1(W)=-n_1(n_1-1)(\varphi-\delta_1)^{n_1-2}-a_1n_1(\varphi-\delta_1)^{n_1-1}-c(\varphi-\delta_1)^{n_1}+cK_1.$$

Therefore the number n_1 can be chosen so that the inequality $L_1(W)<0$ be fulfilled. After that the number K_1 is chosen so that $W(\varphi,H)$ be positive.

Case 3. $\varepsilon<H<\lambda$. In the preceding two cases we have shown that $W>0$ and $L_1(W)<0$ on the intervals $0<H\leq\varepsilon$ and $\lambda\leq H<1$. But the functions W and $L_1(W)$ are continuous at the points $H=\varepsilon$ and $H=\lambda$. Therefore the same inequalities $W>0$ and $L_1(W)<0$ are fulfilled on wider intervals $0<H\leq\varepsilon_1$ and $\lambda_1\leq H<1$, where $\varepsilon<\varepsilon_1<\lambda_1<\lambda$. Hence instead of $\varepsilon<H<\lambda$ it is sufficient to restrict the consideration to the case $\varepsilon_1\leq H\leq\lambda_1$, where $\Phi_i(H)\geq \mathit{const}>0$, $i=1,2$. By a direct calculation we find that

$$L_1(W)=I_1+I_2,$$

where

$$\begin{aligned}I_1=&-n(n-1)(\varphi-\delta)^{n-2}\Phi_1(H)-(1-H)^2H(\varphi-\delta)^n\,\frac{\partial^2\Phi_1(H)}{\partial H^2}\\&-na_1(H,\varphi)(\varphi-\delta)^{n-1}\Phi_1(H)-b_1(H,\varphi)(\varphi-\delta)^n\,\frac{\partial\Phi_1(H)}{\partial H}\\&-c(H,\varphi)(\varphi-\delta)^n\Phi_1(H),\\I_2=&\,\omega_{\varphi\varphi}(\varphi,H)\left[1+\left(\ln\frac{1}{H}\right)^{-\alpha}\right]\Phi_1(H)-n_1(n_1-1)(\varphi-\delta_1)^{n_1-2}\Phi_2(H)\\&+(1-H)^2H\left[\omega(\varphi,H)\left(1+\left(\ln\frac{1}{H}\right)^{-\alpha}\right)\right]\,\frac{\partial^2\Phi_1(H)}{\partial H^2}\end{aligned}$$

$$\begin{aligned}
&+\left\{\omega_H(\varphi,H)\left[1+\left(\ln\frac{1}{H}\right)^{-\alpha}\right]+\frac{\omega(\varphi,H)\alpha}{H}\left(\ln\frac{1}{H}\right)^{-\alpha-1}\right\}\frac{\partial\Phi_1(H)}{\partial H}\\
&+\left\{\omega_{HH}(\varphi,H)\left[1+\left(\ln\frac{1}{H}\right)^{-\alpha}\right]+\frac{2\omega_H(\varphi,H)}{H}\left(\ln\frac{1}{H}\right)^{-\alpha-1}\right.\\
&\left.-\frac{\omega(\varphi,H)}{H^2}\left(\ln\frac{1}{H}\right)^{-\alpha-1}+\alpha(\alpha+1)\left(\ln\frac{1}{H}\right)^{-\alpha-2}\right\}\Phi_1(H)\\
&+\left\{\omega_H(\varphi,H)\left[1+\left(\ln\frac{1}{H}\right)^{-\alpha}\right]+\frac{\omega(\varphi,H)\alpha}{H}\left(\ln\frac{1}{H}\right)^{-\alpha-1}\right\}\frac{\partial\Phi_1(H)}{\partial H}\\
&+\left[-(\varphi-\delta_1)^{n_1}+K_1\right]\frac{\partial^2\Phi_2(H)}{\partial H^2}+a_1(H,\varphi)\omega_\varphi(\varphi,H)\\
&\times\left[1+\left(\ln\frac{1}{H}\right)^{-\alpha}\right]\Phi_1(H)-a_1(H,\varphi)n_1(\varphi-\delta_1)^{n_1-1}\Phi_2(H)\\
&+b_1(H,\varphi)\left\{\omega(\varphi,H)\left[1+\left(\ln\frac{1}{H}\right)^{-\alpha}\right]+K\right\}\frac{\partial\Phi_1(H)}{\partial H}\\
&+b_1(H,\varphi)\left\{\omega_H(\varphi,H)\left[1+\left(\ln\frac{1}{H}\right)^{-\alpha}\right]\right.\\
&\left.+\omega(\varphi,H)\frac{\alpha}{H}\left(\ln\frac{1}{H}\right)^{-\alpha-1}\right\}\Phi_1(H)+b_1(H,\varphi)\left[-(\varphi-\delta_1)^{n_1}+K_1\right]\frac{\partial\Phi_2(H)}{\partial H}\\
&+c\left\{\omega(\varphi,H)\left[1+\left(\ln\frac{1}{H}\right)^{-\alpha}\right]+K\right\}\Phi_1(H)+c\left[-(\varphi-\delta_1)^{n_1}+K_1\right]\Phi_2(H).
\end{aligned}$$

Since in the expression for I_2 all the terms are bounded, while n_1 and K_1 are already fixed, the number n can be take so large that the inequality $L_1(W)<0$ be valid. After that we choose K such that the inequality $W(\varphi,H)>0$ be fulfilled.

The following statement is true by virtue of (8), (9) and the results of [1].

Theorem. *If* $p=1$, $IG^{-1}|_{\partial D}>1$, $I=L(H)-cH$ *and* $\tau=(b_0-1)(H^{1-b_0}-1)^{-1}$, *where* $b_0=b^*(\varphi,0)$, $b^*(\varphi,H)=[H(1-H)+a\cos\varphi+b\sin\varphi](1-H)^2$ *and* φ *is the polar angle corresponding to the point* $(x,y)\in\overline{D}$, *then for any* $f\in C(\partial D)$ *problem* (1), (3) *has a unique regular solution.*

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