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OSCILLATIONS OF HIGHER ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS WITH IMPULSES

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Abstract. A kind of higher order sub-and super-linear FDE with impulses is studied in this paper. Several criteria on the oscillations of solutions are given. In particular, in the case where the coefficients of equations are positive and continuous functions, we find some suitable impulse functions such that all solutions of the equation are oscillatory under the impulse control.

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1. Introduction

Recent years have seen on increasing number of papers dealing with the oscillatory behavior of ODE(FDE) with impulses. There are some good results on the oscillation of first order ODE with impulses [1]–[5]. The oscillation of second order ODE with impulses is studied in [6]–[8], and the oscillation of second order FDE in [9]–[11]. Some results on the oscillation of higher order ODE are obtained in [12], [13]. However papers, where the oscillation of higher order FDE is investigated, are very rare.

In this paper, we mainly study a kind of higher order sub- and super-linear FDE with impulses under conditions (A) and (B). We can always find some suitable impulse functions such that all solutions of the equation can become oscillatory under the impulse control. We believe that the oscillation under the impulse control is significant both in the theory and in applications.

2. Main Results

We consider the system

$$\begin{cases} x^{(2n)}(t) + p(t)|x(t-\tau)|^r \operatorname{sgn}(x(t-\tau)) = 0, \quad t \geq t_0, \quad t \neq t_k, \\ x^{(i)}(t_k^+) = a_k^{(i)} x^{(i)}(t_k), \quad i = 0, 1, \dots, 2n-1, \quad k = 1, 2, \dots, \\ x^{(i)}(t_0^+) = x_0^{(i)}, \quad i = 0, 1, \dots, 2n-1, \\ x(t) = \phi(t), \quad t_0 - \tau \leq t \leq t_0, \end{cases} \tag{1}$$

where

$$x^{(i)}(t_k) = \lim_{h \to 0^-} \frac{x^{(i-1)}(t_k + h) - x^{(i-1)}(t_k)}{h},$$

$$x^{(i)}(t_k^+) = \lim_{h \to 0^+} \frac{x^{(i-1)}(t_k + h) - x^{(i-1)}(t_k^+)}{h},$$

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$\phi : [t_0 - \tau, t_0] \to R$ has at most a finite number of discontinuous points of first kind and is left continuous at these points, $0 \le t_0 < t_1 < \cdots < t_k < \cdots$, $\lim\limits_{k\to\infty} t_k = +\infty, x^{(0)}(t) = x(t)$, n is a natural number. Here we always assume that the following conditions hold:

(A) $a_k^{(i)} > 0$, $i = 0, 1, \ldots, 2n-1$, $\tau \ge 0$, $r > 0$, $t_{k+1} - t_k > \tau$, $p(t)$ is nonnegative and continuous on $[t_0, +\infty)$, and $p(t)$ is not always equal to 0 in $[t, +\infty)$ for $t \ge t_0$;

$$(B) \quad (t_1 - t_0) + \frac{a_1^{(i)}}{a_1^{(i-1)}}(t_2 - t_1) + \frac{a_1^{(i)} a_2^{(i)}}{a_1^{(i-1)} a_2^{(i-1)}}(t_3 - t_2) + \cdots + \frac{a_1^{(i)} a_2^{(i)} \cdots a_m^{(i)}}{a_1^{(i-1)} a_2^{(i-1)} \cdots a_m^{(i-1)}}(t_{m+1} - t_m) + \cdots = +\infty. \tag{2}$$

Definition 1. A function $x : [t_0 - \tau, +\infty) \to R$ is said to be a solution of (1) on $[t_0 - \tau, +\infty)$ starting from $(t_0, \phi, x_0^{(0)}, x_0^{(1)}, \ldots, x_0^{(2n-1)})$ if

(i) $x^{(i)}(t)$ is continuous on $[t_0, +\infty) \setminus \{t_k,\ k \in N\}$, $i = 0, 1, \ldots, 2n-1$;

(ii) $x(t) = \phi(t)$, $t \in [t_0 - \tau, t_0]$, $x^{(i)}(t_0^+) = x_0^{(i)}$, $i = 0, 1, \ldots, 2n-1$;

(iii) $x(t)$ satisfies the first equality of (1) on $[t_0, +\infty) \setminus \{t_k,\ k \in N\}$;

(iv) $x^{(i)}(t)$ has two-side limits and is left continuous at the points t_k, $k = 1, 2, \ldots$, $x^{(i)}(t_k)$ satisfies the second equality of (1), $i = 0, 1, 2, \ldots, 2n-1$.

Remark 1. Let $x_0(t) = x(t)$, $x_1(t) = x'(t), \ldots, x_{2n-1}(t) = x^{(2n-1)}(t)$. Then equation (1) can be changed into

$$\begin{cases} x'_0(t) = x_1(t), \\ x'_1(t) = x_2(t), \\ \cdots\cdots\cdots\cdots \\ x'_{2n-2}(t) = x_{2n-1}(t), \quad t \ge t_0, \quad t \ne t_k, \\ x'_{2n-1}(t) = -p(t)|x_0(t-\tau)|^r \operatorname{sgn} x_0(t-\tau), \\ x_i(t_k^+) = a_k^{(i)} x_i(t_k), \quad i = 0, 1, \ldots, 2n-1, \quad k = 1, 2 \ldots, \\ x_0(t) = \phi(t), \quad t_0 - \tau \le t \le t_0, \\ x^{(i)}(t_0^+) = x_0^{(i)}. \end{cases}$$

The global existence and uniqueness of solution of (1) can be found in [14]–[15]. In the following, we always assume that solutions of (1) exist on $[t_0, +\infty)$.

