

PRINCIPAL MAPS IN TRIANGULATED CATEGORIES AND MAPS BETWEEN 2-STAGE SPECTRA

HANS JOACHIM BAUES

Abstract. Using two natural axioms on track triangles in triangulated track category we show that the mapping cone construction is functorial, i.e., principal maps between mapping cones in a triangulated category can be identified with the homotopy classes of homotopy pairs. This result is applied to obtain an algebraic model of the category of principal maps between 2-stage spectra.

2000 Mathematics Subject Classification: 18D05, 18E30.

Key words and phrases: Triangulated categories, groupoid-enriched categories, homotopy pairs, principal maps.

INTRODUCTION

Consider the following commutative diagram in a triangulated category with translation functor t

$$\begin{array}{ccccccc}
 A & \xrightarrow{d} & B & \longrightarrow & C & \longrightarrow & t(A) \\
 f \downarrow & & \downarrow g & & \downarrow h & & \downarrow t(f) \\
 A' & \xrightarrow{d'} & B' & \longrightarrow & C' & \longrightarrow & t(A')
 \end{array}$$

Here the rows are exact (i.e. distinguished) triangles and C is called a *cone* of d . Then one of the axioms of a triangulated category says: *Given f and g with $gd = d'f$, there exists h such that the diagram commutes.* Conversely we define that a map $h : C \rightarrow C'$ is *principal* with respect to the cone structure of C and C' if there exists (f, g) such that the diagram commutes.

It is an old mystery in triangulated categories how pairs (f, g) and principal maps h are related. For this compare Gelfand–Manin [6] IV.7 who remark: *“This nonfunctoriality of a cone is the first symptom that something is going wrong in the axioms of a triangulated category. Unfortunately, at the moment, we do not have a more satisfactory version.”* We think that triangulated track categories as introduced in [1] are in this respect more satisfactory since we show that in a triangulated track category the category of principal maps is equivalent to the homotopy category of the category of homotopy pairs. This leads (in the case of the stable homotopy category of spectra) to the algebraic computation of all principal maps $C \rightarrow C'$ where A, B, A' and B' are finite direct sums of Eilenberg–MacLane spectra over a prime field \mathbb{F} . In this case we call C and C' *2-stage spectra*, see (1.5) and Section 6.

1. PRINCIPAL MAPS

Let \mathbf{A} be a triangulated category with translation functor $t : \mathbf{A} \rightarrow \mathbf{A}$ which is an additive equivalence. Then a distinguished family of triangles

$$X = (A \xrightarrow{d} B \xrightarrow{d_0} C \xrightarrow{d_1} tA), \tag{1.1}$$

termed *exact triangles*, is given satisfying axioms (Tr0)...(Tr4), see [9], [10]. We also write $C = C_d$. A morphism between exact triangles $X \rightarrow X'$ is a commutative diagram in \mathbf{A} of the form

$$\begin{array}{ccccccc} X & = & (A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & tA) \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \downarrow t(f) \\ X' & = & (A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & tA') \end{array} \tag{1.2}$$

We say that the morphism $(f, g, h) : X \rightarrow X'$ extends the map $h : C \rightarrow C'$ and we say that a map $h : C \rightarrow C'$ in \mathbf{A} is *principal* with respect to X and X' if there exists a morphism $X \rightarrow X'$ extending h . Let \mathbf{ET} be the category of exact triangles (1.1) and morphisms (1.2). Then we have the forgetful functor

$$\mathbf{ET} \rightarrow \mathbf{A} \tag{1.3}$$

which carries X to C and (f, g, h) to h . The image category of this functor is the category \mathbf{Prin} of principal maps. The objects in \mathbf{Prin} are exact triangles X, X' and the morphisms $X \rightarrow X'$ in \mathbf{Prin} are the maps $C \rightarrow C'$ in \mathbf{A} which are principal, i.e. in the image of the functor (1.3). We have the faithful functor $\mathbf{Prin} \rightarrow \mathbf{A}$.

