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TRIANGULATED TRACK CATEGORIES

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Abstract. The concept of a triangulated homotopy category can be canonically lifted to the level of a groupoid-enriched category. This way two natural axioms on track triangles replace the four somewhat obscure axioms of a triangulated category.

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Introduction

The axioms of a triangulated category are deduced from the properties of cofiber sequences (or fiber sequences) in a stable homotopy category $\mathbf{A}$. Here $\mathbf{A}$ is an additive category with a translation functor t as, for example, the homotopy category of chain complexes or the stable homotopy category of spectra. Cofibers or fibers, however, are defined in terms of homotopies and, as observed by Gabriel–Zisman [9], represent "cone functors" obtained by tracks. Here tracks are homotopy classes of homotopies. Therefore it is natural to consider the properties of cofiber sequences in a *track category*, i.e., a groupoid enriched category or a 2-category for which all 2-cells are invertible. Gabriel and Zisman deal with general unstable fiber sequences in a track category. In this paper, however, we consider stable fiber-cofiber sequences in a track category and we lift the classical concept of triangulation of a category (as introduced by Puppe and Verdier) to the new concept of triangulation of a track category. A triangulated category is an additive category with a translation functor and a distinguished class of *exact triangles.* Similarly, a triangulated track category is an additive track category with a translation track functor and a distinguished class of *track triangles* satisfying two natural axioms (TTr1) and (TTr2) which replace the rather obscure axioms of a triangulated category. As a main result we show that a triangulated track category always has a homotopy category which is a triangulated category in the classical sense. This way we also obtain a description of good morphisms between exact triangles in the sense of [11]. Further applications are obtained in [3] where we compute an algebraic model for the category of principal maps between 2-stage Postnikov spectra.

In [6] one finds the cohomological interpretation of the results on track categories in this paper.

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1. TRIANGULATED CATEGORIES

To fix the notation, we repeat some well known concepts. Let $\mathbf{A}$ be an additive category. A *translation functor* $t : \mathbf{A} \to \mathbf{A}$ is an additive automorphism of $\mathbf{A}$. Hence

$$t : \operatorname{Hom}_{\mathbf{A}}(X,Y) \cong \operatorname{Hom}_{\mathbf{A}}(tX,tY) \tag{1.1}$$

is an isomorphism of abelian groups and $t(X \oplus Y) = (tX) \oplus (tY)$. The inverse t^{-1} of t admits natural isomorphisms $tt^{-1}X \cong X$ and $t^{-1}tX \cong X$ in $\mathbf{A}$. We say that t is *strict* if these isomorphisms are the identity of X.

Now let $(\mathbf{A}, t)$ be an additive category with a translation functor. A *triangle* in $\mathbf{A}$ is a diagram in $\mathbf{A}$ of the form

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} tA\,. \tag{1.2}$$

Every such triangle yields an infinite sequence of maps $d_n : X_{n-1} \longrightarrow X_n$, $n \in \mathbb{Z}$, in $\mathbf{A}$ together with isomorphisms $\tau_X : X_n \cong tX_{n-3}$ satisfying $(td_{n-3})\tau_X = \tau_X d_n$ such that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & X_{-3} \arrow[r] & X_{-2} \arrow[r, "d_{-1}"] \arrow[d, equal] & X_{-1} \arrow[r, "d_0"] \arrow[d, equal] & X_0 \arrow[r, "d_1"] \arrow[d, equal] & X_1 \arrow[r] \arrow[d, equal] & X_2 \arrow[r] & \cdots \\
 & & A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, "w"] & tA & &
\end{tikzcd} \tag{1}$$

We define X_n for $n < -2$ by $X_n = t^{-1}X_{n+3}$ and for $n > 1$ by $X_n = tX_{n-3}$. Then $tt^{-1} \cong 1$ yields the isomorphism $X_n \cong tX_{n-3}$ for $n \leq 0$. If $X = (X_n, d_n)$ is a cochain complex, that is $d_n d_{n-1} = 0$ for all n, then X is a *candidate triangle* in $\mathbf{A}$, see [11]. A morphism between triangles is a commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & tA \arrow[d, "tf"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & tA'
\end{tikzcd} \tag{2}$$

This is an isomorphism if f, g, h are isomorphisms in $\mathbf{A}$. In the case of candidate triangles such a morphism induces a chain map $\alpha : X \longrightarrow X'$ between the associated cochain complexes with $\alpha_{-2} = f$, $\alpha_{-1} = g$, $\alpha_0 = h$ and $\alpha_n = t\alpha_{n-3}$ for $n \in \mathbb{Z}$. Following [13] and [14], see also [15] and [8], we have the following notion of a triangulated category.

Definition 1.3. Let $\mathbf{A}$ be an additive category with a translation functor t. Then $\mathbf{A}$ is a *triangulated category* if $\mathbf{A}$ is equipped with a distinguished family $\mathcal{E}$ of triangles (called *exact* triangles) which is subject to the following axioms:

(Tr0) Any triangle isomorphic to an exact triangle is exact. For any object A in $\mathbf{A}$ the identity 1_A of A yields the exact triangle

$$A \xrightarrow{1} A \longrightarrow * \longrightarrow tA,$$

where $*$ is the zero object of $\mathbf{A}$.

(Tr1) Any morphism in $A \longrightarrow B$ in $\mathbf{A}$ is part of an exact triangle:

$$\begin{tikzcd} A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, "w"] & tA. \end{tikzcd}$$

Here the axiom (Tr3) below implies that the isomorphism type of C is well defined by u. We also write $C = C_u$ and we call $v = i_u : B \longrightarrow C_u$ the *inclusion* and $w = q_u : C_u \longrightarrow tA$ the *projection*.

(Tr2) If a triangle $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} tA$ is exact, then the triangles $B \xrightarrow{v} C \xrightarrow{w} tA \xrightarrow{-tu} tB$ and $t^{-1}C \xrightarrow{-t^{-1}w} A \xrightarrow{u} B \xrightarrow{v} C$ are exact.

(Tr3) Any commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] & tA \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & tA' \end{tikzcd}$$

where the rows are exact extends to a morphism of triangles as defined in (1.2).

(Tr4) *The octahedral axiom*: For a composite $A \xrightarrow{u} B \xrightarrow{v} C$ let C_u, C_v and C_{vu} be chosen as in (Tr1). Then there exists an exact triangle

$$\begin{tikzcd} C_{vu} \arrow[r, "\bar{u}"] & C_v \arrow[r, "\bar{v}"] & tC_u \arrow[r, "-t\bar{w}"] & tC_{vu} \end{tikzcd} \tag{\#}$$

with the following properties. The maps $\bar{u}$ and $\bar{w}$ are maps for which the diagram

$$\begin{tikzcd} B \arrow[r, "v"] \arrow[d, "i_u"'] & C \arrow[r, equal] \arrow[d, "i_{uv}"] & C \arrow[d, "i_v"] \\ C_u \arrow[r, "\bar{w}"] \arrow[d, "q_u"'] & C_{vu} \arrow[r, "\bar{u}"] \arrow[d, "q_{vu}"] & C_v \arrow[d, "q_v"] \\ tA \arrow[r, equal] & tA \arrow[r, "tu"] & tB \end{tikzcd} \tag{\#\#}$$

commutes and $\bar{v}$ is the composite

$$\begin{tikzcd} \bar{v} : C_v \arrow[r, "q_v"] & tB \arrow[r, "ti_u"] & tC_u\,. \end{tikzcd}$$

Axiom (Tr4) is a reformulation of Verdier's octahedral axiom as described in Weibel [15]. This axiom does not say that all maps $\bar{u}$, $\bar{w}$ satisfying (##) yield an exact triangle (#). It is only required that there exist such maps. Neeman [11], [12] raises the question to characterize maps $\bar{u}$, $\bar{w}$ for which (#) is an exact triangle, cf. Section 9 below.

It is easy to see that (Tr0), (Tr2) and (Tr3) imply that for all objects U in $\mathbf{A}$ the induced sequences of abelian groups

$$\leftarrow \hom_{\mathbf{A}}(X_0, U) \longleftarrow \hom_{\mathbf{A}}(X_1, U) \longleftarrow \hom_{\mathbf{A}}(X_2, U) \leftarrow,$$

$$\rightarrow \hom_{\mathbf{A}}(U, X_0) \longrightarrow \hom_{\mathbf{A}}(U, X_1) \longrightarrow \hom_{\mathbf{A}}(U, X_2) \rightarrow$$

are long exact sequences.

2. Additive Track Categories

A *track category* is a category enriched with groupoids; in particular, for all of its objects X, Y their hom-groupoid $[\![X,Y]\!]$ is given, whose objects are maps $f : X \to Y$ and whose morphisms, denoted $\alpha : f \Rightarrow f'$, are called tracks. Equivalently, a track category is a 2-category whose all 2-cells are invertible. For a track $\alpha : f \Rightarrow f'$ and maps $g : Y \to Y'$, $e : X' \to X$, the resulting composite tracks are denoted by $g\alpha : gf \Rightarrow gf'$ and $\alpha e : fe \Rightarrow f'e$. Moreover, there is a vertical composition of tracks, i.e., a composition of morphisms in the groupoids $[\![X,Y]\!]$; for $\alpha : f \Rightarrow f'$ and $\beta : f' \Rightarrow f''$ it is denoted by $\beta\square\alpha : f \Rightarrow f'$. The inverse of a track α with respect to this composition is denoted by α . The identity track $f \Rightarrow f$ is denoted by $0_f = 0$ and called the trivial track. A track category has the *homotopy category* which is an ordinary category obtained by identifying *homotopic* maps $f \cong f'$, i.e., maps f, f' for which there exists a track $f \Rightarrow f'$. If $\mathbf{A}$ is the homotopy category, then we denote the associated track category by

$$\mathbf{B} = (\mathbf{B}_1 \rightrightarrows \mathbf{B}_0 \to \mathbf{A}). \tag{2.1}$$

Here $\mathbf{B}_0$ is the underlying ordinary category of 1-cells. We have the quotient functor $\mathbf{B}_0 \to \mathbf{A}$ which is the identity on objects and which carries a 1-cell f to its homotopy class $\{f\}$. Moreover, $\mathbf{B}_1$ denotes the category of 2-cells with the same objects as in $\mathbf{B}_0$ but with morphisms from X to Y being tracks $\alpha : f \Rightarrow f'$ with f, $f' : X \to Y$ in $\mathbf{B}_0$, composite of α and β in the diagram

$$\begin{tikzcd} Z & Y \arrow[l, bend right=40, "f"'] \arrow[l, bend left=40, "f'"] \arrow[l, phantom, "\Downarrow\alpha"] & X \arrow[l, bend right=40, "g"'] \arrow[l, bend left=40, "g'"] \arrow[l, phantom, "\Downarrow\beta"] \end{tikzcd}$$

being

$$\alpha\beta = \alpha g'\square f\beta = f'\beta\square\alpha g : fg \Rightarrow fg'. \tag{2.2}$$

Thus there are two functors $\mathbf{B}_1 \to \mathbf{B}_0$ which are identity on objects and which send a morphism $\alpha : f \Rightarrow f'$ to f, resp. f'.

An object $X \times Y$ in $\mathbf{B}$ is a *strong product* if projections $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are given which induce for all objects U in $\mathbf{B}$ an isomorphism of groupoids

$$(p_{1*}, p_{2*}) : [\![U, X \times Y]\!] \to [\![U,X]\!] \times [\![U,Y]\!].$$

If this is an equivalence of groupoids, then $X \times Y$ is a weak product.