Definition 2. A solution of (1) is said to be non-oscillatory if it is eventually positive or eventually negative. Otherwise, this solution is said to be oscillatory.

Lemma 1. *Let $x(t)$ be a solution of* (1), *and conditions* $(A), (B)$ *be satisfied. Suppose that there exist an $i \in \{1, 2, \ldots, 2n-1\}$ and some $T \ge t_0$ such that $x^{(i)}(t) > 0$ (< 0), $x^{(i+1)}(t) \ge 0$ (≤ 0) for $t \ge T$. Then there exists some $T_1 \ge T$ such that $x^{(i-1)}(t) > 0$ (< 0) for $t \ge T_1$.*

Proof. Without loss of generality, suppose that $T = t_1$, $x^{(i)}(t) > 0$, $x^{(i+1)}(t) \geq 0$ for $t \in (t_k, t_{k+1}]$ $(k = 1, 2, \dots)$. Hence $x^{(i)}(t) > 0$ is monotonically nondecreasing in $(t_k, t_{k+1}]$. For $t \in (t_1, t_2]$, we have

$$x^{(i)}(t) \geq x^{(i)}(t_1^+).$$

Integrating the above inequality, we have

$$x^{(i-1)}(t_2) \geq x^{(i-1)}(t_1^+) + x^{(i)}(t_1^+)(t_2 - t_1). \tag{3}$$

Similarly to (3),

$$x^{(i-1)}(t_3) \geq x^{(i-1)}(t_2^+) + x^{(i)}(t_2^+)(t_3 - t_2). \tag{4}$$

By $x^{(i)}(t_2) \geq x^{(i)}(t_1^+)$ and (3), (4), we have

$$\begin{aligned} x^{(i-1)}(t_3) &\geq x^{(i-1)}(t_2^+) + x^{(i)}(t_2^+)(t_3 - t_2) \\ &= a_2^{(i-1)} x^{(i-1)}(t_2) + a_2^{(i)} x^{(i)}(t_2)(t_3 - t_2) \\ &\geq a_2^{(i-1)} [x^{(i-1)}(t_1^+) + x^{(i)}(t_1^+)(t_2 - t_1)] + a_2^{(i)} x^{(i)}(t_2)(t_3 - t_2) \\ &\geq a_2^{(i-1)} \left[x^{(i-1)}(t_1^+) + x^{(i)}(t_1^+)(t_2 - t_1) + \frac{a_2^{(i)}}{a_2^{(i-1)}} x^{(i)}(t_1^+)(t_3 - t_2) \right]. \end{aligned}$$

Applying mathematical induction for any natural number m we have

$$\begin{aligned} x^{(i-1)}(t_m) \geq a_{m-1}^{(i-1)} \cdots a_3^{(i-1)} a_2^{(i-1)} \Bigg\{ & x^{(i-1)}(t_1^+) + x^{(i)}(t_1^+) \Bigg[(t_2 - t_1) \\ & + \frac{a_2^{(i)}}{a_2^{(i-1)}} (t_3 - t_2) + \cdots + \frac{a_2^{(i)} a_3^{(i)} \cdots a_{m-1}^{(i)}}{a_2^{(i-1)} a_3^{(i-1)} \cdots a_{m-1}^{(i-1)}} (t_m - t_{m-1}) \Bigg] \Bigg\}. \end{aligned} \tag{5}$$

By condition (B) and $a_k^{(i)} > 0$, for all sufficiently large m, we have $x^{(i-1)}(t_m) > 0$. That is, there exists a natural number N such that $t_N \geq T$ and for $m \geq N$, we have $x^{(i-1)}(t_m) > 0$. Since $x^{(i)}(t) > 0$, we have $x^{(i-1)}(t) > x^{(i-1)}(t_k) > 0$ for $t \in (t_k, t_{k+1}]$ where $k \geq N$. Hence for $t \geq t_N$, we have $x^{(i-1)}(t) > 0$. The proof of Lemma 1 is completed. □

Lemma 2. *Let $x(t)$ be a solution of* (1) *and conditions* $(A), (B)$ *be satisfied. Suppose that there exist an $i \in \{1, 2, \dots, 2n\}$ and some $T \geq t_0$ such that $x(t) > 0$, $x^{(i)}(t) \leq 0$ for $t \geq T$, $x^{(i)}(s)$ is not always equal to 0 in $[t, +\infty)$ $(t \geq T)$. Then $x^{(i-1)}(t) > 0$ for all sufficiently large t.*

Proof. Let $T = t_0$. We claim that $x^{(i-1)}(t_k) > 0$ for any $t_k \geq T$.

If this is not true, then there exists some $t_j \geq T$ such that $x^{(i-1)}(t_j) \leq 0$. Since $x^{(i)}(t) \leq 0$, $x^{(i-1)}(t)$ is non-increasing in $(t_k, t_{k+1}]$ for $k \geq j$ and $x^{(i)}(s)$ is not always equal to 0 in $[t, +\infty)$, there exists some $t_l \geq t_j$ such that $x^{(i)}(t)$ is not always equal to 0 in $(t_l, t_{l+1}]$. Without loss of generality, we can assume that $l = j$. So we have

$$x^{(i-1)}(t_{j+1}) < x^{(i-1)}(t_j^+) = a_j^{(i-1)} x^{(i-1)}(t_j) \leq 0.$$

For $t \in (t_{j+1}, t_{j+2}]$ we have

$$x^{(i-1)}(t_{j+2}) \le x^{(i-1)}(t_{j+1}^{+}) = a_{j+1}^{(i-1)} x^{(i-1)}(t_{j+1}) < 0.$$