Lemma 1.4. *A map $h : C \rightarrow C'$ is principal with respect to X and X' if there exists a commutative diagram in \mathbf{A} as in (1) or (2) respectively.*

$$\begin{array}{ccc} B \longrightarrow C & & C \longrightarrow tA \\ \downarrow & (1) \quad \downarrow h & h \downarrow \quad (2) \quad \downarrow \\ B' \longrightarrow C' & & C' \longrightarrow tA' \end{array} \tag{1.5}$$

Axiom (Tr3) says that given (f, g) as above with $gd = df$, there exists $(f, g, h) : X \rightarrow X'$ extending (f, g) . Here, however, the map h is not well defined by the pairs (f, g) . Therefore the axiom yields a mysterious relationship between the pairs (f, g) and the principal maps $h : C \rightarrow C'$ in a triangulated category. In this paper we clarify this relationship in the context of a triangulated track category [1]. This leads in Section 6 below to computation as in the following example.

Example 1.6. Let $\mathbf{A} = \mathbf{Spec}$ be the stable homotopy category of spectra. Then \mathbf{A} is a triangulated category. For a prime field \mathbb{F} we have the full subcategory

2- stage \subset Prin.

The objects of 2-stage are triangles X for which A and B are finite direct sums of Eilenberg–MacLane spectra over \mathbb{F} . Hence for an object $X = C_d$ in

2-stage the map $d : A \rightarrow B$ in \mathbf{A} is algebraically determined by a map between free modules over the Steenrod algebra. Also, for objects $X \rightarrow X'$ in **2-stage** the map $(f, g) : d \rightarrow d'$ is easily expressed algebraically. It is, however, an old problem to compute the principal maps $C_d \rightarrow C_{d'}$ in the category **2-stage**, see [8]. As an application of the main result in this paper we obtain an algebraic description of the category **2-stage** in terms of the pair algebra \mathcal{B} of secondary cohomology operations computed in [3], see Section 6.

2. THE CATEGORY OF HOMOTOPY PAIRS

Let \mathbf{B} be a triangulated track category (see [1]) with strict translation functor $t : \mathbf{B} \rightarrow \mathbf{B}$ and homotopy category \mathbf{A} . Then we have a linear extension of categories

$$D \rightarrow \mathbf{B}_1 \rightrightarrows \mathbf{B}_0 \rightarrow \mathbf{A} \tag{2.1}$$

with coefficient $D(A, B) = Hom_{\mathbf{A}}(tA, B)$. Here $\mathbf{B} = (\mathbf{B}_1 \rightrightarrows \mathbf{B}_0)$ is a track category (i.e., a groupoid enriched category) with hom-groupoids $\llbracket A, B \rrbracket$ for all objects A, B in \mathbf{B} . The objects of the groupoid $\llbracket A, B \rrbracket$ are maps $f : A \rightarrow B$ (or 1-cells) and morphisms, denoted by $F : f \rightrightarrows f'$, are called tracks (or 2-cells). The 1-cells are the morphisms of the category \mathbf{B}_0 . Two 1-cells f, f' are *homotopic* if there exists a track $f \rightrightarrows f'$. The homotopy classes of 1-cells in \mathbf{B}_0 are the morphisms in \mathbf{A} and $\mathbf{B}_0 \rightarrow \mathbf{A}$ is the quotient functor. The track category \mathbf{B} has a strong zero object $*$ with $\llbracket A, * \rrbracket = 0 = \llbracket *, A \rrbracket$ which yields the zero maps $0 : A \rightarrow * \rightarrow B$ in \mathbf{B}_0 . Moreover, we assume that \mathbf{B} has strong coproducts $A \vee B$ (which yield the direct sum $A \oplus B$ in \mathbf{A}) with $\llbracket A \vee B, C \rrbracket = \llbracket A, C \rrbracket \times \llbracket B, C \rrbracket$.