We now assume given a track category $\mathbf{B}$ such that its homotopy category is an additive category like $\mathbf{A}$ from section 1,

$$\mathbf{B}_{\simeq} = \mathbf{A}$$

and moreover that $\mathbf{B}$ has a strong zero object, i.e., an object $*$ such that for every object X of $\mathbf{B}$, $[\![X,*]\!]$ and $[\![*,X]\!]$ are trivial groupoids with a single morphism. It then follows that in each $[\![X,Y]\!]$ there is a distinguished map $0_{X,Y}$ obtained by composing the unique maps $X \to *$ and $* \to Y$. The identity track of this map is denoted by $0\ \ $. Note that $0_{X,Y}$ may also admit non-identity self-tracks; one has

$$0_{Y,Z}\beta = 0 \quad = \alpha 0_{X,Y} \tag{2.3}$$

for any $\alpha : f \Rightarrow f', f, f' : Y \to Z, \beta : g \Rightarrow g', g, g' : X' \to Y$.

A secondary analogue of an additive category is an additive track category defined as follows.

Definition 2.4. A track category $\mathbf{B}$ is called *additive* if it has a strong zero object $*$, the homotopy category $\mathbf{A} = \mathbf{B}_{\simeq}$ is additive and moreover $\mathbf{B}$ is a linear track extension

$$D \to \mathbf{B}_1 \rightrightarrows \mathbf{B}_0 \to \mathbf{A}$$

of $\mathbf{A}$ by a biadditive bifunctor

$$D : \mathbf{A}^{op} \times \mathbf{A} \to \mathbf{Ab}.$$

Explicitly, this means the following: a biadditive bifunctor D as above is given together with a system of isomorphisms

$$\sigma_f : D(X,Y) \to \mathrm{Aut}_{[\![X,Y]\!]}(f) \tag{2.5}$$

for each 1-arrow $f : X \to Y$ in $\mathbf{B}$, such that for any $f : X \to Y$, $g : Y \to Z$, $a \in D(X,Y)$, $b \in D(Y,Z)$, $\alpha : f \Rightarrow f'$ one has

$$\begin{aligned} \sigma_{gf}(ga) &= g\sigma_f(a); \\ \sigma_{gf}(bf) &= \sigma_g(b)f; \\ \alpha\Box\sigma_f(a) &= \sigma_{f'}(a)\Box\alpha. \end{aligned}$$

Remark 2.6. Using 2.5 we can identify the bifunctor D via the natural equation

$$D(X,Y) = \mathrm{Aut}(0_{X,Y}).$$

Here $\mathrm{Aut}(0_{X,Y})$ is easily seen to be a well defined bifunctor.

We use a translation functor t as in (1.1) for the definition of the *coefficient bifunctor*

$$\begin{aligned} &D : \mathbf{A}^{op} \times \mathbf{A} \to \mathbf{Ab}, \\ &D(X,Y) = \mathrm{hom}_{\mathbf{A}}(tX,Y) \end{aligned} \tag{2.7}$$

Then t yields a natural isomorphism

$$t_D : D \xrightarrow{\cong} D(t^{op} \times t) = t^*D$$

which on the objects is given up to a sign by the functor t, i.e.,

$$t_D : D(X,Y) = \mathrm{hom}_{\mathbf{A}}(tX,Y) \cong \mathrm{hom}_{\mathbf{A}}(ttX,tY) = (t^*D)(X,Y)$$

is defined by $t_D(\alpha) = -t(\alpha)$.

Definition 2.8. Let $\mathbf{B}$ be an additive track category with homotopy category $\mathbf{A}$ and coefficient bifunctor $D(X,Y) = \mathrm{hom}_{\mathbf{A}}(tX,Y)$ where t is a strict translation functor on $\mathbf{A}$. Then a *strict translation functor for* $\boldsymbol{B}$ is a track functor (i.e., a functor enriched with groupoids) as in the commutative diagram

$$\begin{tikzcd}
D \arrow[r] \arrow[d, "t_D"'] & \mathbf{B}_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "t^{(1)}"] & \mathbf{B}_0 \arrow[r] \arrow[d, "t^{(0)}"] & \mathbf{A} \arrow[d, "t"] \\
D \arrow[r] & \mathbf{B}_1 \arrow[r, shift left] \arrow[r, shift right] & \mathbf{B}_0 \arrow[r] & \mathbf{A}
\end{tikzcd}$$

such that $t^{(0)}$, $t^{(1)}$ are strict automorphisms of categories inducing t and t_D respectively. We also write $t = t^{(0)}$ and $t = t^{(1)}$.

Example 2.9. The category $Ch(\mathbf{A})$ of chain complexes in an additive category $\mathbf{A}$ is a track category with tracks given by homotopy classes of homotopies. In fact, $Ch(\mathbf{A})$ is an additive track category with a strict translation functor given by the shift functor.

Example 2.10. The category of spectra *Spec* as introduced by Puppe [13] or Adams [10] is a track category with tracks defined by homotopy classes of homotopies. Moreover *Spec* is an additive track category with a strict translation functor given by the shift functor.

Remark 2.11. The main results below are proved in the presence of a strict translation functor for $\mathbf{B}$. It is also possible to deal with translation functors for $\mathbf{B}$ which are not strict. Such (nonstrict) translation functors are given by a pseudofunctor $t : \mathbf{B} \to \mathbf{B}$ inducing t on $\mathbf{A}$ and $-t$ on D as in (2.8). In [6], we study a more general case of nonstrict translation functors. It is however more convenient for the reader to consider first the case of strict translation functors as this is done in this paper.

Example 2.12. Let $\mathbf{C}$ be a cofibration category with a zero object given, for example, by the Quillen model category, see [1]. Assume that the suspension functor

$$\Sigma : \mathbf{C}_{cf}/\simeq \to \mathbf{C}_{cf}/\simeq$$

is an equivalence of categories. Here $\mathbf{C}_{cf}$ is a full subcategory of cofibrant and fibrant objects in $\mathbf{C}$. Then $\mathbf{C}_{cf}$ is an additive track category with a (nonstrict) translation functor Σ as in (2.11). The dual result holds for fibration categories. Tracks are defined as in [1](II.5).

3. Cone Functors in Additive Track Categories

In topology one defines the mapping cone C_f of a map $f : A \to B$ by the push out $B \cup_f CA$ of $CA \supset A \to B$. Here CA is the cone of A. Given a map $\alpha : B \to U$ and a homotopy $\hat{\alpha} : \alpha f \simeq 0$, one gets a map $\alpha \cup \hat{\alpha} : C_f \to U$. Therefore the set of homotopy classes of maps $C_f \to U$ denoted by $\mathrm{Cone}_f(U)$ can be described only in terms of tracks which are homotopy classes of homotopies. This leads to the following definition of cone functors in track categories as introduced by Gabriel–Zisman, see Remark 3.7 below.

Let **B** be an additive track category as in (2.4) with homotopy category **A**. Let **Ab** be the category of abelian groups. Given a morphism $f : X \to Y$ in $\mathbf{B}_0$, we define the *cone functor*

$$\mathrm{Cone}_f : \mathbf{A} \longrightarrow \mathbf{Ab} \tag{3.1}$$

and the *dual cone functor*

$$\mathrm{Cone}^f : \mathbf{A}^{op} \longrightarrow \mathbf{Ab}$$

as follows. For an object U in **A** let $\mathrm{cone}_f(U)$ be the set of pairs $(\alpha, \hat{\alpha})$ where $\alpha : Y \to U$ is a map in $\mathbf{B}_0$ and

$$\hat{\alpha} : \alpha f \Rightarrow 0$$

is a track in **B**. Two such pairs $(\alpha, \hat{\alpha})$ and $(\beta, \hat{\beta})$ in $\mathrm{cone}_f(U)$ are equivalent if there exists a track $H : \alpha \Rightarrow \beta$ such that the pasting of tracks in the diagram

$$\begin{tikzcd}
 & \Uparrow\hat{\alpha} & \\
X \arrow[r, "f"] \arrow[rr, bend left=60, "0"] \arrow[rr, bend right=60, "0"'] & Y \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] \arrow[r, phantom, "\Downarrow H"] & U \\
 & \Downarrow\hat{\beta} &
\end{tikzcd}$$

yields a trivial track 0 . Now let

$$\mathrm{Cone}_f(U) = \mathrm{cone}_f(U)/\sim \tag{1}$$

be the set of equivalence classes $\{\alpha, \hat{\alpha}\}$. Given a map $v : U \to V$ in **A** we choose v_0 in $\mathbf{B}_0$ representing v and we define the induced map

$$v_* : \mathrm{Cone}_f(U) \longrightarrow \mathrm{Cone}_f(V)$$

by $v_* : \{\alpha, \hat{\alpha}\} = \{v_0\alpha, v_0\hat{\alpha}\}$. One readily checks that v_* is well defined. Below we show that $\mathrm{Cone}_f(U)$ is an abelian group.

Similarly, let $\mathrm{cone}^f(U)$ be the set of pairs $(\alpha, \hat{\alpha})$ where $\alpha : U \to X$ is a map in $\mathbf{B}_0$ and $\hat{\alpha} : f\alpha \Rightarrow 0$ is a track in **B**. Now two such pairs $(\alpha, \hat{\alpha})$, $(\beta, \hat{\beta})$ are equivalent if there exists a track $G : \alpha \Rightarrow \beta$ such that the pasting of tracks in the diagram

$$\begin{tikzcd}
 & \Uparrow\hat{\alpha} & \\
U \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] \arrow[r, phantom, "\Uparrow G"] \arrow[rr, bend left=60, "0"] \arrow[rr, bend right=60, "0"'] & X \arrow[r, "f"] & Y \\
 & \Downarrow\hat{\beta} &
\end{tikzcd}$$

yields a trivial track 0 . As above, let

$$\mathrm{Cone}^f(U) = \mathrm{cone}^f(U)/\sim \tag{2}$$

be the set of equivalence classes $\{\alpha, \hat{\alpha}\}$. Then a map $v : V \to U$ in $\mathbf{A}$ represented by v_0 in $\mathbf{B}_0$ induces the map

$$v^* : \mathrm{Cone}^f(U) \longrightarrow \mathrm{Cone}^f(V)$$

by $v^*\{\alpha, \hat{\alpha}\} = \{\alpha v_0, \hat{\alpha} v_0\}$.

A track $H : f' \Rightarrow f$ in $\mathbf{B}$ induces the natural isomorphisms

$$\begin{aligned} &H_* : \mathrm{Cone}_f \cong \mathrm{Cone}_{f'}, \quad \textit{resp.} \\ &H_* : \mathrm{Cone}^f \cong \mathrm{Cone}^{f'} \end{aligned} \tag{3}$$

which carries $\{\alpha, \hat{\alpha}\}$ to $\{\alpha, \hat{\alpha} \square \alpha H\}$, resp. to $\{\alpha, \hat{\alpha} \square H\alpha\}$. Hence cone functors depend up to natural isomorphism only on the homotopy class of f.

Lemma 3.2. *Let $f : X \to Y$ be a map in $\boldsymbol{B}_0$ and let U, W be objects in $\boldsymbol{A}$ with direct sum $U \oplus W$ and projections $p_1 : U \oplus W \to U$, $p_2 : U \oplus W \to W$ in $\boldsymbol{A}$. Then*

$$(p_{1*}, p_{2*}) : \mathrm{Cone}_f(U \oplus W) \to \mathrm{Cone}_f(U) \times \mathrm{Cone}_f(W)$$

is a bijection of sets.

