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FINITELY GENERATED MODULES AND CONNECTIVITY

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Abstract. Let R be a commutative Noetherian ring and let M be a finitely generated R-module. Let $X = \mathrm{Spec}_R(M)$ be the topological space with Zariski topology. Our main goal in this paper is to describe the connectedness dimension of X in terms of Krull dimension of some quotient of M and prove that $c(\mathrm{Spec}_R(M)) = c(\mathrm{Supp}(M))$.

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1. Introduction

Throughout this paper R denotes a commutative ring with an identity and the notation "$\subset$" denotes the strict inclusion. Let M be an R-module. Then a proper submodule N of M is said to be prime if for any $r \in R$ and any $m \in M$ with $rm \in N$ we have $m \in N$ or $r \in (N :_R M)$. Further, the spectrum of M is denoted by $\mathrm{Spec}_R(M)$ and defined by

$$\mathrm{Spec}_R(M) = \{P : P \text{ is a prime submodule of } M\}.$$

For any submodule N of M, we consider two different types of varieties denoted, respectively, by $V(N)$ and $V^*(N)$ defined by (see [3, p. 417])

$$V(N) = \{P \in \mathrm{Spec}_R(M) : (P :_R M) \supseteq (N :_R M)\}$$

and $V^*(N) = \{P \in \mathrm{Spec}_R(M) : P \supseteq N\}$. The sets $V(N)$, where N is a submodule of M, satisfy the axioms for closed sets in a topological space on $X = \mathrm{Spec}(M)$ (see [3, p. 419]). The above resulting topology is called the Zariski topology relative to M. Remark 2.4 gives some further information about this topology.

A topological space X is said to be Noetherian if the open subsets of X satisfy the ascending chain condition (or the maximal condition). For example, if R is a Noetherian ring, then $\mathrm{Spec}(R)$ is a Noetherian topological space with Zariski topology (see [1, Ch. 6, Exc. 5]).

Let X be a nonempty Noetherian topological space. The dimension of X, denoted by $\dim(X)$, is defined by

$$\dim(X) = \sup\{n : Z_0 \supset Z_1 \supset \cdots \supset Z_n \text{ is a chain of irreducible closed subsets of } X\}.$$

The dimension of the empty space is defined to be -1. Note that, for a Noetherian ring R, we have $\dim(\mathrm{Spec}(R)) = \dim(R)$ (see [2, 19]). Further, the

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connectedness dimension of X is denoted by $c(X)$ and defined by

$$c(X) = \min\{\dim(Z) : Z \subseteq X,\ Z \text{ is closed and } X \setminus Z \text{ is disconnected}\}.$$

Also, X is connected if and only if $c(X) \geq 0$.

The study of the topological connectivity of algebraic sets is a fundamental subject in algebraic geometry. There are some well-known results concerning the connectivity of $\mathrm{Spec}(R)$ (with Zariski topology), where R is a commutative Noetherian ring (see [2, 19]). Among these results, the study of $c(\mathrm{Spec}(R))$ is more important and it is specified in terms of Krull dimension of some quotient of R (see [2, 19.2.5]).

In [3], C. P. Lu obtained some nice results concerning the connectivity of $\mathrm{Spec}_R(M)$ (with Zariski topology), where M is an R-module.

Now let R be a commutative Noetherian ring, M a finitely generated R-module and $X = \mathrm{Spec}_R(M)$ with Zariski topology. In this paper we will show, as a generalization, that X is a Noetherian topological space and we will describe $c(X)$ in terms of Krull dimension of some quotient of M. By using this, we will show that $c(\mathrm{Spec}_R(M)) = c(\mathrm{Supp}(M))$ and obtain results 3.7, 3.9 and 3.10 concerning the connectivity of X. These results can be regarded as generalizations of similar classical results proved in [2, 19].

2. AUXILIARY RESULTS

Definition 2.1 (see [4])**.** A proper submodule N of M is said to be prime if for any $r \in R$ and any $m \in M$ with $rm \in N$ we have $m \in N$ or $r \in (N :_R M)$.

Remark 2.2 (see [4]). Let M be a finitely generated R-module. Then

(a) Every proper submodule of M is contained in some maximal submodule of M.
(b) If N is a maximal submodule of M, then N is a prime submudule of M and $(N :_R M)$ is a maximal ideal of R.
(c) Let K be a prime submodule of M. Then $(K :_R M)$ is a prime ideal of R.
(d) Let M be a Noetherian R-module. Then for every prime ideal p of R where $p \supseteq \mathrm{Ann}(M)$ there exists a prime submodule P of M such that $(P :_R M) = p$.