By induction, we have $x^{(i-1)}(t) < 0$, $t \in (t_{j+m}, t_{j+m+1}]$ for all sufficiently large m. Thus we have $x^{(i-1)}(t) < 0$, $x^{(i)}(t) \le 0$, $t \in (t_{j+1}, +\infty)$. By Lemma 1, for all sufficiently large t we have $x^{(i-2)}(t) < 0$. Applying Lemma 1 repeatedly, for all sufficiently large t we have $x(t) < 0$. This is a contradiction to $x(t) > 0 (t \ge T)$! Hence we have $x^{(i-1)}(t_k) > 0$ for any t_k. So we have $x^{(i-1)}(t) > 0$ for all sufficiently large t. The proof of Lemma 2 is completed. □

Lemma 3. *Let $x(t)$ be a solution of (1) and conditions $(A), (B)$ be satisfied. Suppose there exist some $T \ge t_0$ such that $x(t) > 0$ for $t \ge T$. Then there exists some $T' \ge T$ and $l \in \{1, 3, \dots, 2n-1\}$ such that for $t \ge T'$,*

$$\begin{cases} x^{(i)}(t) > 0, \quad i = 0, 1, \dots, l, \\ (-1)^{i-1} x^{(i)}(t) > 0, \quad i = l+1, \dots, 2n-1, \\ x^{(2n)}(t) \le 0. \end{cases} \tag{6}$$

Proof. Let $T = t_0$. Since $x(t) > 0$ $(t \ge t_0)$, by (1) and that $p(t)$ is nonnegative and is not always equal to 0 on any $(t, +\infty)$, we have

$$x^{(2n)}(t) = -p(t)[x(t-\tau)]^r \le 0,$$

and $x^{(2n)}(s)$ is not always equal to 0 in $(t, +\infty)$ for $t \ge t_0$. By Lemma 2, we have $x^{(2n-1)}(t) > 0$ for sufficiently large t. Without loss of generality, let $x^{(2n-1)}(t) > 0$ for $t \ge t_0$. So $x^{(2n-2)}(t)$ is nondecreasing on $(t_k, t_{k+1}]$. If for any t_k, $x^{(2n-2)}(t_k) < 0$, then $x^{(2n-2)}(t) < 0$ $(t \ge t_0)$. If there exists some t_j such that $x^{(2n-2)}(t_j) \ge 0$, by $x^{(2n-2)}(t)$ is nondecreasing on $(t_k, t_{k+1}]$ and $a_k^{(2n-2)} > 0$, we get $x^{(2n-2)}(t) > 0$ for $t > t_j$. So there exists some $T_1 \ge T$ such that one of the following statements holds:

(A_1) $\quad x^{(2n-1)}(t) > 0, \quad x^{(2n-2)}(t) > 0, \quad t \ge T_1;$

(B_1) $\quad x^{(2n-1)}(t) > 0, \quad x^{(2n-2)}(t) < 0, \quad t \ge T_1.$

When (A_1) holds, by Lemma 1, we have $x^{(2n-3)}(t) > 0$ for all sufficiently large t. Applying Lemma 1 repeatedly, for all sufficiently large t, we have $x^{(2n-1)}(t) > 0$, $x^{(2n-2)}(t) > 0, \dots, x'(t) > 0$, $x(t) > 0$, and (6) holds with $l = 2n-1$.

When (B_1)holds, by Lemma 2, we have $x^{(2n-3)}(t) > 0$ for all sufficiently large t. By deducing further, there exists some $T_2 \ge T_1$ such that one of the following statements holds:

(A_2) $\quad x^{(2n-3)}(t) > 0, \quad x^{(2n-4)}(t) > 0, \quad t \ge T_2;$

(B_2) $\quad x^{(2n-3)}(t) > 0, \quad x^{(2n-4)}(t) < 0, \quad t \ge T_2.$

Repeating the above reasoning, we can get that there exists some $T' \ge T$ and

$l \in \{1, 3, \ldots, 2n-3\}$ such that for $t \geq T'$,

$$\begin{cases} x^{(i)}(t) > 0, \quad i = 0, 1, \ldots, l, \\ (-1)^{i-1} x^{(i)}(t) > 0, \quad i = l+1, l+2, \ldots, 2n-1, \\ x^{(2n)}(t) \leq 0. \end{cases}$$

The proof of Lemma 3 is completed. □

Remark 2. If $x(t)$ is an eventually negative solution of (1), we obtain results similar to Lemmas 2 and 3.

Theorem 1. *If conditions* $(A), (B)$ *hold,* $a_k^{(0)} \geq 1$ *for* $k = 1, 2, \ldots$ *and*

$$\int_{t_0}^{t_1} p(t)\,dt + \frac{1}{a_1^{(2n-1)}} \int_{t_1}^{t_2} p(t)\,dt + \frac{1}{a_1^{(2n-1)} a_2^{(2n-1)}} \int_{t_2}^{t_3} p(t)\,dt + \cdots + \frac{1}{a_1^{(2n-1)} a_2^{(2n-1)} \cdots a_m^{(2n-1)}} \int_{t_m}^{t_{m+1}} p(t)\,dt + \cdots = +\infty, \quad (7)$$

then every solution of (1) *is oscillatory.*

Proof. Let $x(t)$ be a non-oscillatory solution of (1). Without loss of generality, let $x(t) > 0$ $(t \geq t_0)$. By Lemma 3 and (1), there exists $T' \geq t_0$ such that for $t \geq T'$ we have

$$x^{(2n)}(t) \leq 0, \quad x^{(2n-1)}(t) > 0, \quad x'(t) > 0, \quad x(t) > 0. \quad (8)$$