Proof. Let $\{\alpha, \hat{\alpha}\} \in \mathrm{Cone}_f(U), \{\beta, \hat{\beta}\} \in \mathrm{Cone}_f(W)$. Then there exists γ together with tracks $Q_1 : p_1\gamma \Rightarrow \alpha$, $Q_2 : p_2\gamma \Rightarrow \beta$, since $U \oplus W$ is a direct sum in $\mathbf{A}$. Moreover, there exists $H : \gamma f \Rightarrow 0$, since $(\alpha, \beta) f = 0$ in $\mathbf{A}$. Now $\hat{\alpha} \square Q_1 f$ and $p_1 H$ differ by $h_1 \in D(X, U)$, while $\hat{\beta} \square Q_2 f$ and $p_2 H$ differ by $H_2 \in D(X, W)$. Therefore we can alter H by $(H_1, H_2) \in D(X, U \oplus W)$ and get $\hat{\gamma}$ such that $(p_1)_*\{\gamma, \hat{\gamma}\} = \{\alpha, \hat{\alpha}\}$ and $(p_2)_*\{\gamma, \hat{\gamma}\} = \{\beta, \hat{\beta}\}$. Hence the map in (6.1) is surjective. Now let $\{\gamma, \hat{\gamma}\}$, $\{\varepsilon, \hat{\varepsilon}\} \in \mathrm{Cone}_f(U \oplus W)$ be given with

$$(p_i)_*\{\gamma, \hat{\gamma}\} = (p_i)_*\{\epsilon, \hat{\epsilon}\} \quad \text{for } i = 1, 2. \tag{1}$$

We have to show $\{\gamma, \hat{\gamma}\} = \{\varepsilon, \hat{\varepsilon}\}$, i.e., there is a track $H : \gamma \Rightarrow \varepsilon$ with $\hat{\varepsilon} \square (Hf) = \hat{\gamma}$. By (1) we obtain tracks $H_i : p_i\gamma \Rightarrow p_i\varepsilon$ with $(p_i\hat{\varepsilon}) \square (H_i f) = p_i\hat{\gamma}$. Hence γ and ε represent the same homotopy class so that there exists a track $G : \gamma \Rightarrow \varepsilon$. Now H_i and $p_i G$ differ by an element D_i with $D_1 \in D(X, U)$, $D_2 \in D(X, W)$. We can alter G by $D = (D_1, D_2) \in D(X, U \oplus W)$ and get H with $p_i H = H_i$. Moreover,

$$p_i(\hat{\varepsilon} \square Hf) = p_i\hat{\gamma}. \tag{2}$$

Now $\hat{\varepsilon} \square Hd$ differs from $\hat{\gamma}$ by an element $A \in D(Y, U \oplus W)$. Then (2) shows $A = 0$. Hence $\hat{\varepsilon} \square Hf = \hat{\gamma}$ is satisfied.

Corollary 3.3. *The sets* $\mathrm{Cone}_f(U)$ *and* $\mathrm{Cone}^f(U)$ *have the structure of abelian groups and the functors* (3.1) *are well defined additive functors*

Proof. Using isomorphism (3.2) we define the addition by the composite

$$\mathrm{Cone}_f(U) \times \mathrm{Cone}_f(U) \cong \mathrm{Cone}_f(U \oplus U) \xrightarrow{(1,1)_*} \mathrm{Cone}_f(U),$$

where $(1, 1) : U \oplus U \to U$ is the folding map. For Cone^f we obtain a result similar to (3.2) and define the addition in $\mathrm{Cone}^f(U)$ in a similar way.

Proposition 3.4. *Let $\boldsymbol{B}$ be an additive category with $\boldsymbol{B}_{\simeq} = \boldsymbol{A}$ and bifunctor $D : \boldsymbol{A}^{op} \times \boldsymbol{A} \to \boldsymbol{Ab}$. Then the cone functors are embedded in natural exact sequences in $\boldsymbol{Ab}$*

$$D(Y,U) \xrightarrow{f^*} D(X,U) \xrightarrow{i} \mathrm{Cone}_f(U) \xrightarrow{q} \hom_{\boldsymbol{A}}(Y,U) \xrightarrow{f^*} \hom_{\boldsymbol{A}}(X,U),$$

$$D(U,X) \xrightarrow{f_*} D(U,Y) \xrightarrow{\bar{i}} \mathrm{Cone}^f(U) \xrightarrow{\bar{q}} \hom_{\boldsymbol{A}}(U,X) \xrightarrow{f_*} \hom_{\boldsymbol{A}}(U,Y),$$

where $f : X \to Y$ is a map in $\boldsymbol{B}_0$. Moreover, i, q and $\bar{i}, \bar{q}$ are compatible with H_* in (3.1)(3), that is, $H_* i = i$, $qH_* = q$.

Proof. We define $q\{\alpha, \hat{\alpha}\} = \{\alpha\}$. Then $f^* q = 0$, since $\hat{\alpha} : \alpha f \Rightarrow 0$. On the other hand, any element $\{\alpha\} \in \hom_{\mathbf{A}}(Y,U)$ with $f^*\{\alpha\} = 0$ admits a track $\hat{\alpha} : \alpha f \Rightarrow 0$ representing an element $\{\alpha, \hat{\alpha}\} \in \mathrm{Cone}_f(U)$ with $q\{\alpha, \hat{\alpha}\} = \{\alpha\}$. We define $i(\delta)$ by the track $\sigma\delta : 0 \Rightarrow 0$ in $\mathbf{B}_1$, see (2.5), so that we get the element $i(\delta) = (0, \sigma\delta)$ by the diagram

$$\begin{tikzcd}
X \arrow[r, "f"'] \arrow[rr, bend left=20, "0"] \arrow[rr, bend left=60, "0"] \arrow[rr, phantom, bend left=42, "\Uparrow\sigma\delta"] & Y \arrow[r, "0"'] & U
\end{tikzcd}$$

with $0f = 0$, since $*$ is a strict zero object of $\mathbf{B}$. Clearly, $iq = 0$. Moreover, assume $q\{\alpha, \hat{\alpha}\} = 0$. Then there exists a track $A : 0 \Rightarrow \alpha$ and $\{\alpha, \hat{\alpha}\} = \{0, \hat{\alpha} \square Af\}$. Moreover, by (2.5) there is δ with $\hat{\alpha} \square Af : 0 \Rightarrow 0$ being $\sigma(\delta)$. Next assume that $\delta = f^*\delta'$ with $\delta' \in D(Y,U)$. Then $\{0, \sigma\delta\} = \{0, 0\ \}$ is the trivial element, since $\sigma\delta$ coincides with

$$\begin{tikzcd}
X \arrow[r] \arrow[rr, bend left=60, ""] \arrow[rr, phantom, bend left=45, "\Uparrow 0"] & Y \arrow[r, "0"'] \arrow[r, bend left=50, "0"] \arrow[r, phantom, bend left=20, "\Uparrow\sigma\delta'"] & U
\end{tikzcd}$$

This shows $if^* = 0$. Moreover, let δ be given with $i\delta = \{0, \sigma\delta\} = 0$. Then there exists a track $A : 0 \Rightarrow 0$ with $\sigma\delta = Af$. Here $A = \sigma\delta'$ by (2.5). Hence $\delta = f^*\delta'$.

In a similar way one proves the exactness of the second sequence in (3.4) with

$$\begin{aligned} \bar{q}\{\alpha, \hat{\alpha}\} &= \{\alpha\} \quad \text{for } \{\alpha, \hat{\alpha}\} \in \mathrm{Cone}^f(U), \\ \bar{i}\delta &= \{0, \sigma\delta\} \quad \text{for } \delta \in D(U,Y). \end{aligned}$$

Corollary 3.5. *Let $\boldsymbol{B}$ be an additive track category with $\boldsymbol{B}_{\simeq} = \boldsymbol{A}$ and with a bifunctor $D(X,Y) = \hom_{\boldsymbol{A}}(tX, Y)$ as in (2.7). Let $f : X \to Y$ be a map in $\boldsymbol{A}$ and let $f(n) : t^n X \to t^n Y$ be a map in $\boldsymbol{B}_0$ representing the homotopy class $t^n\{f\}$. Then one has long exact sequences in $\boldsymbol{Ab}$ as follows, $n \in \mathbb{Z}$:*

$$\begin{aligned} &\hom_{\boldsymbol{A}}(t^{n+1}Y, U) \xrightarrow{f^*} \hom_{\boldsymbol{A}}(t^{n+1}X, U) \xrightarrow{i} \mathrm{Cone}_{f(n)}(U) \\ &\quad \xrightarrow{q} \hom_{\boldsymbol{A}}(t^n Y, U) \xrightarrow{f^*} \hom_{\boldsymbol{A}}(t^n X, U), \end{aligned}$$

$$\begin{aligned} &\mathrm{hom}_{\boldsymbol{A}}(U, t^{n+1}X) \xrightarrow{f^*} \mathrm{hom}_{\boldsymbol{A}}(U, t^{n+1}Y) \xrightarrow{\bar{i}} \mathrm{Cone}^{f(n)}(U) \\ &\xrightarrow{\bar{q}} \mathrm{hom}_{\boldsymbol{A}}(U, t^nX) \xrightarrow{f_*} \mathrm{hom}_{\boldsymbol{A}}(U, t^nY). \end{aligned}$$

Proof.

$$D(X,Y) = \mathrm{hom}_{\mathbf{A}}(tX, Y) \cong \mathrm{hom}_{\mathbf{A}}(X, t^{-1}Y)$$

and we can apply (3.4).

Proposition 3.6. *Let $\boldsymbol{B}$ be an additive track category with $\boldsymbol{B}_{\simeq} = \boldsymbol{A}$ and with a bifunctor $D(X,Y) = \mathrm{hom}_{\boldsymbol{A}}(tX, Y)$ as in* (2.7)*. Let*

$$\begin{tikzcd} A \arrow[r, "u"] & B \arrow[r, "v"] & C \end{tikzcd}$$

be maps in $\boldsymbol{B}_0$ and let be X an object in $\boldsymbol{A}$. Then we have a long exact sequence in $\boldsymbol{Ab}$.

$$\mathrm{Cone}_{vu}(t^{-1}X) \xrightarrow{v^*} \mathrm{Cone}_u(t^{-1}X) \xrightarrow{\delta} \mathrm{Cone}_v(X) \xrightarrow{u^*} \mathrm{Cone}_{vu}(X) \xrightarrow{v^*} \mathrm{Cone}_u(X)$$

Here we set $v^*\{\alpha, \hat{\alpha}\} = \{\alpha v, \hat{\alpha}\}$ and $u^*\{\beta, \hat{\beta}\} = \{\beta, \hat{\beta}u\}$. Moreover, δ is defined by $\delta\{\gamma, \hat{\gamma}\} = it\{\gamma\}$ where $\gamma : B \to t^{-1}X$ and $t\{\gamma\} \in \mathrm{hom}_{\boldsymbol{A}}(tB, X) = D(B, X)$ and i is given as in (3.4)*.*

We point out that the exact sequence (3.6) is an analogue of the exact triangle (#) in the octahedral axiom (1.3). The operators v^*, δ, u^* in (3.6), however, are canonically given, while in (1.3)(#) the maps $\bar{u}, \bar{w}$ are certain choices.