Definition 2.3. Let (M, K) be a Noetherian local R-module with unique maximal submodule K. The topological space $\mathrm{Spec}_R(M) \setminus V(K)$ with the topology induced from the Zariski topology on $\mathrm{Spec}_R(M)$ is called the punctured spectrum of M and denoted by $\mathrm{Spec}^{\circ}_R(M)$. When $M = R$ the definition coincides with [19, 1.3].

Remark 2.4. Let M be an R-module and let N and L be submodules of M. Then we have the following.

(a) If $(N :_R M) = (L :_R M)$, then $V(N) = V(L)$. The converse is also true if both N and L are prime (see [3, Section 2, Result 1]).

(b) $V(N) = V((N :_R M)M)$ and $V(IM) = V^*(IM)$ for every ideal I of R (see [3, Section 2, Result 3]). Also,

$$V(0) = \mathrm{Spec}_R(M) \quad \text{and} \quad V(M) = \varnothing,$$
$$V(N) \cup V(L) = V(N \cap L).$$

Further for any index set Λ (see [3, p. 419])

$$\bigcap_{\lambda \in \Lambda} V(N_\lambda) = V\bigg(\sum_{\lambda \in \Lambda} (N_\lambda : M)M\bigg).$$

Remark 2.5.

(a) Let M be an R-module. For each $f \in R$, we define $X_f = X \setminus V(fM)$. Then every X_f is an open set of X. Further, the set $B = \{X_f : f \in R\}$ forms a base for the Zariski topology on X which may be empty and for any $f, g \in R$, $X_f g = X_f \cap X_g$ (see [3, 4.2 and 4.3]).

(b) Let M be a finitely generated R-module. Then for any $f \in R$, X_f is compact (see [3, 4.5]). In particular, $X_1 = \mathrm{Spec}_R(M)$ is compact.

(c) Let M be a finitely generated R-module. Then Y is an irreducible closed subset of $X = \mathrm{Spec}_R(M)$ if and only if $Y = V(P)$ for some $P \in X$ (see [3, 5.7 and 3.5]).

Remark 2.6. Let M and M' be R-modules. Set $X = \mathrm{Spec}_R(M)$, $X' = \mathrm{Spec}_R(M')$, and let $f : M \to M'$ be an epimorphism. Then the map $g : X' \to X$ defined by $P' \mapsto f^{-1}(P')$ is continuous (see [3, 3.9]).

3. Main Results

Throughout this section R will denote a commutative Noetherian ring and for an R-module M, $c(\mathrm{Supp}(M))$ and $c(\mathrm{Spec}(R))$ will be denoted respectively by $c(M)$ and $c(R)$.

Theorem 3.1. *Let M be a finitely generated R-module. Then $\mathrm{Spec}_R(M)$ is a Noetherian topological space.*

Proof. Set $X = \mathrm{Spec}_R(M)$. By [1, Ch.6, Exe. 6], it is enough to show that every open subset of X is compact. To see this, let G be an open subset of X. Since $\{X_r = X \setminus V(rM) : r \in R\}$ forms a base for the Zariski topology on X (see [3, 4.3]), then by 2.4,

$$G = \bigcup_{i \in I} X_{r_i} = X \setminus \bigcap_{i \in I} V(r_i M) = X \setminus V\bigg(\sum_{i \in I} r_i M\bigg) = X \setminus V\bigg(\Big(\sum_{i \in I} (r_i)\Big) M\bigg).$$

Since R is a Noetherian ring, $\sum_{i \in I} (r_i) = \sum_{i=1}^{n} R x_i$, where $x_i \in R$ for $i = 1, 2, \ldots, n$. Hence we have

$$G = X \setminus V\bigg(\sum_{i=1}^{n} x_i M\bigg) = \bigcup_{i=1}^{n} X_{x_i}.$$

Since M is a finitely generated R-module, each X_{x_i}, $i = 1, 2, \ldots, n$, is compact by [3, 4.4]. Hence G is a compact set and the proof is completed. ☐

Lemma 3.2. *Let M be a finitely generated R-module. Set $\bar{R} := R/\operatorname{Ann}_R(M)$ and*

$$A = \{V(pM) : \bar{p} \in \operatorname{Min}(\bar{R})\}.$$

Then A is equal to the set of all irreducible components of $\operatorname{Spec}_R(M)$.