Since $a_k^{(0)} \geq 1$ $(k = 1, 2, \ldots)$, there exists some natural number l such that $x(t)$ is nondecreasing in $[t_l, +\infty)$, $x(t_l^+) \leq x(t_{l+1}) \leq x(t_{l+1}^+) \leq x(t_{l+2}) \leq x(t_{l+2}^+) \leq \cdots$. Let j be some natural number such that $[t_j - \tau, +\infty) \subset [t_l, +\infty)$. Then (1) yields

$$x^{(2n)}(t) = -p(t) x^r(t-\tau), \quad t > t_j, \quad t \neq t_k. \quad (9)$$

Integrating (9) from t_j to t_{j+1}, we have

$$x^{(2n-1)}(t_{j+1}) - x^{(2n-1)}(t_j^+) = -\int_{t_j}^{t_{j+1}} p(t) x^r(t-\tau) dt. \quad (10)$$

By (10) and that $x(t)$ is increasing, we have

$$\begin{aligned} x^{(2n-1)}(t_{j+1}) &= x^{(2n-1)}(t_j^+) - \int_{t_j}^{t_{j+1}} p(t) x^r(t-\tau) dt \\ &\leq x^{(2n-1)}(t_j^+) - [x(t_j - \tau)]^r \int_{t_j}^{t_{j+1}} p(t) dt \end{aligned}$$

$$= a_j^{(2n-1)} x^{(2n-1)}(t_j) - [x(t_j - \tau)]^r \int_{t_j}^{t_{j+1}} p(t)dt. \tag{11}$$

Similarly to (11), we have

$$x^{(2n-1)}(t_{j+2}) = a_{j+1}^{(2n-1)} x^{(2n-1)}(t_{j+1}) - [x(t_{j+1} - \tau)]^r \int_{t_{j+1}}^{t_{j+2}} p(t)dt$$
$$\leq a_{j+1}^{(2n-1)} \left[a_j^{(2n-1)} x^{(2n-1)}(t_j) - [x(t_j - \tau)]^r \int_{t_j}^{t_{j+1}} p(t)dt \right] - [x(t_j - \tau)]^r \int_{t_{j+1}}^{t_{j+2}} p(t)dt$$
$$\leq a_{j+1}^{(2n-1)} \left\{ a_j^{(2n-1)} x^{(2n-1)}(t_j) - [x(t_j - \tau)]^r \left[\int_{t_j}^{t_{j+1}} p(t)dt \right.\right.$$
$$\left.\left. + \frac{1}{a_{j+1}^{(2n-1)}} \int_{t_{j+1}}^{t_{j+2}} p(t)dt \right] \right\}. \tag{12}$$

By mathematical induction we have for any natural number $m \geq 2$,

$$x^{(2n-1)}(t_{j+m}) \leq a_{j+m-1}^{(2n-1)} a_{j+m-2}^{(2n-1)} \cdots a_{j+1}^{(2n-1)} \left\{ a_{j+1}^{(2n-1)} x^{(2n-1)}(t_j) \right.$$
$$- [x(t_j - \tau)]^r \left[\int_{t_j}^{t_{j+1}} p(t)dt + \frac{1}{a_{j+1}^{(2n-1)}} \int_{t_{j+1}}^{t_{j+2}} p(t)dt + \frac{1}{a_{j+2}^{(2n-1)} a_{j+1}^{(2n-1)}} \int_{t_{j+2}}^{t_{j+3}} p(t)dt \right.$$
$$\left.\left. + \cdots + \frac{1}{a_{j+m-1}^{(2n-1)} a_{j+m-2}^{(2n-1)} \cdots a_{j+1}^{(2n-1)}} \int_{t_{j+m-1}}^{t_{j+m}} p(t)dt \right] \right\}. \tag{13}$$

By (7), (13) and $a_k^{(i)} > 0$, for all sufficiently large m, we have

$$x^{(2n-1)}(t_{j+m}) < 0.$$

This contradicts the fact that $x^{(2n-1)}(t) > 0$ for all sufficiently large t. So every solution of (1) is oscillatory. The proof of Theorem 1 is completed. □

Corollary 1. *Assume that conditions* (A) *and* (B) *hold, and* $a_k^{(0)} \geq 1$, $a_k^{(2n-1)} \leq 1$ *for* $k = 1, 2, \ldots$. *If* $\int^{+\infty} p(t)dt = +\infty$, *then every solution of* (1) *is oscillatory.*

Proof. By $a_k^{(2n-1)} \leq 1$, we have

$$\frac{1}{a_1^{(2n-1)} a_2^{(2n-1)} \cdots a_k^{(2n-1)}} \int_{t_k}^{t_{k+1}} p(t)dt \geq \int_{t_k}^{t_{k+1}} p(t)dt.$$

As $m \to +\infty$, $\int_{t_0}^{t_{m+1}} p(t)dt \to +\infty$, condition (7) of Theorem 1 holds. By Theorem 1, we know that every solution of (1) is oscillatory. □

Corollary 2. *Assume that conditions* (A) *and* (B) *hold and there exists an* $\alpha > 0$ *such that* $a_k^{(0)} \geq 1$, $\frac{1}{a_k^{(2n-1)}} \geq (\frac{t_{k+1}}{t_k})^\alpha$ *for* $k = 1, 2, \ldots$. *If* $\int^{+\infty} t^\alpha p(t)dt = +\infty$, *then every solution of* (1) *is oscillatory.*

Proof. By $\frac{1}{a_k^{(2n-1)}} \geq (\frac{t_{k+1}}{t_k})^\alpha$, we have

$$\frac{1}{a_1^{(2n-1)} a_2^{(2n-1)} \cdots a_k^{(2n-1)}} \int_{t_k}^{t_{k+1}} p(t)dt \geq \left(\frac{1}{t_1}\right)^\alpha \int_{t_k}^{t_{k+1}} t^\alpha p(t)dt.$$