Proof of (3.6). For $\{\beta, \hat{\beta}\} \in \mathrm{Cone}_v(X)$ with $\hat{\beta} : \beta v \Rightarrow 0$ we get $\hat{\beta}u : \beta vu \Rightarrow 0$ so that $v^*u^*\{\beta, \hat{\beta}\} = \{\beta vu, \hat{\beta}u\} = 0$ in $\mathrm{Cone}_u(X)$.

Now let $\{\alpha, \hat{\alpha}\} \in \mathrm{Cone}_{vu}(X)$ with $\hat{\alpha} : \alpha vu \Rightarrow 0$ and assume $u^*\{\alpha, \hat{\alpha}\} = \{\alpha v, \hat{\alpha}\} = 0$ in $\mathrm{Cone}_u(X)$. Then there exists $z : \alpha v \Rightarrow 0$ with $zu = \hat{\alpha}$. Hence $\{\alpha, \hat{\alpha}\} = \{\alpha, zu\} = u^*\{\alpha, z\}$. Hence we proved that $im(u^*) = ker(v^*)$.

Next let $\{\gamma, \hat{\gamma}\} \in \mathrm{Cone}_u(t^{-1}X)$ with $\hat{\gamma} : \gamma u \Rightarrow 0$. Let $z : 0 \Rightarrow 0$ be given by $\sigma t\{\gamma\}$ where $0 : B \to * \to X$. Then $u^*\delta\{\gamma, \hat{\gamma}\} = u^*\{0, z\} = \{0, zu\} = \{0, \sigma(t\{\gamma u\})\} = 0$ since $\gamma u = 0$ in $\mathbf{A}$ by the track $\hat{\gamma}$. Now let $\{\beta, \hat{\beta}\} \in \mathrm{Cone}_v(X)$ with $u^*\{\beta, \hat{\beta}\} = \{\beta, \hat{\beta}u\} = 0$ in $\mathrm{Cone}_{vu}(X)$. Then there exists $z : \beta \Rightarrow 0$ with $\hat{\beta}u = zvu$. Hence we get an element

$$\sigma(y) = \hat{\beta}\Box(zv) \quad : 0 \Rightarrow 0$$

with $y \in D(B, X)$ and $u^*y = 0$. Thus we can find an element $\bar{y} \in \mathrm{Cone}_u(t^{-1}X)$ with $\delta\bar{y} = \{\beta, \hat{\beta}\}$. This completes the proof that $im(\delta) = ker(u^*)$.

Finally, let $\{\rho, \hat{\rho}\} \in \mathrm{Cone}_{vu}(t^{-1}X)$ with $\hat{\rho} : \rho vu \Rightarrow 0$ and $\rho : C \to t^{-1}X$. Then $\delta v^*\{\rho, \hat{\rho}\} = \delta\{\rho v, \hat{\rho}\} = it\{\rho v\} = \{0, \sigma(t(\rho))v\} = 0$ in $\mathrm{Cone}_v(X)$.

Moreover let $\{\gamma, \hat{\gamma}\} \in \mathrm{Cone}_u(t^{-1}X)$ with $\hat{\gamma} : \gamma u \Rightarrow 0$ and $\delta\{\gamma, \hat{\gamma}\} = 0$ in $\mathrm{Cone}_v(X)$. Then there exists $z \in D(C, X)$ such that in the diagram

$$\begin{tikzcd}
X & & C \arrow[ll, "0"'] \arrow[ll, bend left=50, "0"] \arrow[ll, phantom, bend left=25, "\Downarrow\sigma z"] & & B \arrow[ll, "v"'] \arrow[llll, bend right=50, "0"'] \arrow[llll, phantom, bend right=30, "\Uparrow\sigma t\{\gamma\}"]
\end{tikzcd}$$

we have $\sigma(z)v = \sigma t\{\gamma\}$. Let $\rho : C \to t^{-1}X$ be a map in $\mathbf{B}_0$ which represents $t^{-1}z$. Then there exists $\hat{\rho} : \rho v u \Rightarrow 0$ since $\gamma u = 0$ in $\mathbf{A}$. This completes the proof that $im(v^*) = ker(\delta)$.

Remark 3.7. Gabriel and Zisman consider cone functors Cone^f in a track category $\mathbf{B}$ with zero object $*$ as follows. Given an object U and a map $f : X \longrightarrow Y$ in $\mathbf{B}$, one gets the morphism

$$f^U : \llbracket U, X\rrbracket \longrightarrow \llbracket U, Y\rrbracket$$

between the pointed groupoids induced by f. Then the groupoid $\Gamma(f^U)$ is defined by the objects $(\alpha, \hat{\alpha})$ and by the tracks $G : (\beta, \hat{\beta}) \Longrightarrow (\alpha, \hat{\alpha})$ satisfying condition in (3.1)(2). Hence

$$\mathrm{Cone}^f(U) = \pi_0(\Gamma(f^U))$$

is the set of path components of $\Gamma(f^U)$. Here the groupoid $\Gamma(f^U)$ coincides with the construction V.3.1 [9] and the axiom C in V.3.2 [9] implies that Cone^f is representable as in (4.1) below. Gabriel and Zisman use the axiom C for the proof of an “exact fiber sequence”, see V.4.2 [9], which in the context of this paper is easily achieved by the dual of (3.5) and (4.3).

4. Representable Cone Functors

Let $\mathbf{B}$ be an additive track category with homotopy category $\mathbf{A}$ and with bifunctor $D(X,Y) = \mathrm{hom}_{\mathbf{A}}(tX, Y)$ as in (2.7). An additive functor $F : \mathbf{A} \to \mathbf{Ab}$ is *representable* if there exists an object X in $\mathbf{A}$ and a natural isomorphism of functors

$$\chi : F \cong \mathrm{hom}_{\mathbf{A}}(X, -) \tag{4.1}$$

in $\mathbf{Ab}$. Then X is well defined up to an isomorphism in $\mathbf{A}$ and called a *representation* of F. We now consider the representations of cone functors.

If $C = C_f$ represents Cone_f with $f : A \to B$ in $\mathbf{B}_0$, then we obtain the triangle

$$A \xrightarrow{f} B \xrightarrow{i_f} C_f \xrightarrow{q_f} tA \tag{4.2}$$

in $\mathbf{A}$ as follows. For $U = C_f$ we have the operator

$$\mathrm{hom}_{\mathbf{A}}(C_f, C_f) \xrightarrow{\chi_f^{-1}} \mathrm{Cone}_f(C_f) \xrightarrow{q} \mathrm{hom}_{\mathbf{A}}(B, C_f)$$

by (3.4) which carries the identity of C_f to i_f. Moreover for $U = tA$ we have the operator

$$\hom_{\mathbf{A}}(tA, tA) \xrightarrow{i} \mathrm{Cone}_f(tA) \xrightarrow{\chi_f} \hom_{\mathbf{A}}(C_f, tA)$$

by (3.4) which carries the identity of tA to q_f.

Of course i_f and q_f depend on the representation (C_f, χ_f) of Cone_f. Using the exact sequence (3.4) we get by a representation the following diagram in **Ab**

$$\begin{tikzcd}
\hom_{\mathbf{A}}(tB,U) \arrow[r, "f^*"] \arrow[d, equal] & \hom_{\mathbf{A}}(tA,U) \arrow[r, "i"] \arrow[d, equal] & \mathrm{Cone}_f(U) \arrow[r, "q"] \arrow[d, "\chi_f", "\cong"'] & \hom_{\mathbf{A}}(B,U) \arrow[r, "f^*"] \arrow[d, equal] & \hom_{\mathbf{A}}(A,U) \arrow[d, equal] \\
\hom_{\mathbf{A}}(tB,U) \arrow[r, "f^*"] & \hom_{\mathbf{A}}(tA,U) \arrow[r, "q_f^*"] & \hom_{\mathbf{A}}(C_f,U) \arrow[r, "i_f{}^*"] & \hom_{\mathbf{A}}(B,U) \arrow[r, "f^*"] & \hom_{\mathbf{A}}(A,U)
\end{tikzcd} \tag{4.3}$$

Here we have $i_f^*\chi_f = q$, since

$$\begin{aligned} q\chi_f^{-1}(\alpha) &= q\chi_f^{-1}(\alpha 1_{C_f}) = \alpha_* q\chi_f^{-1}(1_{C_f}) \\ &= \alpha_* i_f = i_f^*(\alpha). \end{aligned}$$

Similarly, we get $\chi_f i = q_f^*$ by

$$\begin{aligned} \chi_f i(\beta) &= \chi_f i(\beta 1_{t_A}) = \beta_* \chi_f i(1_{tA}) \\ &= \beta_* q_f = q_f^* \beta. \end{aligned}$$

Hence diagram (4.3) commutes and therefore the bottom of (4.3) is an exact sequence, cf. also (1.5).

Proposition 4.4. *If* Cone_f *is representable by* (C_f, χ_f) *then there exists a diagram in* $\boldsymbol{B}$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[rr, bend left=50, "0"] \arrow[rr, phantom, bend left=25, "\Uparrow\hat{i}_f"] & B \arrow[r, "i_f"] \arrow[rr, bend right=50, "0"'] \arrow[rr, phantom, bend right=25, "\Downarrow\hat{q}_f"] & C_f \arrow[r, "q_f"] & tA
\end{tikzcd}$$

with the properties

$$\chi_f\{i_f, \hat{i}_f\} = 1_{C_f},$$

$$\sigma^{-1}(q_f\hat{i}_f\Box(\hat{q}_f\,)f) = 1_{tA} \in D(A, tA).$$

Proof. We have

$$i(1_{tA}) = \chi_f^{-1}q^*(1_{tA}) = \chi_f^{-1}(q_f 1_{C_f}) = q_{f*}\{i_f, \hat{i}_f\}.$$

Hence by definition of i and the equivalence relations in Cone_f there exists $\hat{q}_f$ such that the following diagram represents a trivial track:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[rrr, bend left=40, "0"] \arrow[drr, bend right=40] & B \arrow[rr, "0"] \arrow[dr, "i_f"] \arrow[r, phantom, "\Uparrow \sigma 1_{tA}" above=12pt] \arrow[d, phantom, "\Downarrow \hat{i}_f"] & & tA \\
& & C_f \arrow[ur, "q_f"'] \arrow[u, phantom, "\Uparrow \hat{q}_f"] &
\end{tikzcd}$$

This yields the second formula in (4.4).

The converse of (4.4) is also true. We get the following criterion for the representability of cone functors.

Theorem 4.5. *Let $f : A \to B$ be a map in $\boldsymbol{B}_0$. Then* Cone_f *is representable if and only if there exists a diagram in $\boldsymbol{B}$*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[rr, bend left=50, "0"] & B \arrow[r, "i_f"] \arrow[rr, bend right=50, "0"'] \arrow[u, phantom, "\Uparrow \hat{i}"] & C_f \arrow[r, "q_f"] \arrow[d, phantom, "\Uparrow \hat{q}"] & tA
\end{tikzcd}$$

which by pasting yields $\varepsilon 1_{tA} \in D(A, tA)$, with $\varepsilon = 1$ or $\varepsilon = -1$ and for which the bottom row in (4.3) is exact for all U.

Proof. If Cone_f is representable, then a diagram as in (4.5) exists by (4.4). Conversely, given a diagram as in (4.5), we define the function

$$\chi_f^{-1} : \hom_{\mathbf{A}}(C_f, U) \to \mathrm{Cone}_f(U)$$

which carries α to $\alpha_*\{i_f, \hat{i}\}$. Now we obtain a diagram as in (4.3) satisfying $i_f^*\chi_f = q$. Moreover, $\chi_f^{-1} q_f^* = \varepsilon i$ since the diagram in (4.5) represents $\varepsilon 1_{tA}$, cf. the proof of (4.4). Hence diagram (4.3) commutes up to the sign and therefore χ_f is a bijection by the five lemma.