Proof. Let Y be an irreducible component of $\operatorname{Spec}_R(M)$. Then $Y = V(P)$, for some $P \in \operatorname{Spec}_R(M)$ by 2.5(c). Set $p = (P :_R M)$. Then $p \supseteq \operatorname{Ann}_R(M)$ and p is a prime ideal of R by 2.2 (c). Since Y is an irreducible component, it follows that $\bar{p} = p/\operatorname{Ann}_R(M) \in \operatorname{Min}(\bar{R})$ by [3, 5.8]. By 2.4 (b), $V(P) = V((P : M)M) = V(pM)$. Hence $Y \in A$. Conversely, let $Y \in A$. Then there exists $\bar{p} \in \operatorname{Min}(\bar{R})$ such that $Y = V(pM)$. Now $p \supseteq \operatorname{Ann}_R(M)$ and there exists $P \in \operatorname{Spec}_R(M)$ such that $(P :_R M) = p$ by 2.2 (d). Hence $Y = V(pM) = V((P : M)M) = V(P)$. It follows that $V(pM)$ is an irreducible component of $\operatorname{Spec}_R(M)$ by [3, 5.8]. □

Theorem 3.3. *Let M be a finitely generated R-module. Then*

$$\dim(\operatorname{Spec}_R(M)) = \dim_R(M).$$

Proof. Set $\bar{R} := R/\operatorname{Ann}_R(M)$. One can see that $\dim_R(M) = \dim_{\bar{R}}(M)$ and $\dim(\operatorname{Spec}_R(M)) = \dim(\operatorname{Spec}_{\bar{R}}(M))$. Hence we may assume that M is a faithful R-module. Now let $\dim_R(M) = s$. Then there exists a chain

$$p_0 \subset p_1 \subset \cdots \subset p_s,$$

of prime ideals of R such that $p_i \in \operatorname{Supp}_R(M)$. By 2.2.(d), for every $i = 0, 1, \ldots, s$, there exists a prime submodule P_i of M such that $p_i = (P_i : M)$. Hence we have the chain

$$(P_0 :_R M) \subset (P_1 :_R M) \subset \cdots \subset (P_s :_R M).$$

By 2.4, we have

$$V(P_0) \supset V(P_1) \supset \cdots \supset V(P_s).$$

But each $V(P_i)$, $i = 0, 1, \ldots, s$, is an irreducible closed subset of $\operatorname{Spec}_R(M)$ by 2.5 (c). Hence $\dim(\operatorname{Spec}_R(M)) \geq s$. To see the reverse inclusion assume that $\dim(\operatorname{Spec}_R(M)) = t$. Then there is a chain

$$V(P_0) \supset V(P_1) \supset \cdots \supset V(P_t)$$

of irreducible closed subsets of $\operatorname{Spec}_R(M)$. This induces the chain

$$(P_0 :_R M) \subset (P_1 :_R M) \subset \cdots \subset (P_t :_R M)$$

of prime ideals of R such that $(P_i :_R M) \in \operatorname{Supp}_R(M)$ for $i = 0, 1, \ldots, s$ by 2.4 (a). It turns out that $\dim_R(M) \geq t$. □

Lemma 3.4. *Let M be a finitely generated R-module. Then for every submodule N of M, $\operatorname{Spec}_R(M/N)$ is homeomorphic to $V^*(N)$. In particular, $\operatorname{Spec}_R(M/IM)$ is homeomorphic to $V(IM)$ for every ideal I of R.*

Proof. Let N be a submodule of M and let $f : M \to M/N$ be a natural homomorphism. One can easily see that

$$\operatorname{Spec}_R(M/N) = \{P/N : P \in \operatorname{Spec}_R(M) \quad \text{and} \quad P \supseteq N\}.$$

Let $g : \operatorname{Spec}_R(M/N) \to \operatorname{Spec}_R(M)$ be the map defined by $P/N \mapsto f^{-1}(P/N) = P$. Hence g is continuous by 2.6. Also g is one-to-one and

$$\operatorname{Im}(g) = \{P : P \in \operatorname{Spec}_R(M) \quad \text{and} \quad P \supseteq N\} = V^*(N).$$

Further, for any submodule L of M with $L \supseteq N$ we have $f(V(L/N)) = V(L)$. This implies that f^{-1} is a continuous map. Hence $\operatorname{Spec}_R(M/N)$ is homeomorphic to $V^*(N)$. Now by 2.4 (b), $V(IM) = V^*(IM)$ for every ideal I of R. Hence $\operatorname{Spec}_R(M/IM)$ is homeomorphic to $V(IM)$ and the proof is completed. $\square$