Then

$$\int_{t_0}^{t_1} p(t)dt + \frac{1}{a_1^{(2n-1)}} \int_{t_1}^{t_2} p(t)dt + \frac{1}{a_1^{(2n-1)} a_2^{(2n-1)}} \int_{t_2}^{t_3} p(t)dt \geq \left(\frac{1}{t_1}\right)^\alpha \int_{t_1}^{t_{m+1}} t^\alpha p(t)dt.$$

As $m \to +\infty$, $\int_{t_1}^{t_{m+1}} p(t)dt \to +\infty$, condition (7) of Theorem 1 holds. Theorem 1 implies that every solution of (1) is oscillatory. □

Theorem 2. *Suppose that conditions* $(A), (B)$ *hold and for any natural number* k, $t_k - t_{k-1} > \tau > 0$. *Let either*

$$(a):\quad \int_{t_0+\tau}^{t_1} p(t)dt + \frac{[a_1^{(0)}]^r}{a_1^{(2n-1)}} \int_{t_1+\tau}^{t_2} p(t)dt + \frac{[a_1^{(0)}]^r [a_2^{(0)}]^r}{a_1^{(2n-1)} a_2^{(2n-1)}} \int_{t_2+\tau}^{t_3} p(t)dt + \cdots + \frac{[a_1^{(0)}]^r [a_2^{(0)}]^r \cdots [a_m^{(0)}]^r}{a_1^{(2n-1)} a_2^{(2n-1)} \cdots a_m^{(2n-1)}} \int_{t_m+\tau}^{t_{m+1}} p(t)dt + \cdots = +\infty \qquad (14)$$

or

$$(b):\quad \int_{t_0}^{t_0+\tau} p(t)dt + \frac{1}{a_1^{(2n-1)}} \int_{t_1}^{t_1+\tau} p(t)dt + \frac{[a_1^{(0)}]^r}{a_1^{(2n-1)} a_2^{(2n-1)}} \int_{t_2}^{t_2+\tau} p(t)dt + \cdots + \frac{[a_1^{(0)}]^r [a_2^{(0)}]^r \cdots [a_{m-1}^{(0)}]^r}{a_1^{(2n-1)} a_2^{(2n-1)} \cdots a_m^{(2n-1)}} \int_{t_m}^{t_m+\tau} p(t)dt + \cdots = +\infty \qquad (15)$$

hold. Then every solution of (1) *is oscillatory.*

Proof. Suppose that (1) has a non-oscillatory solution $x(t)$. Without the loss of generality suppose that $T' = t_0$ and $x(t) > 0$ $(t \geq t_0)$. Lemma 3 and (1) imply that there exists $T' = t_0$ such that for $t \geq T'$,

$$x^{(2n)}(t) \leq 0, \quad x^{(2n-1)}(t) > 0, \quad x'(t) > 0, \quad x(t) > 0. \qquad (16)$$

Then

$$\begin{aligned} x^{(2n-1)}(t_{k+m}) &\le x^{(2n-1)}(t_{k+m-1}^+) = a_{k+m-1}^{(2n-1)} x^{(2n-1)}(t_{k+m-1}) \\ &\le a_{k+m-1}^{(2n-1)} x^{(2n-1)}(t_{k+m-2}^+) \\ &= a_{k+m-1}^{(2n-1)} a_{k+m-2}^{(2n-1)} x^{(2n-1)}(t_{k+m-2}) \\ &\le \cdots\cdots\cdots\cdots \\ &\le a_{k+m-1}^{(2n-1)} a_{k+m-2}^{(2n-1)} \cdots a_3^{(2n-1)} a_2^{(2n-1)} a_1^{(2n-1)} x^{(2n-1)}(t_1), \\ x(t_k^+) &= a_k^{(0)} x(t_k) \ge a_k^{(0)} x(t_{k-1}^+) \ge a_k^{(0)} a_{k-1}^{(0)} x(t_{k-1}) \\ &\ge \cdots\cdots\cdots\cdots \\ &\ge a_k^{(0)} a_{k-1}^{(0)} \cdots a_2^{(0)} a_1^{(0)} x(t_1). \end{aligned}$$

If condition (a) holds, then for $t \in (t_1, t_2]$,

$$x^{(2n)}(t) = -p(t)[x(t-\tau)]^r. \tag{17}$$

Integrating (17) from t_1 to t_2, we have

$$\begin{aligned} x^{(2n-1)}(t_2) &= x^{(2n-1)}(t_1^+) - \int_{t_1}^{t_2} p(t)[x(t-\tau)]^r dt \\ &\le a_1^{(2n-1)} x^{(2n-1)}(t_1) - [x(t_1^+)]^r \int_{t_1+\tau}^{t_2} p(t)dt \\ &\le a_1^{(2n-1)} x^{(2n-1)}(t_1) - [a_1^{(0)}]^r [x(t_1)]^r \int_{t_1+\tau}^{t_2} p(t)dt. \end{aligned}$$

Similarly,

$$\begin{aligned} x^{(2n-1)}(t_3) &\le a_2^{(2n-1)} x^{(2n-1)}(t_2) - [x(t_2^+)]^r \int_{t_2+\tau}^{t_3} p(t)dt \\ &\le a_2^{(2n-1)} \left[a_1^{(2n-1)} x^{(2n-1)}(t_1) - [a_1^{(0)}]^r [x(t_1)]^r \int_{t_1+\tau}^{t_2} p(t)dt \right] \\ &\quad - [a_1^{(0)}]^r [a_2^{(0)}]^r [x(t_1)]^r \int_{t_2+\tau}^{t_3} p(t)dt \\ &= a_2^{(2n-1)} a_1^{(2n-1)} x^{(2n-1)}(t_1) \\ &\quad - [x(t_1)]^r \left[[a_1^{(0)}]^r a_2^{(2n-1)} \int_{t_1+\tau}^{t_2} p(t)dt + [a_1^{(0)}]^r [a_2^{(0)}]^r \int_{t_2+\tau}^{t_3} p(t)dt \right]. \end{aligned}$$