A *track-triangle* $X = (H_0, H_1, H_2)$ in \mathbf{B} is a diagram of the form

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & \curvearrowright & & \curvearrowleft & & \curvearrowright & \\
 & & \uparrow H_0 & & \uparrow H_2 & & \\
 A & \xrightarrow{d} & B & \xrightarrow{\quad} & C & \xrightarrow{\quad} & tA \xrightarrow{td} tB \\
 & & & \curvearrowright & & \curvearrowleft & \\
 & & & \downarrow H_1 & & & \\
 & & & & 0 & &
 \end{array} \tag{2.2}$$

satisfying certain properties [1]. In a triangulated track category \mathbf{B} one has a distinguished class of track triangles subject to two axioms (TTr1) and (TTr2)'. We show in [1] that the two axioms imply that \mathbf{A} is a triangulated category in which exact triangles are induced by track triangles via the quotient functor $\mathbf{B}_0 \rightarrow \mathbf{A}$.

Definition 2.3. Let \mathbf{B} be a track category. Then the *category HP of homotopy pairs in B* is defined as follows. Objects are the morphisms $d : A \rightarrow B$ in \mathbf{B}_0 and the morphisms $F : d \rightarrow d'$ in \mathbf{HP} are diagrams

$$\begin{array}{ccc}
 A & \xrightarrow{f} & A' \\
 d \downarrow & \xrightarrow{F} & \downarrow d' \\
 B & \xrightarrow{g} & B'
 \end{array}$$

in **B**. The composition in **HP** of the morphisms

$$\begin{array}{ccccc}
 A & \xrightarrow{f} & A' & \xrightarrow{f'} & A'' \\
 d \downarrow & \Downarrow F & \downarrow & \Downarrow F' & \downarrow d'' \\
 B & \xrightarrow{g} & B' & \xrightarrow{g'} & B''
 \end{array}$$

is defined by ff', gg' and

$$F' \square F = F'f \square g'F.$$

Here $F \square G$ denotes the composition of tracks. Let F^\square be the inverse of the track F .

Proposition 2.4. *Let \mathbf{B} be a triangulated track category. Then there is a functor*

$$\psi : \mathbf{HP} \longrightarrow \mathbf{Prin}$$

from the category of homotopy pairs in \mathbf{B} to the category of principal maps in \mathbf{A} such that this functor is full and representative.

This result shows that there is a natural equivalence relation \sim on **HP** such that the quotient category \mathbf{HP}/\sim is equivalent to the category **Prin**. We shall compute the equivalence relation in section 4. The functor ψ in (2.4) corresponds to the construction of principal maps between mapping cones in [2].

Addendum 2.5. *If \mathbf{B} has strong coproducts then there is a functor ψ such that ψ carries coproducts in **HP** to coproducts in \mathbf{A} .*

Proof of (2.4). For each object $d : A \rightarrow B$ we choose a distinguished track triangle $X = X_d$ as in (2.2) extending d , this corresponds to axiom (TTr1) in **B**. We define the functor ψ on the object d by $\psi(d) = C_d$ where $C = C_d$ is given by X as in (2.2), i.e., $\psi(d)$ is the triangle induced by X_d in \mathbf{A} . Let $(f, g, F) : d \rightarrow d'$ be a morphism in **HP**. Then we can choose by the axiom (TTr2)' in **B** a map $\varphi : X_d \rightarrow X_{d'}$ between the track triangles extending (f, g, F) . The map φ defines a map $h : C_d \rightarrow C_{d'}$ and a track G as in the diagram

