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FOURIER SERIES WITH SMALL GAPS

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Abstract. Let f be a 2π-periodic function in $L^1[-\pi,\pi]$ and $\sum\limits_{k=-\infty}^{\infty}\widehat{f}(n_k)e^{in_kx}$ be its lacunary Fourier series with small gaps. We have estimated Fourier coefficients of f if it is of $\varphi\bigwedge BV$ locally. We have also obtained a precise interconnection between the lacunarity in such series and the localness of the hypothesis to be satisfied by the generic function which allows us to the interpolate the results concerning lacunary series and non-lacunary series.

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Let f be a 2π periodic function in $L^1[0,2\pi]$ and $\widehat{f}(n)$, $n\in Z$, be its Fourier coefficients. The series

$$\sum_{k\in Z}\widehat{f}(n_k)e^{in_kx} \tag{1}$$

with $n_{-k}=-n_k$, $n_0=0$, where $\{n_k\}_1^\infty$ is a strictly increasing sequence of natural numbers satisfy the inequality

$$n_{k+1}-n_k\geq q\geq 1 \quad \text{for all} \;\; k=0,1,2,\dots\,, \tag{2}$$

is called the lacunary Fourier series of f with "small" gaps.

Obviously, if $n_k=k$ for all k (i.e. $n_{k+1}-n_k=q=1$, for all k), then we get a non-lacunary Fourier series and if $\{n_k\}$ is such that

$$n_{k+1}-n_k\to\infty \quad \text{as} \quad k\to\infty, \tag{3}$$

then (1) is said to be the lacunary Fourier series of f.

For the small gap (for $q>1$ in (2)) P. Isaza and D. Waterman [1] estimated the order of magnitude of Fourier coefficients of functions of $\bigwedge BV$ and φBV. Here we have generalized these results. By applying the Wiener–Ingham result [6, Vol. I, p. 222] for finite trigonometric sums with small gap (2) we have estimated the order of magnitude of Fourier coefficients of a function of $\varphi\bigwedge BV$. We have also obtained a precise and beautiful interconnection between the type of lacunarity (as determined by q in (2)) and the localness of the hypothesis to be satisfied by the generic function (as determined by the q-dependent length of I) which allows us to interpolate the results for lacunary and non-lacunary series.

Definition. Given an interval I, a sequence of non-decreasing positive real numbers $\bigwedge=\{\lambda_m\}$ $(m=1,2,\dots)$ such that $\sum\limits_m\frac{1}{\lambda_m}$ diverges and nonnegative

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convex function φ(x) defined on [0,∞) such that $\varphi(0) = 0$. We say that f$\in$ $\varphi\bigwedge$BV(I) (that is f is a function of $\varphi\bigwedge$-bounded variation over (I)) if

$$V_{\Lambda_\varphi}(f, I) = \sup_{\{I_m\}}\{V_{\Lambda_\varphi}(\{I_m\}, f, I)\} < \infty,$$

where

$$V_{\Lambda_\varphi}(\{I_m\}, f, I) = \left(\sum_m \frac{\varphi(|f(b_m) - f(a_m)|)}{\lambda_m}\right),$$

and $\{I_m\}$ is a sequence of non-overlapping subintervals $I_m = [a_m, b_m] \subset I = [a, b]$.

Note that if $\varphi(x) = x^p$ $(1 \le p < \infty)$, then one gets the class $\bigwedge BV^{(p)}(I)$; if $\lambda_m \equiv 1$ for all m, then one gets the class φBV; if $\varphi(x) = x$ and $\lambda_m \equiv 1$ for all m, then one gets the class $BV(I)$.

For an integrable function f (on $[0, 2\pi]$) of $\varphi\bigwedge BV$ locally, we have obtained the sufficient condition [5, Theorem 1.1], in terms of the integral modulus of continuity, for the β-absolute convergence $(0 < \beta \le 2)$ of the lacunary Fourier series (1) with small gaps (2).

We propose the following theorem.

Theorem. *Let $f \in L[-\pi, \pi]$ possess a lacunary Fourier series with small gaps* (2) *and I be a subinterval of length $\delta > \frac{2\pi}{q}$. If $f \in \varphi\bigwedge BV(I)$, then $\widehat{f}(n) = O\left(\varphi{-1}\left(\frac{1}{\sum_{j=1}^{|n|}\frac{1}{\lambda_j}}\right)\right)$.*

Remark. Observe that the interval I considered in the Theorem for the gap condition (2) is of length $> \frac{2\pi}{q}$ so that when $n_k = k$ for all k, I is of length 2π. Thus it gives a precise and beautiful interconnection between lacunary and non-lacunary Fourier series. Also under the gap condition (3) I can be taken as an arbitrary nontrivial subinterval of $[-\pi, \pi]$. Hence the theorem gives Kennedy's result [2] for $f \in BV(I)$ as a particular case.