Suppose that, for any natural number k

$$x^{(2n-1)}(t_k) \le a_1^{(2n-1)} a_2^{(2n-1)} \cdots a_{k-2}^{(2n-1)} a_{k-1}^{(2n-1)} x^{(2n-1)}(t_1) - [x(t_1)]^r \Big\{ \sum_{i=1}^{k-1} [a_1^{(0)}]^r \cdots [a_i^{(0)}]^r a_{i+1}^{(2n-1)} \cdots a_{k-1}^{(2n-1)} \int_{t_i+\tau}^{t_{i+1}} p(t)dt \Big\}.$$

Integrating (17) from t_k to t_{k+1}, we have

$$\begin{aligned} x^{(2n-1)}(t_{k+1}) &\le a_k^{(2n-1)} x^{(2n-1)}(t_k) - [x(t_k^+)]^r \int_{t_k+\tau}^{t_{k+1}} p(t)dt \\ &\le a_k^{(2n-1)} a_1^{(2n-1)} a_2^{(2n-1)} \cdots a_{k-1}^{(2n-1)} x^{(2n-1)}(t_1) \\ &\quad - [x(t_1)]^r \Big\{ \sum_{i=1}^{k-1} [a_1^{(0)}]^r \cdots [a_i^{(0)}]^r a_{i+1}^{(2n-1)} \cdots a_{k-1}^{(2n-1)} \int_{t_i+\tau}^{t_{i+1}} p(t)dt \Big\} \\ &\quad - [a_1^{(0)}]^r [a_2^{(0)}]^r \cdots [a_{k-1}^{(0)}]^r [a_k^{(0)}]^r [x(t_1)]^r \int_{t_k+\tau}^{t_{k+1}} p(t)dt \\ &= a_1^{(2n-1)} a_2^{(2n-1)} \cdots a_{k-1}^{(2n-1)} a_k^{(2n-1)} x^{(2n-1)}(t_1) \\ &\quad - [x(t_1)]^r \Big\{ \sum_{i=1}^{k} [a_1^{(0)}]^r \cdots [a_i^{(0)}]^r a_{i+1}^{(2n-1)} \cdots a_k^{(2n-1)} \int_{t_i+\tau}^{t_{i+1}} p(t)dt \Big\}. \end{aligned}$$

By mathematical induction, we know that for any natural number m

$$x^{(2n-1)}(t_m) \le a_1^{(2n-1)} a_2^{(2n-1)} \cdots a_{m-1}^{(2n-1)} x^{(2n-1)}(t_1) - [x(t_1)]^r \Big\{ \sum_{i=1}^{m-1} [a_1^{(0)}]^r \cdots [a_i^{(0)}]^r a_{i+1}^{(2n-1)} \cdots a_{m-1}^{(2n-1)} \int_{t_i+\tau}^{t_{i+1}} p(t)dt \Big\}.$$

Therefore

$$x^{(2n-1)}(t_m) \le a_1^{(2n-1)} a_2^{(2n-1)} \cdots a_{m-1}^{(2n-1)} [x(t_1)]^r \Big\{ \frac{x^{(2n-1)}(t_1)}{[x(t_1)]^r} - \Big\{ \sum_{i=1}^{m-1} \frac{[a_1^{(0)}]^r \cdots [a_i^{(0)}]^r}{a_1^{(2n-1)} \cdots a_i^{(2n-1)}} \int_{t_i+\tau}^{t_{i+1}} p(t)dt \Big\} \Big\}.$$

The above inequality and (14) imply that $x^{(2n-1)}(t_m) \le 0$ for m large enough. Thus $x^{(2n-1)}(t) \le 0$ for all sufficiently large t. This contradicts $x^{(2n-1)}(t) > 0$ for $t \ge T'$. So every solution of (1) is oscillatory.

If condition (b) holds, integrating (17) from t_1 to t_2, we have

$$\begin{aligned} x^{(2n-1)}(t_2) &= x^{(2n-1)}(t_1^+) - \int_{t_1}^{t_2} p(t)[x(t-\tau)]^r dt \\ &\le x^{(2n-1)}(t_1^+) - \int_{t_1}^{t_1+\tau} p(t)[x(t-\tau)]^r dt \\ &\le a_1^{(2n-1)} x^{(2n-1)}(t_1) - [x(t_1-\tau)]^r \int_{t_1}^{t_1+\tau} p(t)dt. \end{aligned}$$

Similarly,

$$\begin{aligned} x^{(2n-1)}(t_3) &\le a_2^{(2n-1)} x^{(2n-1)}(t_2) - [x(t_2-\tau)]^r \int_{t_2}^{t_2+\tau} p(t)dt \\ &\le a_2^{(2n-1)} x^{(2n-1)}(t_2) - [x(t_1^+)]^r \int_{t_2}^{t_2+\tau} p(t)dt \\ &\le a_2^{(2n-1)} \Big\{ a_1^{(2n-1)} x^{(2n-1)}(t_1) - [x(t_1-\tau)]^r \int_{t_1}^{t_1+\tau} p(t)dt \Big\} \\ &\quad - [a_1^{(0)}]^r [x(t_1-\tau)]^r \int_{t_2}^{t_2+\tau} p(t)dt \\ &= a_2^{(2n-1)} a_1^{(2n-1)} x^{(2n-1)}(t_1) \\ &\quad - [x(t_1-\tau)]^r \Big\{ a_2^{(2n-1)} \int_{t_1}^{t_1+\tau} p(t)dt + [a_1^{(0)}]^r \int_{t_2}^{t_2+\tau} p(t)dt \Big\}. \end{aligned}$$