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \uparrow H_0 & & \\
 A & \xrightarrow{\quad} & B & \xrightarrow{\quad} & C \\
 f \downarrow & \Downarrow F & \downarrow g & \Downarrow G & \downarrow h \\
 A' & \xrightarrow{\quad} & B' & \xrightarrow{\quad} & C' \\
 & & \Downarrow H'_0 & & \\
 & & 0 & &
 \end{array} \tag{1}$$

which represents the trivial track $0 \implies 0$. The homotopy class $\{h\}$ of h in \mathbf{A} corresponds via the representation

$$\{h\} \in \text{hom}_{\mathbf{A}}(C, C') \cong \text{Cone}_d(C') \tag{2}$$

in [1] (4.4) to the element $\{hd, hH_0\}$ in $\text{Cone}_d(C')$. By the equivalence relation defining the cone functor in [1] (3.1) (1) we see that for any other map $\varphi' : X_d \longrightarrow X_{d'}$ extending (f, g, F) we have

$$\begin{aligned} \{hd, hH_0\} &= \{d'g, hH_0 \square G^{\square} d\} \stackrel{(*)}{\cong} \{d'g, H_0'f \square d'F\} \\ &\stackrel{(*)}{\cong} \{d'g, h'H_0 \square G'^{\square} d\} = \{h'd, h'H_0\} \end{aligned} \tag{3}$$

Here (h', G') is given by φ' and in $(*)$ we use the fact that diagram (1) represents the trivial track. This shows that the homotopy class $\{h\}$ of h does not depend on the choice of φ . We now define the functor ψ on morphisms by $\psi(f, g, F) = \{h\}$. It is clear that ψ is a well defined functor. By [1] (6.6) we see that ψ is representative, i.e., for each exact triangle (1.1) in **Prin** there is an object d in **HP** with $\psi(d) \cong C$ in **Prin**. Finally, we check that the functor ψ is full. For this let $h : \psi(d) \longrightarrow \psi(d')$ be a map in **Prin**. Then we can choose (g, G) as in (1) such that $(g, h, G) : d_0 \longrightarrow d'_0$ is a map in **HP** with $d_0 = d : B \longrightarrow C$. We can find a map $\varphi' : X_{d_0} \longrightarrow X_{d'_0}$ extending (g, h, G) and we define (f, F) in such a way that the strict translation functor t on **B** yields (tf, tg, tF) as part of φ_0 . Then we get $\psi(f, g, F) = \{h\}$. This completes the proof of (2.4). \square

For the proof of the addendum we observe that the coproduct of track triangles is a track triangle.

3. COGROUPS AND Γ -TRACKS

Let **gr** be the category of finitely generated free groups $\langle E \rangle$.

Definition 3.1. Let \mathbf{C} be a category with coproducts $X \vee Y$. Then a *cogroup* X in \mathbf{C} is given by a coproduct preserving functor

$$\mathbf{gr} \longrightarrow \mathbf{C}$$

which carries \mathbb{Z} to X . If \mathbf{B} is a track category with strong coproducts $X \vee Y$, then a *pseudo cogroup* X in \mathbf{B} is a pseudo functor

$$\mathbf{gr} \longrightarrow \mathbf{B}$$

which preserves coproducts. Hence we have for $\alpha : \langle E \rangle \longrightarrow \langle E' \rangle$ in **gr** a map $\alpha_* : \vee_E X \longrightarrow \vee_{E'} X$ and for $\beta : \langle E \rangle \longrightarrow \langle E'' \rangle$ we have a track

$$\begin{array}{ccccc} \vee_E X & \xrightarrow{\alpha_*} & \vee_{E'} X & \xrightarrow{\beta_*} & \vee_{E''} X \\ & & \downarrow \tau & & \uparrow \\ & & (\beta\alpha)_* & & \end{array}$$

We call τ a *linearity track* for X .

For $-1 : \mathbb{Z} \rightarrow \mathbb{Z}$ we write $-1_X = (-1)_* : X \rightarrow X$ and for $i_1 + i_2 : \mathbb{Z} \rightarrow \mathbb{Z} \vee \mathbb{Z}$ in \mathbf{gr} we write $(i_1 + i_2)_* = i_1 + i_2 : X \rightarrow X \vee X$. This is the *comultiplication* of X .