5. Toda Complexes

As in (2.4), let $\mathbf{B}$ be an additive track category with homotopy category $\mathbf{A}$ and bifunctor D. A *Toda pair* (H, G) is a diagram in $\mathbf{B}$ of the form

$$\begin{tikzcd}
Z \arrow[r, "\gamma"] \arrow[rr, bend left=50, "0"] & Y \arrow[r, "\beta"] \arrow[rr, bend right=50, "0"'] \arrow[u, phantom, "\Uparrow H"] & X \arrow[r, "\alpha"] \arrow[d, phantom, "\Downarrow G"] & W
\end{tikzcd} \tag{5.1}$$

We say that (H,G) is *associated* to the triple (a,b,c) where $a=\{\alpha\}$, $b=\{\beta\}$ and $c=\{\gamma\}$ are homotopy classes in $\mathbf{A}$. Moreover, we say that (H,G) represents $\xi \in D(Z,W)$ or $[H|G] \in \mathrm{Aut}_{[\![z,w]\!]}(\alpha\beta\gamma)$ if the equation

$$\sigma(\xi)=[H|G]=\alpha H\square G\ \ \gamma \tag{1}$$

holds where we use σ in (2.4). We point out that the convention in (1) uses the positive direction of H to define ξ. Given a triple (a,b,c) in $\mathbf{A}$ with $ab=0$ and $bc=0$, the *Toda bracket*

$$\langle a,b,c\rangle \subset D(Z,W) \tag{2}$$

consists of all elements ξ represented by Toda pairs associated to (a,b,c). This is a coset of

$$\gamma^* D(Y,W)+\alpha_* D(Z,X)\subset D(Z,W).$$

We now define a *Toda complex* by a sequence of maps $d=d_n : X_{n-1}\to X_n$, $n\in\mathbb{Z}$, in $\mathbf{B}_0$ together with a sequence of tracks H_n as in the diagram

(5.2)

$$\begin{tikzcd}
X_{n-3} \arrow[r, "d_{n-2}"'] \arrow[rr, bend left=40, "0"] \arrow[rr, bend left=20, phantom, "\Uparrow H_{n-1}"] & X_{n-2} \arrow[r, "d_{n-1}"'] \arrow[rr, bend right=40, "0"'] \arrow[rr, bend right=20, phantom, "\Downarrow H_n"] & X_{n-1} \arrow[r, "d_n"'] & X_n
\end{tikzcd}$$

Hence (H_{n-1},H_n) is a Toda pair representing $\alpha_n\in D(X_{n-3},X_n)$ for $n\in\mathbb{Z}$.

Let $X=(X_n,d_n,H_n,n\in\mathbb{Z})$ and $X'=(X'_n,d'_n,H'_n,n\in\mathbb{Z})$ be Toda complexes. A *morphism between Toda complexes* $f:X\to Y$ is a sequence $f=(f_n,F_n,n\in\mathbb{Z})$ of maps f_n in $\mathbf{B}_0$ and tracks F_n in $\mathbf{B}_1$ as in the diagram

(5.3)

$$\begin{tikzcd}
X_{n-2} \arrow[r] \arrow[rr, bend left=40, "0"] \arrow[rr, bend left=20, phantom, "\Uparrow H_n"] \arrow[d, "f_{n-2}"] \arrow[dr, phantom, "\Downarrow F_{n-1}"] & X_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] \arrow[dr, phantom, "\Downarrow F_n"] & X_n \arrow[d, "f_n"] \\
X'_{n-2} \arrow[r] \arrow[rr, bend right=40, "0"'] \arrow[rr, bend right=20, phantom, "\Downarrow H'_n"] & X'_{n-1} \arrow[r] & X'_n
\end{tikzcd}$$

such that this diagram represents a trivial track for all n, i.e.,

$$f_n H_n = H'_n f_{n-2}\square d'_n F_{n-1}\square F_n d_{n-1}.$$

Let $\mathbf{Tod}$ be the category of Toda complexes and morphisms as in (5.3). We have the *shift functor*

$$sh:\mathbf{Tod}\to\mathbf{Tod} \tag{5.4}$$

which is a strict isomorphism of categories defined as follows. For $X = (X_n, d_n, H_n, n \in \mathbb{Z})$ in **Tod** let $sh(X) = (X'_n, d'_n, H'_n, n \in \mathbb{Z})$ be given by $X'_n = X_{n+1}, d'_n = d_{n+1}, H'_n = H_{n+1}$.

We also define the category of rows denoted by **row** as follows. A row in $\mathbf{B}_0$ is a sequence of maps $(X_n, d_n : X_{n-1} \to X_n, n \in \mathbb{Z})$. A map $f : (X_n, d_n) \to (X'_n, d'_n)$ between the rows is a sequence $f = (f_n, F_n, n \in \mathbb{Z})$ of diagrams in $\mathbf{B}$

$$\begin{tikzcd}
X_{n-1} \arrow[r, "d_n"] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow F_n"] & X_n \arrow[d, "f_n"] \\
X'_{n-1} \arrow[r, "d'_n"'] & X'_n
\end{tikzcd} \tag{5.5}$$

Two such maps $f, f' : X \to X'$ are *homotopic* if there exist tracks $G_n : f_n \Rightarrow f'_n$ such that for $n \in \mathbb{Z}$

$$(d'_n G_{n-1}) \square F_n = F'_n \square (G_n d_n).$$

A *weak equivalence* in **row** is a map f for which all f_n are isomorphisms in the homotopy category $\mathbf{A}$.

We have the shift functor also on the category **row** and the faithful forgetful functor

$$\mathbf{Tod} \longrightarrow \mathbf{row} \tag{5.6}$$

carries $X = (X_n, d_n, H_n)$ to (X_n, d_n). Then we say that the Toda complex X *extends* the row (X_n, d_n).

6. Triangulated Track Categories

Let $\mathbf{B}$ be an additive track category with a strict translation functor $t : \mathbf{B} \to \mathbf{B}$ as in (2.8). Let $\mathbf{A}$ be the homotopy category of $\mathbf{B}$ and $t : \mathbf{A} \to \mathbf{A}$ the induced translation functor on $\mathbf{A}$.

A *track triangle* is a diagram in $\mathbf{B}$

$$\begin{tikzcd}
 & \Uparrow H_0 & & \Uparrow H_2 & \\
A \arrow[r, "f"'] \arrow[rr, bend left=50, "0"] & B \arrow[r, "i_f"'] \arrow[rr, bend right=50, "0"'] & C \arrow[r, "q_f"'] \arrow[rr, bend left=50, "0"] & tA \arrow[r, "tf"'] & tB \\
 & & \Downarrow H_1 & &
\end{tikzcd} \tag{6.1}$$

for which the Toda pairs (H_0, H_1), (H_1, H_2) and (H_2, tH_0) represent $1_{tA}, -1_{tB}$ and 1_{tC}, respectively, see (5.1). As in (1.2)(1), a track triangle can be equivalently described as follows, cf. the notation in Section 5.

Definition 6.2. A *track triangle* X is a Toda complex in $\mathbf{B}$ satisfying the following properties:

$$sh^3(X) = tX, \tag{1}$$

i.e., for $X = (X_n, d_n, H_n, n \in \mathbb{Z})$ we have $X_{n+3} = tX_n, d_{n+3} = td_n, H_{n+3} = tH_n$. Moreover, the Toda pairs (H_{n-1}, H_n) represent for $n \in \mathbb{Z}$ the element

$$\begin{aligned}\sigma[H_{n-1}|H_n] = (-1)^{n+1}1_{X_n} \in D(X_{n-3}, X_n) &= \hom_{\mathbf{A}}(tX_{n-3}, X_n) \\ &= \hom_{\mathbf{A}}(X_n, X_n) \qquad (2)\end{aligned}$$

given by the identity 1_{X_n} of X_n. A *map* between the track triangles X, X' is a map $f : X \to X'$ between Toda complexes such that

$$sh^3(f) = tf, \qquad (3)$$

i.e., for $f = (f_n, F_n, n \in \mathbb{Z})$ we have $f_{n+3} = tf_n,\ F_{n+3} = tF_n$.

Remark 6.3. If **B** has a (nonstrict) translation functor, we define a track triangle X as in (6.1), but we replace the track tH_0 by the obvious track $(ti_f)(tf) \Rightarrow 0$ given by tH_0 and the pseudofunctor track $t_{i_f,f}$. Similarly we define maps between such track triangles. For details see [6]. It is easy to generalize the results and proofs in this paper to the nonstrict case.

The definition of a track triangle in **B** is motivated by the representability result for cone functors in (4.5) which implies:

Proposition 6.4. *Given a track triangle X, one gets, for $n \in \mathbb{Z}$, representations of cone functors*

$$\hom_{\boldsymbol{A}}(X_n, U) \cong \mathrm{Cone}_{d_{n-1}}(U)$$

which carry α to $\alpha_*\{d_n, H_n\}$. Moreover, the exact sequences (4.3) *for these representations coincide up to the sign with the long exact sequence*

$$\cdots \to \hom_{\boldsymbol{A}}(X_{n+1}, U) \xrightarrow{d^*} \hom_{\boldsymbol{A}}(X_n, U) \xrightarrow{d^*} \hom_{\boldsymbol{A}}(X_{n-1}, U)$$

induced by X with $n \in \mathbb{Z}$.

Proof. Using (4.5) we only have to show that the sequence is exact. We do this for $n = 1$ so that by convention (1.2)(1) we have $X_1 = tA$. The track H_2 implies that $d^*d^* = 0$. Hence it remains to show $ker(d^*) \subset image(d^*)$. Let $\alpha : tA \to U$ be a map with $G : \alpha q_f \Rightarrow 0$. We then construct a map $\beta : tB \to U$ in **A** with $\beta(tf) = \pm t\alpha$. Let β be given by the Toda pair (H_1, G), i.e., $\sigma(\beta) = [H_1|G]$ as in (5.1). Then we get

$$\sigma((tf)^*\beta) = f^*[H_1|G] = \pm\alpha_*[H_0|H_1] = \pm \quad \alpha_*\sigma(1_{tA}) = \sigma(\pm\alpha).$$

This completes the proof.

Definition 6.5. Let **B** be an additive track category with a (strict) translation functor $t : \mathbf{B} \to \mathbf{B}$ as in (2.8) or (2.11). Then **B** is a *triangulated track category* if **B** is equipped with a distinguished family of track triangles which is subject to the following axioms:

(TTr1) For each morphism $f : A \to B$ in $\mathbf{B}_0$ there exists a distinguished track triangle X extending f, i.e., $d_{-1} = f$.

(TTr2) Let X and X' be distinguished track triangles extending $f : A \to B$ and $f' : A' \to B'$, respectively, and let

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow H"] & B \arrow[d, "b"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$$

be a diagram in $\mathbf{B}$. Then there exists a morphism $h = (h_n, H_n, n \in \mathbb{Z}) : X \to X'$ between the track triangles extending (a, b, H), i.e., $h_{-2} = a$, $h_{-1} = b$, $H_{-1} = H$.

The category $\mathbf{Ch}(\mathbf{A})$ of chain complexes in (2.9) and the category **Spec** in (2.10) are examples of triangulated track categories with a strict translation functor. Moreover, a stable cofibration category as in (2.12) is an example of a triangulated track category with a nonstrict translation functor, see (2.11).