Theorem 3.5. *Let M be a finitely generated R-module and let $\bar{R} = R/\operatorname{Ann}_{(} M)$. For a prime ideal p of R with $p \supseteq \operatorname{Ann}(M)$, set $\bar{p} := p/\operatorname{Ann}_R(M)$. Then*

$$c(\operatorname{Spec}_R(M)) = \min\Big\{ \dim \Big(M/\big((\bigcap_{\bar{p}\in C} p) + (\bigcap_{\bar{p}\in D} p)\big) M\Big) : C \cup D = \operatorname{Min}(\bar{R})\Big\}.$$

Proof. By [2, 19.1.15], for a Noetherian topological space T we have

$$c(T) = \min\Big\{ \dim \Big((\bigcup_{i\in A} T_i) \cap (\bigcup_{j\in B} T_j)\Big) : (A,B) \in \phi(n)\Big\}$$

where

$$\phi(n) = \{(A,B) : A \text{ and } B \text{ are nonempty subsets of } \{1,2,\ldots,n\} \\ \text{and } A \cup B = \{1,2,\ldots,n\}\}$$

and $T_1, T_2, \ldots, T_n$ are irreducible components of T. Hence by 3.1 and 3.2, we have

$$c(\operatorname{Spec}_R(M)) = \min\Big\{ \dim \Big((\bigcup_{\bar{p}\in C} V(pM)) \cap (\bigcup_{\bar{p}\in D} V(pM))\Big) : C \cup D = \operatorname{Min}(\bar{R})\Big\}.$$

But by 2.4 (b), $V(N) \cup V(L) = V(N \cap L)$. This implies that

$$c(\operatorname{Spec}_R(M)) = \min\Big\{ \dim \Big(V(\bigcap_{\bar{p}\in C} pM) \cap V(\bigcap_{\bar{p}\in D} pM)\Big) : C \cup D = \operatorname{Min}(\bar{R})\Big\}.$$

Thus by using 2.4 (b),

$$c(\operatorname{Spec}_R(M)) = \min\Big\{ \dim \Big(V\big((\bigcap_{\bar{p}\in C} pM :_R M) M + (\bigcap_{\bar{p}\in D} pM :_R M) M\Big) : \\ C \cup D = \operatorname{Min}(\bar{R})\Big\}.$$

But for a subset H of $\operatorname{Min}(\bar{R})$ we have

$$\Big(\bigcap_{\bar{p}\in H} pM :_R M\Big) = \bigcap_{\bar{p}\in H} p.$$

To see this let $x \in \Big(\bigcap_{\bar{p}\in H} pM :_R M\Big)$. Then for every $\bar{p} \in H$, $xM \subseteq pM$. Thus $x \in \operatorname{Ann}_R(M/pM) \subseteq \sqrt{\operatorname{Ann}_R(M/pM)}$ for every $\bar{p} \in H$. But for every $\bar{p} \in H$, $p \supseteq \operatorname{Ann}_R(M)$ and $\sqrt{\operatorname{Ann}_R(M/pM)} = \sqrt{(\operatorname{Ann}_R(M) + p)}$ by [5, Exc. 2.2]. Hence for every $\bar{p} \in H$, $x \in \sqrt{(\operatorname{Ann}_R(M) + p)} = p$. It follows that $x \in \bigcap_{\bar{p}\in H} p$. The reverse inclusion is clear, since $x \in \bigcap_{\bar{p} \in H} p$ implies that $xM \subseteq pM$ for all $\bar{p} \in H$. This in turn implies that $x \in \Big(\bigcap_{\bar{p} \in H} pM :_R M\Big)$. Hence we have

$$c(\mathrm{Spec}_R(M)) = \min\Big\{ \dim\Big(V\Big(\Big((\bigcap_{\bar{p}\in C} p) + (\bigcap_{\bar{p}\in D} p)\Big)M\Big)\Big) : C \cup D = \mathrm{Min}(\bar{R})\Big\}.$$

But for an ideal I of R, $\dim(V(IM)) = \dim(\mathrm{Spec}_R(M/IM))$ by 3.4. It follows that

$$c(\mathrm{Spec}_R(M)) = \min\Big\{ \dim \mathrm{Spec}\Big(M/\Big((\bigcap_{\bar{p}\in C} p) + (\bigcap_{\bar{p}\in D} p)\Big)M\Big) : \\ C \cup D = \mathrm{Min}(\bar{R})\Big\}.$$