Now let \mathbf{B} be an additive track category with strong coproducts and a strong zero object $*$. Then, by the use of the linearity tracks above, we obtain as in [1] (8.1) well defined Γ -Tracks for each $f : X \rightarrow Y$ in \mathbf{B}_0 .

$$\begin{array}{ccc}
 X \vee X & \xrightarrow{f \vee f} & Y \vee Y \\
 \uparrow i_1 + i_2 & \xRightarrow{\Gamma} & \uparrow i_1 + i_2 \\
 X & \xrightarrow{f} & Y
 \end{array} \tag{3.2}$$

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 \uparrow -1 & \xRightarrow{\Gamma} & \uparrow -1 \\
 X & \xrightarrow{f} & X
 \end{array} \tag{1}$$

We use these tracks to define, for $-i_1 + i_2 : X \rightarrow X \vee X$, also the Γ -track

$$\Gamma : (-i_1 + i_2)f \xRightarrow{\Gamma} -i_1f + i_2f \tag{2}$$

given by pasting in the diagram

$$\begin{array}{ccc}
 X \vee X & \xrightarrow{f \vee f} & Y \vee Y \\
 \uparrow -1 \vee 1 & \xRightarrow{\Gamma \vee 0_f^\square} & \uparrow -1 \vee 1 \\
 X \vee X & \xrightarrow{f \vee f} & Y \vee Y \\
 \uparrow i_1 + i_2 & \xRightarrow{\Gamma} & \uparrow i_1 + i_2 \\
 X & \xrightarrow{f} & Y
 \end{array} \tag{3}$$

Here $0_f^\square : f \xRightarrow{\Gamma} f$ is the trivial track. In [5] we prove the following result.

Lemma 3.3. *Let \mathbf{B} be an additive track category with strong coproducts. Then each object X in \mathbf{B} admits the structure of a pseudo cogroup.*

In some of the proofs below we assume that objects X in \mathbf{B} are actually cogroups so that we omit this way the linearity tracks for X . We leave it to the reader to generalize the argument in case the linearity tracks are nontrivial.

Example 3.4. Let $\mathbf{S}_{\mathbf{P}_F}$ be the category of simplicial group spectra FX where X is a simplicial spectrum, see [7] (5.2). Then $\mathbf{S}_{\mathbf{P}_F}$ has a cylinder functor and the homotopy category $\mathbf{S}_{\mathbf{P}_F}/\simeq = \mathbf{Spec}$ is the stable homotopy category of the

spectra. Moreover the associated track category for $\mathbf{S}_{\mathbf{P}_F}$ is a triangulated track category \mathbf{B} with a strict translation functor t in which each object is a cogroup in \mathbf{B}_0 . Hence the results of this paper hold in particular in the stable homotopy theory.

4. THE HOMOTOPY RELATION FOR HOMOTOPY PAIRS

Let \mathbf{B} be an additive track category with strong coproducts and a strong zero object. For each object X in \mathbf{B} we choose the structure of a pseudo cogroup as in Lemma 3.3.

Definition 4.1. For a morphism $d : A \rightarrow B$ in \mathbf{B}_0 we define the *cylinder object* $I(d)$ by the diagram in \mathbf{B}

$$\begin{array}{ccc}
 A & \xrightarrow{\partial_2} & A \vee B \vee A & \xrightarrow{\partial_1} & B \vee B \\
 & \searrow & \Downarrow \partial_{12} & \nearrow & \\
 & & 0 & &
 \end{array}$$

Here we set $\partial_1 = (i_1d, -i_1 + i_2, i_2d)$ and $\partial_2 = i_1 + i_2d - i_3$. Moreover the track ∂_{12} is given by Γ in (3.2)(2) and by linearity tracks, i.e., for $\partial_1\partial_2 = i_1d + (-i_1 + i_2)d - i_2d$ and $\delta = i_1d + d(-i_1 + i_2) - i_2d = (d \vee d)(i_1 + (-i_1 + i_2) - i_2)$ let $\partial_1\partial_2$ be the composite

$$\partial_1\partial_2 \xrightarrow{i_1d + \Gamma - i_2d} \delta \xrightarrow{(d \vee d)\tau} 0.$$

Here τ is the appropriate linearity track.