The author does not know any example of a triangulated category which is not derived from a triangulated track category. We shall use triangulated track categories in [3].

We are ready to formulate the main result of this paper.

Theorem 6.6. *Let* $(\mathbf{B}, t)$ *be a triangulated track category and let* $\mathbf{A}$ *be the homotopy category of* $\mathbf{B}$ *and* $t\colon \mathbf{A} \to \mathbf{A}$ *be induced by* $t\colon \mathbf{B} \to \mathbf{B}$. *Let* $\mathcal{E}$ *be the class of triangles in* $(\mathbf{A}, t)$ *which are induced by track triangles in* $(\mathbf{B}, t)$ *via the quotient functor* $\mathbf{B}_0 \to \mathbf{A}$. *Then* $(\mathbf{A}, t, \mathcal{E})$ *is a triangulated category.*

More precisely, we prove the following addendum.

Addendum 6.7. Let $\mathbf{B}$ be an additive track category with a strict translation functor $t\colon \mathbf{B} \to \mathbf{B}$ inducing t on the homotopy category $\mathbf{A}$ of $\mathbf{B}$. Let $\mathcal{E}$ be the class of triangles in $(\mathbf{A}, t)$ which are induced by track triangles in $(\mathbf{B}, t)$. Then $(\mathbf{A}, t, \mathcal{E})$ satisfies axioms (Tr0), (Tr2) and (Tr3). Moreover, if $(\mathbf{B}, t)$ is a triangulated track category, i.e., $(\mathbf{B}, t)$ satisfies (TTr1) and (TTr2), then $(\mathbf{A}, t, \mathcal{E})$ is a triangulated category, i.e., (TTr1) obviously implies (Tr1) and (TTr1), (TTr2) both imply (Tr4).

The theorem has a generalization for an additive track category with a nonstrict translation functor $t\colon \mathbf{B} \to \mathbf{B}$, see [6]. In fact, in [6] we show that $(\mathbf{B}, t)$ determines a characteristic cohomology class

$$\nabla = \langle \mathbf{B}, t \rangle \in H^3(\mathbf{A}, t)$$

in the translation cohomology of $(\mathbf{A}, t)$ defined in [5]. Moreover, the triangulated structure $\mathcal{E}$ of $(\mathbf{A}.t)$ depends only on this class, i.e., $\mathcal{E} = \mathcal{E}_\nabla$.

We point out that the family of exact triangles in $\mathbf{A}$ is well defined only by the additive track category $\mathbf{B}$ and the translation functor t on $\mathbf{B}$, i.e., the distinguished track triangles in (6.5) are not used in the definition of exact triangles in $\mathbf{A}$. The next result shows that each exact triangle in $\mathbf{A}$ is isomorphic to an exact triangle induced by a distinguished track triangle. For this we use (TTr1).

Proposition 6.8. *Let X and X' be track triangles both extending $f : A \to B$. Then there exists a commutative diagram in* $\boldsymbol{A}$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r] \arrow[d, equal] & C \arrow[r] \arrow[d, "h"] & tA \arrow[d, equal] \\
A \arrow[r, "f"] & B \arrow[r] & C' \arrow[r] & tA
\end{tikzcd}$$

where the rows are the exact triangles induced by X and X' and where h is an isomorphism.

Proof. We have a commutative diagram of natural transformations, see (4.3) and (4.4),

$$\begin{tikzcd}
 & & \hom_{\mathbf{A}}(C,U) \arrow[drr] & & \\
\hom_{\mathbf{A}}(tA,U) \arrow[urr] \arrow[drr] \arrow[rr, "i"] & & \mathrm{Cone}_f(U) \arrow[rr, "q"] \arrow[u, phantom, "\wr\|"] \arrow[d, phantom, "\wr\|"] & & \hom_{\mathbf{A}}(B,U) \\
 & & \hom_{\mathbf{A}}(C',U) \arrow[urr] & &
\end{tikzcd}$$

Now the isomorphism in the middle carries the identity of C' to h and carries the identity of C to the inverse of h.

Proof of (6.6). We show (Tr0), (Tr2) and (Tr4) in the following sections. We here only consider (Tr1) and (Tr3). It is clear that (TTr1) implies (Tr1). Moreover, given track triangles X and X' extending f and f', we find by (TTr1) distinguished track triangles Y and Y' extending f and f' and we can choose (a, b) as in (TTr2) inducing the square in (Tr3). Then (TTr2) yields a map $Y \to Y'$ between distinguished track triangles extending (a, b). Using this map and (6.8) we get (Tr3).

Remark 6.9. Grandis [7] describes triangulated categories derived from an abstract setting of homotopical algebra. This setting axiomatizes the behaviour of continuous maps and homotopies between them and does not use tracks as above which are homotopy classes of homotopies.

7. Induced Toda Complexes and Proof of (Tr0)

Lemma 7.1. *Let X be a Toda complex extending the row (X_n, d_n) and let $f : (X_n, d_n) \to (X'_n, d'_n)$ be weak equivalence between rows. Then there exists a unique Toda complex X' extending (X'_n, d'_n) such that $f : X \to X'$ is a morphism between Toda complexes.*

We denote by $X' = f_*X$ the *Toda complex induced by a weak equivalence* f.

Proof. Since f_n is an isomorphism in $\mathbf{A}$, we can choose a track $G_n : d'_n d'_{n-1} \Rightarrow 0$. Then the tracks (H_n, F_{n-1}, F_n, G_n) represent an element $\xi_n \in D(X_{n-2}, X'_n)$ and hence we can alter G_n by $(f^*_{n-2})^{-1}(\xi) \in D(X'_{n-2}, X'_n)$ so that we get H'_n. Then $(H_n, F_{n-1}, F_n, H'_n)$ represents a trivial track and hence $f : X \to X'$ is a morphism between the Toda complexes with $X' = (X'_n, d'_n, H'_n, n \in \mathbb{Z})$.

Lemma 7.2. *Let X be a track triangle in $\boldsymbol{B}$ with row (X_n, d_n) and let $f : (X_n, d_n) \to (X'_n, d'_n)$ be a weak equivalence of rows with $sh^3 f = tf$. Then the induced Toda complex f_*X is a track triangle.*

Proof. We only show that the Toda pair H'_0, H'_1 represents $1_{tA'}$. Let $F = F_{-1},\ G = F_0,\ H = F_1$ and $a = f_{-2},\ b = f_{-1},\ c = f_0,\ e = f_1$ be given by $f = (f_n, H_n, n \in \mathbb{Z})$. Moreover, we write $d = d_i$ and $d = d'_i$. Assume the Toda pair (H'_0, H'_1) represents $\xi \in D(A', tA')$. Then we get the equations

$$\begin{aligned}
\sigma(\alpha^*\xi) &= (dH'_0\Box(H'_1 d)\ \ )a\\
&= dH'_0 a\Box(H'_1)\ \ da\\
&= d(cH_0\Box G\ \ d\Box dF\ \ )\Box((H'_1)\ \ bd)\\
&= d(cH_0\Box G\ \ d\Box dF\ \ )\Box(dG\Box Hd\Box eH_1\ \ )d\\
&= dcH_0\Box dG\ \ d\Box ddF\ \ \Box dGd\Box Hdd\Box eH_1\ \ )d\\
&= edH_0\Box dG\ \ d\Box dGd\Box eH_1\ \ d\\
&= e(dH_0\Box H_1\ \ d)\\
&= \sigma((t\alpha)_* 1_{tA})\\
&= \sigma(1_{tA'} t\alpha)\\
&= \sigma(\alpha^* 1_{tA'}).
\end{aligned}$$

Here $\alpha : A \to A'$ is the isomorphism induced by a. We know that e induces $t\alpha$. The equations above show that $\xi = 1_{tA'}$.

Proof of (Tr0) *in* (6.6). The identity 1_A yields a track triangle by considering

$$\begin{tikzcd}
A \arrow[r, "1"'] \arrow[rr, bend left=60] & A \arrow[r] \arrow[rr, bend right=60, "0"'] \arrow[r, phantom, shift left=4ex, "\Uparrow H_0"] \arrow[r, phantom, shift right=4ex, "\Downarrow H_1"'] & * \arrow[r] \arrow[rr, bend left=60] \arrow[r, phantom, shift left=4ex, "\Uparrow H_2"] & tA \arrow[r, "1"'] & tA
\end{tikzcd}$$

Here $H_0 = 0$, $H_2 = 0$, since $*$ is a strict zero object and $H_1 = \sigma(-1_{tA})$. Therefore

$$A \xrightarrow{1} A \longrightarrow * \longrightarrow tA$$

is exact. Next let $\alpha : T \cong T'$ be an isomorphism of triangles where T is induced by the track triangles X. We obtain for T' a cochain complex (X'_n, d'_n) in $\mathbf{A}$ as in (2.1) and we choose a row (X'_n, d'_n) in $\mathbf{B}$ representing this cochain complex.

Then α shows that there is a weak equivalence of rows $f : (X_n, d_n) \to (X'_n, d'_n)$ with $sh^3 f = tf$ inducing α so that the track triangle $f_* X$ is defined by (7.2). Hence T' is induced by $f_* X$ and therefore T' is exact.

8. NEGATIVE TODA COMPLEXES AND THE PROOF OF (TR2)

Let $\mathbf{B}$ be an additive track category with $\mathbf{B}_\simeq = \mathbf{A}$. In this section we use the existence of strong products $X \times Y$ in $\mathbf{B}$. The case of strong coproducts is treated in a dual way. For each object X we can choose maps in $\mathbf{B}_0$

$$\mu : X \times X \longrightarrow X,$$

$$-1_X : X \longrightarrow X$$

which induce $(1_X, 1_X)$ and -1_X in the additive category $\mathbf{A}$. Moreover, we choose tracks

$$\begin{aligned} \tau_X &: \mu_X(1_X, -1_X) \Rightarrow 0, \\ \tau'_X &: \mu_X(0_X, 1_X) \Rightarrow 1_X, \\ \tau''_X &: \mu_X(1_X, 0_X) \Rightarrow 1_X. \end{aligned}$$

Here $0_X : X \to * \to X$ is given by the strict zero object $*$ in $\mathbf{B}$. For each morphism $f : X \to Y$ in $\mathbf{B}_0$, we now define tracks Γ_f and Γ_f^+ as in the diagrams (cf. [2]):

$$\begin{tikzcd}
X \times X \arrow[r, "f \times f"] \arrow[d, "\mu_X"'] \arrow[dr, phantom, "\overset{\Gamma_f^+}{\Rightarrow}"] & Y \times Y \arrow[d, "\mu_Y"] & & X \arrow[r, "f"] \arrow[d, "-1_X"'] \arrow[dr, phantom, "\overset{\Gamma_f}{\Rightarrow}"] & Y \arrow[d, "-1_Y"] \\
X \arrow[r, "f"'] & Y & & X \arrow[r, "f"'] & Y
\end{tikzcd} \tag{8.1}$$

Here Γ_f^+ is a unique track for which the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i_\varepsilon"'] \arrow[dd, bend right=60, "1"'] & Y \arrow[d, "i_\varepsilon"] \arrow[dd, bend left=60, "1"] \\
\Leftarrow X \times X \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{\Gamma_f^+}{\Rightarrow}"] & Y \times Y \Rightarrow \arrow[d] \\
X \arrow[r] & Y
\end{tikzcd}$$

represents the trivial track $0 \quad : f \Rightarrow f$ for $\varepsilon = 1$ and $\varepsilon = 2$ with $i_1 = (0, 1)$ and $i_2 = (1, 0)$. Moreover, Γ_f is a unique track for which the following diagram represents a trivial track $0\quad : 0 \Rightarrow 0$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "{(1,-1)}"'] \arrow[dd, bend right=60] & Y \arrow[d, "{(1,-1)}"] \arrow[dd, bend left=60, "0"] \\
\Leftarrow X \times X \arrow[r] \arrow[d] \arrow[ru, phantom, "\overset{(0\ ,\Gamma_f)}{\Rightarrow}"] & Y \times Y \Rightarrow \arrow[d] \\
X \arrow[r] \arrow[ru, phantom, "\overset{\Gamma_f^+}{\Rightarrow}"] & Y
\end{tikzcd}$$

The uniqueness of Γ_f^+ and Γ_f follows from the biadditivity of the coefficient functor D in the additive track category $\mathbf{B}$.