Further $\dim(\mathrm{Spec}_R(M)) = \dim(M)$ by 3.3. Hence

$$c(\mathrm{Spec}_R(M)) = \min\Big\{ \dim(M/\Big((\bigcap_{\bar{p}\in C} p) + (\bigcap_{\bar{p}\in D} p)\Big)M\Big) : C \cup D = \mathrm{Min}(\bar{R})\Big\}. \quad \square$$

Corollary 3.6. *Let M be a finitely generated R-module. Then*

$$c(\mathrm{Spec}_R(M)) = c(M).$$

Proof. Set $\bar{R} = R/\mathrm{Ann}_R(M)$. Since $\mathrm{Supp}(M) = V(\mathrm{Ann}_R(M))$ and $V(\mathrm{Ann}_R(M))$ is homeomorphic to $\mathrm{Spec}(\bar{R})$ by [5, p. 29, Exc. 4.4], it follows that $\mathrm{Supp}(M)$ is homeomorphic to $\mathrm{Spec}(\bar{R})$. Hence we have $c(\bar{R}) = c(\mathrm{Supp}(M)) = c(M)$. Also $c(\mathrm{Spec}_{\bar{R}}(M)) = c(\mathrm{Spec}_R(M))$ by 3.5. So we may assume that M is a faithful R-module. Then with respect to 3.5, we have

$$c(\mathrm{Spec}_R(M)) = \min\Big\{ \dim\Big(M/\Big((\bigcap_{p\in C} p) + (\bigcap_{p\in D} p)\Big)M\Big) : C \cup D = \mathrm{Min}(R)\Big\}.$$

Also by [2, 19.2.5],

$$c(\bar{R}) = c(R) = \min\Big\{ \dim\Big(R/\Big((\bigcap_{p\in C} p) + (\bigcap_{p\in D} p)\Big) : C \cup D = \mathrm{Min}(R)\Big\}.$$

But if I is an ideal of R, then by using [5, Exc. 2.2], we have

$$\begin{aligned}\dim(M/IM) &= \dim(R/\mathrm{Ann}(M/IM)) = \dim(R/\sqrt{(}\mathrm{Ann}(M/IM))) \\ &= \dim(R/\sqrt{(}I + \mathrm{Ann}(M))).\end{aligned}$$

Since M is faithful, it follows that

$$\dim(M/IM) = \dim(R/I).$$

Hence by the above arguments, $c(\mathrm{Spec}_R(M)) = c(\bar{R}) = c(M)$. $\square$

Corollary 3.7. *Let (R, m) be a local ring and let M be a finitely generated R-module. Then*

$$c(\mathrm{Spec}_R(M)) \geq c(\mathrm{Spec}_{\hat{R}}(\hat{M})),$$

where $\hat{M}$ (resp. $\hat{R}$) is the completion of M (resp. R) with respect to the m-adic topology.

Proof. By [2, 19.3.1], for a local ring (R, m) we have $c(R) \geq c(\hat{R})$. As we mentioned in the proof of 3.6,

$$c(\hat{M}) = c(\mathrm{Supp}(\hat{M})) = c(\hat{R}/(\mathrm{Ann}_{\hat{R}} \hat{M})).$$

This implies that $c(M) \geq c(\hat{M})$. Hence by 3.6,

$$c(\mathrm{Spec}_R(M)) \geq c(\mathrm{Spec}_{\hat{R}}(\hat{M})). \qquad \square$$

Example 3.8. Let (M, K) be a Noetherian local R-module with a unique maximal submodule K. Let $V(N)$ and $V(L)$ be two nonempty closed subsets of M. Then we have $K \in V(N) \cap V(L)$ by Remark 2.2 (a). This implies that $\mathrm{Spec}_R(M)$ is connected so that $c(\mathrm{Spec}_R(M)) \geq 0$.