Definition 4.2. Let $d : A \rightarrow B$ and $d' : A' \rightarrow B'$ be objects in the category of homotopy pairs. Then two morphisms

$$(d, g, F), (f', g', F') : d \rightarrow d'$$

in \mathbf{HP} are *homotopic* if and only if there exists a map $a : B \rightarrow A$ and there exist tracks

$$\begin{aligned}
 \alpha &: (g, g')(-i_1 + i_2) \implies d'a, \\
 \beta &: (f, a, f')\partial_2 \implies 0
 \end{aligned}$$

such that pasting in the following diagram yields the trivial track $0 \implies 0$.

$$\begin{array}{ccccc}
 & & A & & \\
 & & \downarrow \partial_2 & & \searrow 0 \\
 & & A \vee B \vee A & \xrightarrow{(f, a, f')} & A' \\
 & \swarrow 0 & \downarrow \partial_1 & \xrightarrow{(F, \alpha, F')} & \downarrow d' \\
 & & B \vee B & \xrightarrow{(g, g')} & B'
 \end{array}$$

Here the left-hand side of the diagram is given by the cylinder $I(d)$ in (4.1).

Remark 4.3. A Toda complex $X = (X_n, d_n, H_n, n \in \mathbb{Z})$ is a sequence of maps $d_n : X_{n+1} \rightarrow X_n$ and tracks $H_n : d_{n+1}d_n \rightrightarrows 0$ in \mathbf{B} . This is a *secondary chain complex* if all Toda pairs (H_{n+1}, H_n) represent the zero element, see [1] (5.1). Let \mathbf{SCh} be the category of secondary chain complexes with morphisms defined as maps between Toda complexes, see [1] (5.3). We consider an object $d : A \rightarrow B$ in \mathbf{HP} as an object X in \mathbf{SCh} concentrated in degrees 0 and 1, that is, $X_n = *$ for $n \neq 0, 1$. This way we have the full inclusion of categories $\mathbf{HP} \subset \mathbf{SCh}$. In [4] we shall describe a cylinder $I(X)$ in the category \mathbf{SCh} which generalizes the cylinder $I(d)$ in (4.1). The cylinder $I(X)$ yields a homotopy relation \simeq on \mathbf{SCh} such that one has the full inclusion of homotopy categories $\mathbf{HP}/\simeq \subset \mathbf{SCh}/\simeq$.

Theorem 4.4. *Let \mathbf{B} be a triangulated track category with strict coproducts $X \vee Y$. Then the functor ψ in (2.4) induces an equivalence of categories*

$$\psi : \mathbf{HP}/\simeq \xrightarrow{\sim} \mathbf{Prin}.$$

Hence the homotopy category of the homotopy pairs in \mathbf{B} is equivalent to the category of principal maps in $\mathbf{A} = \mathbf{B}_0/\simeq$. Results of this kind under connectivity and dimensional restrictions are described in [2]. We prove the theorem in the next section.