Lemma 8.2. *The pasting in the diagram*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "-1"'] & Y \arrow[r, "g"] \arrow[d] & Z \arrow[d, "-1"] \\
X \arrow[r] \arrow[ru, phantom, "\overset{\Gamma_f}{\Rightarrow}"] & Y \arrow[r] \arrow[ru, phantom, "\overset{\Gamma_g}{\Rightarrow}"] & Z
\end{tikzcd} \tag{1}$$

yields the track Γ_{gf}. *Moreover, for each track* $H : f \Rightarrow g$ *the pasting in the diagram*

$$\begin{tikzcd}
X \arrow[r, bend left=60, "f"] \arrow[r, phantom, bend left=30, "\Uparrow H"] \arrow[r, "g"] \arrow[d, "-1"'] & Y \arrow[d, "-1"] \\
X \arrow[r, "g"] \arrow[r, bend right=60, "f"'] \arrow[r, phantom, bend right=30, "\Uparrow H"] \arrow[ru, phantom, "\overset{\Gamma_g}{\Rightarrow}"] & Y
\end{tikzcd} \tag{2}$$

yields Γ_f.

Using Γ-tracks (8.1) we can define the *negative of a Toda complex*

$$neg(X) = X^- = (X_n^-, d_n^-, H_n^-, n \in \mathbb{Z}) \tag{8.3}$$

with $X_n^- = X_n$ and $d_n^- = d_n$ for $n \equiv 0, 1\ mod\ 3$ and $d_n^- = (-1_{X_n})d_n$ for $n \equiv 2\ mod\ 3$. Moreover H_n^- is obtained by $H_n^- = H_n$ for $n \equiv 1\ mod\ 3$ and

$H_n^- = (-1_{X_n})H_n$ for $n \equiv 2 \; mod \; 3$ and for $n \equiv 0 \; mod \; 3$ the track H_n^- is given by pasting the diagram

$$\begin{tikzcd}
& \Uparrow H_n & \\
& X_{n-1} \arrow[r, "d"] \arrow[d, "-1"] \arrow[dr, phantom, "\overset{\Gamma_d}{\Rightarrow}"] & X_n \arrow[d, "-1"] \\
X_{n-2} \arrow[ur, "d"] \arrow[uurr, bend left=60, "0"] \arrow[r, "d_{n-1}^-"'] & X_{n-1} \arrow[r, "d_n^-"'] & X_n
\end{tikzcd} \tag{8.4}$$

Here we have $d_{n-1}^- = (-1)d_{n-1}$ and $d_n^- = d_n$.

Proposition 8.5. *Toda pairs* (H_{n-1}^-, H_n^-) *in the negative Toda complex* $neg(X) = X^-$ *satisfy the formula*

$$\sigma^{-1}[H_{n-1}^- | H_n^-] = -\sigma^{-1}[H_{n-1}|H_n] \in D(X_{n-3}, X_n).$$

Proof of (8.5). We check three cases

$$\begin{aligned}
[H_1^-|H_2^-] &= [H_1|(-1)H_2] = (-1)d_2H_1\square(-1)H_2 \; d_0 \\
&= (-1)(d_2H_1\square H_2 \; d_0) = (-1)[H_1, H_2], \\
[H_2^-|H_3^-] &= d_3(-1)H_2\square\Gamma_d \; d_2d_1\square(-1)H_3 \; d_1 = (-1)d_3H_2\square(-1)H_3 \; d_1 \\
&= (-1)(d_3H_2\square H_3 \; d_1) = (-1)[H_2|H_3], \\
[H_3^-|H_4^-] &= d_4((-1)H_3\square\Gamma_{d_3}d_2)\square H_4 \; (-1)d_2 \\
&= (-1)d_4H_3\square(\Gamma_{d_4}d_3d_2\square d_4\Gamma_{d_3}d_2)\square H_4 \; (-1)d_2 \\
&= (-1)d_4H_3\square\Gamma_{d_4d_3}d_2\square H_4 \; (-1)d_2, \;\; \text{see } (8.2)(1), \\
&= (-1)d_4H_3\square(\Gamma_{d_4d_3}\square H_4 \; (-1))d_2 \\
&= (-1)d_4H_3\square((-1)H_4 \; \square\Gamma_0)d_2, \;\; \text{see } (8.2)(1), \\
&= (-1)(d_4H_3\square H_4 \; d_2)\square\Gamma_0d_2 = (-1)[H_3|H_4].
\end{aligned}$$

Here Γ_0 for $0 : X \to * \to Z$ is trivial by (8.2)(1).

The shift functor sh in (5.4) does not carry track triangles X to track triangles since $sh(X)$ does not satisfy condition (6.2)(2) on Toda pairs. We get, however, by (8.5) and (8.2)(3) the next result.

Proposition 8.6. *If* $X = (X_n, d_n, H_n, i \in \mathbb{Z})$ *is a track triangle then* (H_1^-, H_2^-, H_3^-) *and* (H_{-1}^-, H_0^-, H_2^-) *are track triangles in the sense of* (6.1).

Proof of (Tr2). Let X be a track triangle in $\mathbf{B}$ inducing (u, v, w) in (Tr2). Then by (8.6) we get track triangles (H_1^-, H_2^-, H_3^-) and (H_{-1}^-, H_0^-, H_2^-) inducing $(v, w, -tu)$ and $(-t^{-1}w, u, v)$, respectively. Here we assume that $\mathbf{B}$ has strong products. If this assumption is not satisfied, we can choose by [4] a track category $\mathbf{B}'$ with strong products and a pseudofunctor $\lambda \colon \mathbf{B} \to \mathbf{B}'$ with homotopy inverse $\rho\colon \mathbf{B}' \to \mathbf{B}$ inducing the identity on $\mathbf{A}$ and coefficients D. For X in $\mathbf{B}$, we get the Toda complex λX in $\mathbf{B}'$ and $neg(\lambda X) = (X_n^-, d_n^-, H_n^-, n \in \mathbb{Z})$ is defined as above. Then $(\rho H_1^-, \rho H_2^-, \rho H_3^-)$ and $(\rho H_{-1}^-, \rho H_0^-, \rho H_2^-)$ are track triangles in $\mathbf{B}$ inducing $(v, w, -tu)$ and $(-t^{-1}w, u, v)$, respectively.

Proof of (6.6). Let $\mathbf{B}$ be an additive track category with (strict) translation functor t. Then track triangles in $\mathbf{B}$ yield exact triangles in $\mathbf{A}$ which satisfy (Tr0) and (Tr2) by the proof above. Moreover, by an argument as in the proof of (6.7) we see that (Tr3) holds, cf. [6] 7.5. If $\mathbf{B}$ is a triangulated track category, then (TTr1) shows that $(\mathbf{A}, t, \mathcal{E})$ satisfies (Tr1). Moreover $(\mathbf{A}, t, \mathcal{E})$ satisfies (Tr4) by the proof in the next section. Hence $(\mathbf{A}, t, \mathcal{E})$ is a triangulated category.

9. Proof of (Tr4)

Using (6.8) we see that (Tr4)(#) is isomorphic to a sequence of the form (Tr4)(#) where, however, C_u, C_v and C_{uv} are chosen by distinguished track triangles. Then (Tr0) already proved above shows that it suffices to prove (Tr4) for such distinguished choices. We consider the following diagram in $\mathbf{B}$.

$$\begin{tikzcd}
 & A \arrow[r, equal] \arrow[d, "u"] & A \arrow[r, "u"] \arrow[d, "vu"] & B \arrow[d, "v"] & \\
 & B \arrow[r, "v"] \arrow[d, "d"] \arrow[dr, phantom, "\overset{F}{\Rightarrow}"] & C \arrow[r, equal] \arrow[d, "d'"] \arrow[dr, phantom, "\overset{F'}{\Rightarrow}"] & C \arrow[d, "d''"] & \\
 & C_u \arrow[r, "\bar{w}"] \arrow[d, "d"] \arrow[dr, phantom, "\overset{G}{\Rightarrow}"] & C_{vu} \arrow[r, "\bar{u}"] \arrow[d, "d'"] \arrow[dr, phantom, "\overset{G'}{\Rightarrow}"] & C_v \arrow[d, "d''"] \arrow[dd, bend left=60, "0"] & \\
 & tA \arrow[r, equal] \arrow[d, "tu"] \arrow[dd, bend right=60, "0"'] & tA \arrow[r, "tu"] \arrow[d, "tvu"] & tB \arrow[d, "tv"] \arrow[r, phantom, "\overset{H_2''}{\Rightarrow}"] & {} \\
 {} & tB \arrow[r, "tv"] \arrow[d, "d"] \arrow[l, phantom, "\overset{tH_0}{\Leftarrow}"] \arrow[dr, phantom, "\overset{tF}{\Rightarrow}"] & tC \arrow[r, equal] \arrow[d, "d'"] & tC & \\
 & tC_u \arrow[r, "t\bar{w}"] & tC_{vu} & &
\end{tikzcd}$$

The subdiagrams without tracks are commutative. Let $\bar{C}_u = (H_0, H_1, H_2)$ be a distinguished track triangle extending the left column, let $\bar{C}_{vu} = (H_0', H_1', H_2')$ be a distinguished track triangle extending the column in the middle and let $\bar{C}_v = (H_0'', H_1'', H_2'')$ be a distinguished track triangle extending the right-hand column. Then (TTr2) shows that we can find tracks F, G and F', G' which determine maps between the track triangles $\bar{C}_u \to \bar{C}_{vu}$ and $\bar{C}_{vu} \to \bar{C}_v$, respectively. This implies the equations

$$(t\bar{w})(tH_0) = (tH_0')\Box(tF)(tu), \tag{1}$$

$$H_2' = (H_2''\bar{u})\Box(tv)G'. \tag{2}$$

Moreover, we know (H_2', tH_0') represents

$$\sigma(1_{tC_{vu}}) = d'H_2'\Box(tH_0')\ \ d'. \tag{3}$$

Therefore we get the equation

$$\sigma(1_{tC_{vu}}) = d'H_2''\bar{u}\Box d'(tv)G'\Box(tF)(tu)d'\Box(t\bar{w})(tH_0)\ \ d'. \tag{4}$$

We now choose a distinguished track triangle for the map $\bar{u} : C_{vu} \to C_v$ which exists by (TTr1). This yields $C_{\bar{u}}$ and the following diagram in the homotopy category $\mathbf{A}$ where the bottom row corresponds to (2.3)(#).

$$\begin{tikzcd}
 & & C_{\bar{u}} \arrow[d, "\varphi"] \arrow[dr, "-q_{\bar{u}}"] & \\
C_{vu} \arrow[r, "\bar{u}"'] & C_v \arrow[ur, "i_{\bar{u}}"] \arrow[r, "dd''"'] & tC_u \arrow[r, "t\bar{w}"'] & tC_{vu}
\end{tikzcd} \tag{5}$$