Theorem 3.9. *Let (M, K) be a Noetherian local R-module with a unique maximal submodule K. Then for the Noetherian topological space* $\mathrm{Spec}^{\circ}_R(M)$

$$c(\mathrm{Spec}^{\circ}_R(M)) = c(\mathrm{Spec}_R(M)) - 1.$$

Proof. Let $c(\mathrm{Spec}^{\circ}_R(M)) = n$ and $c(\mathrm{Spec}_R(M)) = t$. Set $T := (\mathrm{Spec}^{\circ}_R(M))$. Then there exists a closed subset Z of T such that $T \setminus Z$ is disconnected and $\dim Z = n$. Hence there exists a chain

$$Z_0 \supset Z_1 \supset \cdots \supset Z_n$$

of irreducible closed subsets of Z. But $Z = T \cap V(N)$, where N is a submodule of M. Then by 2.2 (a), $Z = (\mathrm{Spec}_R(M) \setminus \{K\}) \cap V(N) = V(N) \setminus \{K\}$. Further, for each $i = 1, 2, \ldots, n$, there exists a prime submodule N_i of M such that $Z_i = Z \cap V(N_i)$. Set $E_i = ((N :_R M) + (N_i :_R M))M$. Then $V(E_i) = V(N) \cap V(N_i)$ by 2.4 (b). This implies that

$$Z_i = Z \cap V(N_i) = (V(N) \setminus \{K\}) \cap V(N_i) = (V(N) \cap V(N_i)) \setminus \{K\} = V(E_i) \setminus \{K\}.$$

By the above arguments, we obtain the chain

$$V(E_0) \setminus \{K\} \supset V(E_1) \setminus \{K\} \supset \cdots \supset V(E_n) \setminus \{K\}.$$

It follows that

$$V(E_0) \supset V(E_1) \supset \cdots \supset V(E_n).$$

Now we have

$$\begin{aligned} T \setminus Z = (\mathrm{Spec}^{\circ}_R(M)) \setminus (V(N) \setminus \{K\}) &= (\mathrm{Spec}_R(M) \setminus \{K\}) \setminus (V(N) \setminus \{K\}) \\ &= \mathrm{Spec}_R(M) \setminus V(N). \end{aligned}$$

Hence

$$V(E_0) \supset V(E_1) \supset \cdots \supset V(E_n) \supset V(K)$$

is a chain of irreducible closed subsets of $V(N)$ such that $\mathrm{Spec}_R(M) \setminus V(N)$ is disconnected. This implies that $c(\mathrm{Spec}_R(M)) \geq n + 1$ so that $n \leq t - 1$. Also, a similar argument shows that $t - 1 \leq n$. $\square$

Theorem 3.10. *Let M be a finitely generated R-module. Set $\bar{R} = R/\mathrm{Ann}_R(M)$. Then for each of the following cases we have $c(\mathrm{Spec}_R(M)) = \dim M$.*

(a) $\mathrm{Min}(\bar{R})$ *consists of a single prime $\bar{p}$, where $p \in \mathrm{Spec}(R)$.*

(b) $\mathrm{Min}(\mathrm{Spec}_R(M))$ *consists of a single prime* P.

Proof. (a) Let $\mathrm{Min}(\bar{R}) = \{\bar{p}\}$. By 2.2 (d), there exists a prime submodule P of M such that $(P :_R M) = p$. We show that $\mathrm{Spec}_R(M) = V(P)$. To see this, let $Q \in \mathrm{Spec}_R(M)$. Set $q = (Q :_R M)$. Then $\bar{q} = q/\mathrm{Ann}_R(M) \in \mathrm{Spec}(\bar{R})$ by 2.2 (c). Hence $p \subseteq q$. This implies that $Q \in V(P)$. Hence $\mathrm{Spec}_R(M)$ is an irreducible space by 2.5 (c). Thus $c(\mathrm{Spec}_R(M)) = \dim(\mathrm{Spec}_R(M))$ by [2, 19.1.10]. But by 3.3, $\dim(\mathrm{Spec}_R(M)) = \dim(M)$. It follows that $c(\mathrm{Spec}_R(M)) = \dim M$.

(b) Since $\mathrm{Min}(\mathrm{Spec}_R(M))$ consists of a single prime P, then $\mathrm{Spec}_R(M) = V(P)$ so that $\mathrm{Spec}_R(M)$ is an irreducible space by 2.5. ☐

Example 3.11. Let $M = Z \oplus Z$ and let $\bar{Z} = Z/\mathrm{Ann}_Z(M)$. Then M is a finitely generated faithful Z-module and $\mathrm{Min}(\bar{Z}) = \{0\}$. Now by 3.10, $c(\mathrm{Spec}_Z(M)) = \dim(M) = \dim(\bar{Z}) = \dim(Z) = 1$. This implies that $\mathrm{Spec}_Z(M)$ is a connected topological space.

Question 3.12. Let (M, K) be a Noetherian local R-module with unique maximal submodule K. Is $\mathrm{Spec}^{\circ}_R(M)$ connected?

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