5. PROOF OF THE EQUIVALENCE (4.4)

Let $d : A \rightarrow B$ be a map of \mathbf{B}_0 and let $C = C_d$ be given by a distinguished track triangle extending d , see (2.2). We consider the following diagram in \mathbf{B}

$$\begin{array}{ccccccc}
 A & \xrightarrow{d} & B & \xlongequal{\quad} & B & & \\
 \downarrow & \xrightarrow{H_0^\square} & \downarrow d_0 & \xrightarrow{\Gamma} & \downarrow \partial & & \\
 * & \xrightarrow{0} & C & \xrightarrow{-i_1+i_2} & C \vee C & \xrightarrow{(\varphi, \varphi')} & X \\
 \downarrow & & \downarrow j & & \downarrow i & \nearrow u & \\
 tA & \xrightarrow{v} & C_{d_0} & \xrightarrow{w} & C_\partial & &
 \end{array}$$

Here $\partial = i_1d_0 + i_2d_0$ and Γ is defined by (3.2)(2). Moreover H_0 is given by (2.2). The tracks in the diagram describe homotopy pairs in \mathbf{HP} and wv is a map in \mathbf{B}_0 representing $\psi(\Gamma \square H_0^\square)$ where ψ is the functor in (2.4) and $w = \psi(\Gamma)$ and $v = \psi(H_0^\square)$. Here v is an isomorphism in \mathbf{A} by (2.2).

Lemma 5.1. *Let $\varphi, \varphi' : C \rightarrow X$ in \mathbf{B}_0 . Then $\varphi \simeq \varphi'$ if and only if there exists $u : C_\partial \rightarrow X$ in \mathbf{B}_0 with $ui = (\varphi, \varphi')$ and $uw = 0$ in \mathbf{A} .*

Proof. We have $\varphi \simeq \varphi'$ if and only if $(\varphi, \varphi')(-i_1 + i_2) \simeq 0$. Hence $\varphi \simeq \varphi'$ implies that $(\varphi, \varphi')\partial = 0$ in \mathbf{A} and hence there is u' with $u'i = (\varphi, \varphi')$ in \mathbf{A} . Then $u'wj = 0$ so that $u'w$ factors in \mathbf{A} through the cofiber tB of j . This cofiber is isomorphic to the cofiber of i . Therefore we can alter u' by a map $tB \rightarrow X$ in such a way that we get u with $uw = 0$. Conversely assume u is given. Then $uv = 0$ and hence $(\varphi, \varphi')(-i_1 + i_2) = uwj = 0$ in \mathbf{A} . □

Next we consider the following diagram in \mathbf{B} where we use the cylinder object $I(d)$ in (4.1)

$$\begin{array}{ccccc}
 A & \xrightarrow{\partial_2} & A \vee B \vee A & \xrightarrow{(0,1,0)} & B \\
 \downarrow & \xrightarrow{\partial_{12}^\square} & \downarrow \partial_1 & \xrightarrow{V} & \downarrow \partial \\
 * & \longrightarrow & B \vee B & \xrightarrow{d_0 \vee d_0} & C \vee C \\
 \downarrow & & \downarrow & & \downarrow \\
 tA & \xrightarrow{w'} & C_{\partial_1} & \xrightarrow{v'} & C_{\partial}
 \end{array}$$

Here V is given by $(i_1 H_0, 0^\square, i_2 H_0)$. We consider V and ∂_{12}^\square as morphisms in \mathbf{HP} . Let v' and w' be maps which represent $\psi(V)$ and $\psi(\partial_{12}^\square)$ respectively. We observe v' is an isomorphism in \mathbf{A} .

Lemma 5.2. *We have $v'w' = wv$ in \mathbf{A} .*

Proof. We have $wv = \psi(\Gamma \bar{\square} H_0^\square)$ and $v'w' = \psi(V \bar{\square} \partial_{12}^\square)$. Therefore it suffices to show that in \mathbf{HP} we have the equation

$$\Gamma \bar{\square} H_0^\square = V \bar{\square} \partial_{12}^\square. \tag{1}$$

Let $p_1 = (1, 0) : C \vee C \rightarrow C$ and $p_2 = (0, 1) : C \vee C \rightarrow C$ be the projections. Then (1) holds if and only if (2) holds.