By mathematical induction, for any natural number m

$$\begin{aligned} x^{(2n-1)}(t_m) &\le a_1^{(2n-1)} a_2^{(2n-1)} \cdots a_{m-1}^{(2n-1)} x^{(2n-1)}(t_1) \\ &\quad - [x(t_1-\tau)]^r \Big\{ \sum_{i=1}^{m-1} [a_1^{(0)}]^r \cdots [a_{i-1}^{(0)}]^r a_{i+1}^{(2n-1)} \cdots a_{m-1}^{(2n-1)} \int_{t_i}^{t_i+\tau} p(t)dt \Big\}. \end{aligned}$$

Hence

$$x^{(2n-1)}(t_m) \le a_1^{(2n-1)} a_2^{(2n-1)} \cdots a_{m-1}^{(2n-1)} [x(t_1-\tau)]^r \Big\{ \frac{x^{(2n-1)}(t_1)}{[x(t_1-\tau)]^r}$$

$$-\left\{\sum_{i=1}^{m-1}\frac{[a_1^{(0)}]^r\cdots[a_{i-1}^{(0)}]^r}{a_1^{(2n-1)}\cdots a_i^{(2n-1)}}\int_{t_i}^{t_i+\tau}p(t)dt\right\}\Bigg\}.$$

The above inequality and (15) imply that $x^{(2n-1)}(t_m)\le 0$ for m large enough. So $x^{(2n-1)}(t)\le 0$ for all sufficiently large t. This contradicts $x^{(2n-1)}(t)>0$ for $t\ge T'$. So every solution of (1) is oscillatory.

Summing up the above discussion, we can see that every solution of (1) is oscillatory. The proof is completed. □

Consider

$$\begin{cases} x^{(2n)}(t)+p(t)|x(t-\tau)|^r\operatorname{sgn}(x(t-\tau))=0, & t\ge t_0,\\ x(t)=\phi(t), \quad t_0-\tau\le t\le t_0 \end{cases}\tag{18}$$

and

$$\begin{cases} x^{(2n)}(t)+p(t)|x(t-\tau)|^r\operatorname{sgn}(x(t-\tau))=0,\\ x^{(i)}(t_k^+)=I_{k(i)}(x^{(i)}(t_k)) \quad \text{for} \quad i=0,1,\dots,2n-1,\\ x^{(i)}(t_0^+)=x_0^{(i)} \quad \text{for} \quad k=0,1,\dots,\\ x(t)=\phi(t), \quad t_0-\tau\le t\le t_0. \end{cases}\tag{19}$$

where $I_{k(i)}(x)$ is continuous on $(-\infty,+\infty)$ and $xI_{k(i)}(x)>0$ $(x\neq 0)$, $k=1,2,\dots$, $\varphi(t):[t_0-\tau,t_0]\to R$ has at most a finite number of discontinuous points of first kind and is left continuous at those points.

Theorem 3. *If $p(t)>0$ is continuous on $[0,+\infty)$, $r>0$, $r\neq 1$, then for any $\{t_m\}:0<t_1<t_2<\cdots<t_m<\cdots$, $t_m-t_{m-1}>\tau>0$, one can find suitable impulsive functions $I_{k(i)}(x)$ such that under the impulsive effects* (18) *transforms to* (19)*, all solutions of* (19) *are oscillatory.*

Proof. Let $c_k=\int_{t_k+\tau}^{t_{k+1}}p(t)dt$, $c_0=1$ and $I_{k(i)}(x)=d_kx$ $(a_k^{(i)}=d_k)$, where $d_k=(\frac{c_{k-1}}{c_k})^{\frac{1}{r-1}}$, $k=1,2,\dots$. Then

$$d_1^{r-1}=\frac{1}{c_1},\quad (d_1d_2)^{r-1}=\frac{1}{c_2},\ \dots,\ (d_1d_2\cdots d_m)^{r-1}=\frac{1}{c_m}.$$

Hence

$$\int_{t_0+\tau}^{t_1}p(t)dt+\frac{[a_1^{(0)}]^r}{a_1^{(2n-1)}}\int_{t_1+\tau}^{t_2}p(t)dt+\frac{[a_1^{(0)}]^r[a_2^{(0)}]^r}{a_1^{(2n-1)}a_2^{(2n-1)}}\int_{t_2+\tau}^{t_3}p(t)dt$$
$$+\cdots+\frac{[a_1^{(0)}]^r[a_2^{(0)}]^r\cdots[a_m^{(0)}]^r}{a_1^{(2n-1)}a_2^{(2n-1)}\cdots a_m^{(2n-1)}}\int_{t_m+\tau}^{t_{m+1}}p(t)dt$$
$$\ge d_1^{r-1}c_1+(d_1d_2)^{r-1}c_2+\cdots+(d_1d_2\cdots d_m)^{r-1}c_m=m.$$

Therefore the condition (a) of Theorem 2 holds. We can see that every solution of (19) is oscillatory. □

Remark 3. Though the condition on $p(t)$ cannot guarantee that all solutions of (18) are oscillatory, we can see from Theorem 3 that if we give some suitable impulsive effects to it, all solutions can become oscillatory.