We need the following Lemma to prove the result.

Lemma ([3, Lemma 4])**.** *Let f and I be as in Theorem. If $f \in L^2(I)$, then*

$$\sum |\widehat{f}(n_k)|^2 \le A_\delta |I|^{-1} \parallel f \parallel_{2,I}^2, \tag{4}$$

where A_δ depends on δ.

Proof of the theorem. Observe that

$$|f(x)| \le |f(a)| + |f(x) - f(a)| \le |f(a)| + \lambda_1 \left(\frac{|f(x) - f(a)|}{\lambda_1}\right)$$
$$\le |f(a)| + \lambda_1 \varphi^{-1}(V_{\wedge_\varphi}(f, I))$$

so that $f \in \varphi\bigwedge BV(I)$ implies that f is bounded over I and hence $f \in L^2(I)$. Since the Fourier series of f has gaps (2), the inequality (4) holds and therefore $f \in L^2[-\pi, \pi]$.

Let $I = [a, b] = [x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}]$ for some x_0 and δ_2 such that $0 < \frac{2\pi}{q} < \delta_2 < \delta$. Put $\delta_3 = \delta - \delta_2$ and $J = [x_0 - \frac{\delta_2}{2}, x_0 + \frac{\delta_2}{2}]$.

Suppose the integers m and j satisfy

$$|n_m| > \frac{4\pi}{\delta_3} \quad \text{and} \quad 0 \le j \le \frac{\delta_3 |n_m|}{4\pi}. \tag{1.1}$$

If we put $F(x) = \sum\limits_j f_j(x)$, where $f_j = \frac{T_{2j\pi/|n_m|}f - T_{(2j-1)\pi/|n_m|}f}{\lambda_j}$ $(\lambda_0 = 1)$ and $T_h f(x) = f(x+h)$, then

$$\widehat{F}(n_k) = \sum_j \widehat{f}_j(n_k),$$

$$\widehat{f}_j(n_k) = \frac{2i\widehat{f}(n_k) e^{i(2j-1/2)\pi n_k/|n_m|} \sin(\frac{\pi n_k}{2|n_m|})}{\lambda_j}$$

implies

$$|\widehat{F}(n_m)| = \left| \sum_j \widehat{f}_j(n_m) \right| = \left| \sum_j \frac{2\widehat{f}(n_m)}{\lambda_j} \right| = 2|\widehat{f}(n_m)| \left(\sum_j \frac{1}{\lambda_j} \right). \tag{1.2}$$

Clearly, the Fourier series of F has gaps (2) and $f \in L^2[-\pi, \pi]$ implies that so does F. By the choice of δ_2 and J, in view of the lemma, one has

$$|\widehat{F}(n_k)|^2 \le \sum_{-\infty}^{\infty} |\widehat{F}(n_k)|^2 = O(1) \|F\|_{2,J}^2. \tag{1.3}$$

Thus relations (1.2) and (1.3) imply

$$|\widehat{f}(n_m)|^2 = O(1) \left(\frac{1}{\sum_j \frac{1}{\lambda_j}} \|F\|_{2,J} \right)^2. \tag{1.4}$$

Observe that for a sufficiently small constant $\alpha > 0$

$$\begin{aligned} \varphi\left(\frac{\alpha |F(x)|}{\sum_j^{|n_m|} \frac{1}{\lambda_j}} \right) &\le \left(\frac{\alpha}{\sum_j^{|n_m|} \frac{1}{\lambda_j}} \right) \left(\sum_j \varphi\left(\frac{|(T_{2j\pi/|n_m|}f - T_{(2j-1)\pi/|n_m|}f)(x)|}{\lambda_j} \right) \right) \\ &\le \left(\frac{\alpha V_{\wedge\varphi}(f, I)}{\sum_j^{|n_m|} \frac{1}{\lambda_j}} \right) \le \left(\frac{1}{\sum_j^{|n_m|} \frac{1}{\lambda_j}} \right) \end{aligned} \tag{1.5}$$

because for any $x \in J$ and j, m satisfying (1.1) the intervals $[x + \frac{(2j-1)\pi}{|n_m|}, x + \frac{2j\pi}{|n_m|}]$ are non-overlapping subintervals of I and $f \in \varphi \bigwedge BV(I)$.

Hence

$$\frac{|F(x)|}{\sum_j^{|n_m|} \frac{1}{\lambda_j}} = O\left(\varphi^{-1}\left(\frac{1}{\sum_j^{|n_m|} \frac{1}{\lambda_j}} \right) \right),$$

which together with (1.4) and (1.5)) implies

$$|\widehat{f}(n_m)| = O\left(\varphi^{-1}\left(\frac{1}{\sum_j^{|n_m|} \frac{1}{\lambda_j}}\right)\right). \qquad \square$$

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