We shall construct a map φ for which this diagram commutes. For this we use the representation $\chi : \mathrm{Cone}_{\bar{u}}(U) \to \hom_{\mathbf{A}}(C_{\bar{u}}, U)$ defining $i_{\bar{u}}$, $q_{\bar{u}}$ as in (8.5) with $U = tC_u$. Moreover, we use the following diagram of tracks:

$$\begin{tikzcd}
C_{vu} \arrow[r, "\bar{u}"] \arrow[d, "d'"'] \arrow[dr, phantom, "\overset{G'}{\Rightarrow}"] & C_v \arrow[d, "d''"'] \arrow[dr, "0"] \arrow[dr, phantom, "\overset{H_2''}{\Rightarrow}"'] & \\
tA \arrow[r, "tu"] \arrow[dr, "0"'] \arrow[dr, phantom, "\overset{(tH_0)}{\Rightarrow}"] & tB \arrow[r, "tv"] \arrow[d, "d"] \arrow[dr, phantom, "\overset{tF}{\Rightarrow}"] & tC \arrow[d, "d'"] \\
 & tC_u \arrow[r, "t\bar{w}"'] & tC_{vu}
\end{tikzcd} \tag{6}$$

According to (4), this diagram represents $\sigma(1_{tC_{vu}})$. Now let

$$\varphi = \chi\{\alpha, \hat{\alpha}\} \text{ with } \alpha = dd'' \textit{ and } \hat{\alpha} = (tH_0)d'\Box d(G')\ . \tag{7}$$

Then we have $\varphi i_{\bar u} = dd'$ and we get $(t\bar w)\varphi = -\bar q_v$, since using $R = (d'H_2''\square(tF)d'')$ yields the equation

$$\chi((t\bar w)\varphi) = (t\bar w)_*\{\alpha, \hat\alpha\} = \{(t\bar w)\alpha, (t\bar w)\hat\alpha\} = \{0, \sigma(-1_{tC_{vu}})\} = \chi(-q_{\bar u}). \tag{8}$$

Hence the proof of the commutativity of (5) is complete.

Finally, we observe that there is a commutative diagram

$$\begin{tikzcd}
\hom_{\mathbf{A}}(tC_{vu}, X) \arrow[r, "(t\bar w)^*"] & \hom_{\mathbf{A}}(tC_u, X) \arrow[r, "(dd'')^*"] & \hom_{\mathbf{A}}(C_v, X) \arrow[r, "\bar u^*"] & \hom_{\mathbf{A}}(C_{vu}, X) \\
\mathrm{Cone}_{vu}(t^{-1}X) \arrow[r, "v^*"] \arrow[u, "t\chi"] & \mathrm{Cone}_u(t^{-1}X) \arrow[r, "\delta"] \arrow[u, "t\chi"] & \mathrm{Cone}_v(X) \arrow[r, "u^*"] \arrow[u, "\chi"] & \mathrm{Cone}_{vu}(X) \arrow[u, "\chi"]
\end{tikzcd} \tag{9}$$

Here the bottom row is the exact sequence in (3.6). The vertical arrows are isomorphisms as in (4.4) by the track triangles $\bar C_v$, $\bar C_u$ and $\bar C_{vu}$ respectively. One readily checks by the properties of maps between track triangles that the diagram commutes. Hence the top row is exact by (3.6) and therefore φ in (5) is an isomorphism. Now (Tr0) shows that the bottom row of (5) is an exact triangle. This completes the proof of (Tr4).

10. The Dual of a Triangulated Track Category

We have seen in (1.5) that, for each object U of $\mathbf{A}$, an exact triangle $A \xrightarrow{f} B \to C \to tA$ in a triangulated category $\mathbf{A}$ induces the long exact sequence

$$\begin{aligned} &\hom_{\mathbf{A}}(U, t^{-1}A) \to \hom_{\mathbf{A}}(U, t^{-1}B) \\ &\to \hom_{\mathbf{A}}(U, t^{-1}C) \to \hom_{\mathbf{A}}(U, A) \xrightarrow{f_*} \hom_{\mathbf{A}}(U, B) \end{aligned} \tag{10.1}$$

which resembles an exact sequence for the dual cone functor Cone^f in (3.4). In fact we get

Proposition 10.2. *Let $\boldsymbol{B}$ be a triangulated track category. Then for each $f : A \to B$ in $\boldsymbol{B}_0$ the dual cone functor* Cone^f *has a canonical representation*

$$\chi : \mathrm{Cone}^f(U) \cong \hom_{\boldsymbol{A}}(U, t^{-1}C_f),$$

where $A \xrightarrow{f} B \to C_f \to tA$ is induced by a track triangle extending f. Moreover, χ yields an isomorphism of the exact sequences (3.4) *and* (10.1).

Proof. We have the tracks

$$\begin{tikzcd}
t^{-1}A \arrow[r, "t^{-1}f"] \arrow[rr, bend right=50, ""{name=L}, "\Downarrow H_{-3}" description, phantom] \arrow[rr, bend right=50] & t^{-1}B \arrow[r, "u"] \arrow[rr, bend left=50] \arrow[rr, bend left=50, phantom, "\Uparrow H_{-2}" description] & t^{-1}C_f \arrow[r, "w"] \arrow[rr, bend right=50] \arrow[rr, bend right=50, phantom, "\Downarrow H_{-1}" description] & A \arrow[r, "f"] & B
\end{tikzcd}$$

which define an element

$$\{w, H_{-1}\} \in \mathrm{Cone}^f(t^{-1}C_f).$$

Now we define the inverse of χ in (10.1) by

$$\chi^{-1}(\alpha) = \alpha^*\{w, H_{-1}\}.$$

We claim that the following diagram commutes up to the sign

$$\begin{tikzcd}
& \mathrm{Cone}^f(U) \arrow[dr, "\bar{q}"] & \\
\hom_{\mathbf{A}}(U, t^{-1}B) \arrow[ur, "\bar{i}"] \arrow[dr, "u_*"'] & & \hom_{\mathbf{A}}(U, A) \\
& \hom_{\mathbf{A}}(U, t^{-1}C_f) \arrow[uu, "\chi^{-1}"] \arrow[ur, "w_*"'] &
\end{tikzcd}$$

where $\bar{i}$ and $\bar{q}$ are defined as in (3.4). In fact $\bar{q}\chi^{-1} = w_*$ since

$$\bar{q}\chi^{-1}(\alpha) = \bar{q}\{w\alpha, H_{n-1}\alpha\} = w\alpha = w_*(\alpha).$$

Moreover, for $\delta \in \hom_{\mathbf{A}}(U, t^{-1}B)$ we have to show that $\bar{i}\delta = \chi^{-1}u_*\delta = \chi^{-1}(u\delta)$ where

$$\bar{i}\delta = \{0, \sigma\delta\} = \delta^*\{0, \sigma 1_{t^{-1}B}\},$$
$$\chi^{-1}(u\delta) = \delta^*\chi^{-1}(u) = \delta^*\{wu, H_{-1}u\}.$$

Hence it suffices to prove that in $\mathrm{Cone}^f(t^{-1}B)$ we have $\{0, \sigma 1_{t^{-1}B}\} = -\{wu, H_{-1}u\}$. But this follows since the pair (H_{-2}, H_{-1}) represents $\sigma 1_B$. Hence we have proved $\chi^{-1}u_x = -\bar{i}$. Using the exact sequences (3.4) and (10.1) we see that χ^{-1} is an isomorphism and Proposition 10.2 is proved.

Proposition (10.1) corresponds to the following result:

Theorem 10.3. *The categorical dual $\boldsymbol{B}^{op}$ of a triangulated track category $\boldsymbol{B}$ is again a triangulated track category with a translation functor given by the inverse t^{-1} of the strict translation functor t for $\boldsymbol{B}$. The distinguished track triangles in $\boldsymbol{B}^{op}$ are the duals X^{op} of distinguished track triangles X in $\boldsymbol{B}$ with $X^{op}_n = X_{-n}$.*

11. THE NATURAL NOTION OF GOOD MORPHISM

This section should be compared with section 3 in [11] which has the same title. Let $\mathbf{B}$ be a triangulated track category with homotopy category $\mathbf{A}$ and quotient functor $p : \mathbf{B}_0 \to \mathbf{A}$. Then p induces the functor

$$p : \mathbf{TTr}_{\mathbf{B}} \to \mathbf{Tri}_{\mathbf{A}}. \tag{11.1}$$

Here $\mathbf{Tri}_{\mathbf{A}}$ is the category of triangles in $\mathbf{A}$ and morphisms as in (2.1). Moreover, $\mathbf{TTr}_{\mathbf{B}}$ is the category of distinguished track triangles in $\mathbf{B}$ and maps as in (6.2).

A *special* isomorphism in $\mathbf{Tri_A}$ is an isomorphism of the form

$$\begin{tikzcd}
A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d, equal] & C \arrow[r] \arrow[d] & tA \arrow[d, equal] \\
A \arrow[r] & B \arrow[r] & C' \arrow[r] & tA
\end{tikzcd} \tag{11.2}$$

For example, (6.8) yields such special isomorphisms. We use the functor p in (11.1) and special isomorphisms for our next

Definition 11.3. A *good triangle* is a triangle T in $\mathbf{Tri_A}$ together with an equivalence class of special isomorphisms

$$T \cong p(X),$$

where X is an object in $\mathbf{TTr_B}$. Two such isomorphisms $T \cong p(X)$, $T \cong p(Y)$ are equivalent if there exists a map $X \to Y$ in $\mathbf{TTr_B}$ inducing composite $p(X) \cong T \cong p(Y)$. A *good morphism* between the good triangles T, T' is a morphism $f : T \to T'$ in $\mathbf{Tri_A}$ for which there exists a commutative diagram in $\mathbf{Tri_A}$

$$\begin{tikzcd}
p(X) \arrow[r, "p(\varphi)"] & p(X') \\
T \arrow[u, "\alpha"] \arrow[r, "f"'] & T' \arrow[u, "\alpha'"']
\end{tikzcd}$$

where α and α' represent the equivalence classes of isomorphisms for T and T' respectively and where $p(\varphi)$ is induced by a map $\varphi : X \to X'$ in $\mathbf{TTr_B}$. Let $\mathbf{GTr}$ be the category of good triangles and good morphisms. We have the faithful forgetful functor

$$F : \mathbf{GTr} \longrightarrow \mathbf{Tri_A}.$$

Theorem 11.4. *For a triangulated track category* $\boldsymbol{B}$*, the homotopy category* $\boldsymbol{A}$ *of* $\boldsymbol{B}$ *together with the functor* $F : \mathbf{GTr} \longrightarrow \mathbf{Tri}_{\boldsymbol{A}}$ *satisfy the axioms* (GTR1),...,(GTR10) *of a good triangulated category defined in* [11](3.4).

We do not want to recall 10 axioms of Neeman and leave the proof of the theorem to the reader. The author does not know any examples of triangulated categories or good triangulated categories which are not given by triangulated track categories. Therefore it seems reasonable to replace the complicated set of axioms (Trn) for $n = 0, \dots, 4$ or (GTRn) for $n = 0, \dots, 10$ by two very natural axioms (TTr1) and (TTr2) of a triangulated track category. Many of the numerous results on triangulated categories in the literature have analogues in a triangulated track category yielding natural and interesting extension, of such results.

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