3. Examples

Example 1. Consider the system

$$\begin{cases} x^{(2n)}(t) + x^{2n-1}(t-\frac{1}{2}) = 0, \quad t \geq t_0 = 1, \quad t \neq 2^k, \quad k = 1, 2, \dots, \\ x((2^k)^+) = 2x(2^k), \quad x^{(i)}((2^k)^+) = \frac{k}{k+1} x^{(i)}(2^k), \quad i = 1, \dots, 2n-1, \\ x(1) = x_0, \quad x^{(i)}(1^+) = x_0^{(i)}, \\ x(t) = \phi(t), \quad t \in [\frac{1}{2}, 1]. \end{cases} \tag{20}$$

where $a_k^{(0)} = 2 > 1$, $a_k^{(i)} = \frac{k}{k+1}$, $i = 1, 2, \dots, 2n-1$, $p(t) = 1$, $t_k = 2^k$, $\tau = \frac{1}{2} > 0$, $t_{k+1} - t_k = 2^{k+1} - 2^k = 2^k > \frac{1}{2}$, $t_0 = 1$, $\gamma = 2n-1$. It is obvious that condition (A) is satisfied. As to condition (B), for $i > 1$, $a_k^{(i)} = \frac{k}{k+1}$ we have

$$(t_1 - t_0) + (t_2 - t_1) + (t_3 - t_2) + \cdots + (t_{m+1} - t_m) + \cdots = 1 + 2^1 + 2^2 + \cdots + 2^m + \cdots = +\infty.$$

For $i = 1$, $a_k^{(0)} = 2$, $a_k^{(1)} = \frac{k}{k+1}$ we have

$$\begin{aligned} (t_1 - t_0) &+ \frac{1}{2 \times 2}(t_2 - t_1) + \frac{1}{3 \times 2^2}(t_3 - t_2) + \cdots + \frac{1}{(m+1) \times 2^m}(t_{m+1} - t_m) + \cdots \\ &= 1 + \frac{1}{2 \times 2} 2 + \frac{1}{3 \times 2^2} 2^2 + \cdots + \frac{1}{(m+1) \times 2^m} 2^m + \cdots \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m+1} + \cdots = +\infty. \end{aligned}$$

So, condition (B) holds.

Since $a_k^{(2n-1)} = \frac{k}{k+1}$, we have

$$\begin{aligned} &\int_{t_0}^{t_1} p(t)dt + \frac{1}{a_1^{(2n-1)}} \int_{t_1}^{t_2} p(t)dt + \frac{1}{a_1^{(2n-1)} a_2^{(2n-1)}} \int_{t_2}^{t_3} p(t)dt \\ &\qquad + \cdots + \frac{1}{a_1^{(2n-1)} a_2^{(2n-1)} \cdots a_m^{(2n-1)}} \int_{t_m}^{t_{m+1}} p(t)dt + \cdots \\ &\quad = \int_{t_0}^{t_1} 1dt + 2 \int_{t_1}^{t_2} 1dt + 3 \int_{t_2}^{t_3} 1dt + \cdots + (m+1) \int_{t_m}^{t_{m+1}} 1dt + \cdots \\ &\quad = (t_1 - t_0) + 2(t_2 - t_1) + 3(t_3 - t_2) + \cdots + (m+1)(t_{m+1} - t_m) + \cdots \\ &\quad = 1 + 2 \times 2 + 3 \times 2^2 + \cdots + (m+1) \times 2^m + \cdots = +\infty. \end{aligned}$$

Therefore we get that the conditions of Theorem 1 hold. So we can see that every solution of (20) defined on $[t_0, +\infty)$ is oscillatory.

Example 2. Consider the sublinear system:

$$\begin{cases} x^{(2n)}(t) + \frac{2}{t}x^{\frac{1}{9}}(t-\ln 2) = 0, \quad t \geq t_0 = \frac{1}{2}, \quad t \neq k, \quad k = 1, 2, \dots, \\ x(k^+) = x(k), \quad x^{(i)}(k^+) = \frac{k}{k+1}x^{(i)}(k), \quad i = 1, \dots, 2n-1, \\ x(\dfrac{1}{2}) = x_0, \quad x^{(i)}(\frac{1}{2}^+) = x_0^{(i)}, \\ x(t) = \phi(t), \quad t \in [\frac{1}{2} - \ln 2, \frac{1}{2}], \end{cases} \tag{21}$$

where $a_k^{(0)} = 1$, $a_k^{(i)} = \frac{k}{k+1}$, $i = 1, 2, \dots, 2n-1$, $\tau = \ln 2$, $t_k = k$, $t_{k+1} - t_k = 1 > \ln 2$, $p(t) = \frac{2}{t}$, $\gamma = \frac{1}{9}$, $t_0 = \frac{1}{2}$. It is obvious that condition (A) is satisfied. As to condition (B), for $i > 1$, $a_k^{(i)} = \frac{k}{k+1}$ we have

$$(t_1 - t_0) + (t_2 - t_1) + (t_3 - t_2) + \cdots + (t_{m+1} - t_m) + \cdots = +\infty.$$

For $i = 1$, $a_k^{(0)} = 1$, $a_k^{(1)} = \frac{k}{k+1}$ we obtain

$$(t_1 - t_0) + \frac{1}{2}(t_2 - t_1) + \frac{1}{3}(t_3 - t_2) + \cdots + \frac{1}{m+1}(t_{m+1} - t_m) + \cdots$$
$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m+1} + \cdots = +\infty.$$

So, condition (B) holds.

Let $\alpha = 1$, $a_k^{(0)} = 1$, $\frac{1}{a_k^{(2n-1)}} = \frac{k+1}{k} \geq (\frac{t_{k+1}}{t_k})^\alpha = \frac{k+1}{k}$, $\int^{+\infty} t \cdot p(t)dt = +\infty$. Therefore the conditions of Corollary 2 are satisfied. Then every solution of (21) defined on $[t_0, +\infty)$ is oscillatory.

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