$$p_i(\Gamma \bar{\square} H_0^\square) = p_i(V \bar{\square} \partial_{12}^\square) \quad \text{for } i = 1, 2. \tag{2}$$

This follows from the bilinearity of the coefficients D in (2.1). For convenience we now assume that additive tracks τ in (3.1) are trivial tracks. Then we get $p_2 V = (0^\square, H_0)$, $p_2 \partial_{12}^\square = 0^\square$ and $p_2 \Gamma = 0^\square$. Hence (2) for $i = 2$ is equivalent to

$$H_0^\square = d_0 d - H_0. \tag{3}$$

This holds since for any track $H : f \Rightarrow 0$ one has $0^\square = H - H = H \square (f - H)$.

Next we consider the case $i = 1$ in (2). Then we have $p_1 V = (H_0, 0^\square)$, $p_1 \partial_{12}^\square = d + \bar{\Gamma}$ with $\bar{\Gamma} : d(-1) \Rightarrow (-1)d$, $p_1 \Gamma = \bar{\Gamma}_0^\square : (-1)d_0 \Rightarrow d_0(-1)$. Therefore for $i = 1$, (2) is equivalent to

$$\bar{\Gamma}_0^\square \bar{\square} H_0^\square = (H_0^\square, 0^\square) \bar{\square} (d + \bar{\Gamma}). \tag{4}$$

Indeed, we have

$$\begin{aligned}
 (H_0, 0^\square) \bar{\square} (d + \bar{\Gamma}) &= (H_0, 0^\square)(i_1 + i_2 d) \square d_0(d + \bar{\Gamma}) \\
 &= (H_0 + d_0(-1)d) \square (d_0 d + d_0 \bar{\Gamma}) \\
 &= H_0 + d_0 \bar{\Gamma} : 0 \implies d_0(-1)d.
 \end{aligned} \tag{5}$$

On the other hand, we get

$$\begin{aligned}
 \bar{\Gamma}_0^\square \bar{\square} H_0^\square &= \bar{\Gamma}_0^\square d \square (-1)H_0^\square \\
 &= d_0 \bar{\Gamma} \square (-H_0^\square), && \text{see (7),} \\
 &= d_0 \bar{\Gamma} \square (H_0 - d_0 d), && \text{see (3),} \\
 &= H_0 + d_0 \bar{\Gamma}.
 \end{aligned} \tag{6}$$

Here we use the fact that by the rules for Γ -tracks the following diagram represents a trivial track.

$$\begin{array}{ccccc}
 & & 0 & & \\
 & \curvearrowright & \uparrow H_0 & \curvearrowleft & \\
 \bullet & \xrightarrow{d} & \bullet & \xrightarrow{d_0} & \bullet \\
 \downarrow -1 & \xrightarrow{\bar{r}} & \downarrow -1 & \xrightarrow{\bar{r}_0} & \downarrow -1 \\
 \bullet & \xrightarrow{d} & \bullet & \xrightarrow{d_0} & \bullet \\
 & \curvearrowleft & \downarrow H_0 & \curvearrowright & \\
 & & 0 & &
 \end{array} \tag{7}$$

This completes the proof of (5.2). □

Proof of (4.4). Let $d : A \rightarrow B$ and $d' : A' \rightarrow B'$ be objects in **HP** and let $F = (f, g, F)$ and $F' = (f', g', F')$ be morphisms $d \rightarrow d'$ in **HP**. We have to show that F is homotopic to F' as in (4.2) if and only if $\psi(F) = \psi(F')$. By (4.2) we obtain the following diagram in **B**

$$\begin{array}{ccccc}
 & & 0 & & \\
 & \curvearrowright & \uparrow \beta & \curvearrowleft & \\
 A & \xrightarrow{\partial_2} & A \vee B \vee A & \xrightarrow{(f, a, f')} & A' \\
 \downarrow & \xrightarrow{\partial_{12}^{\square}} & \downarrow \partial_1 & \xrightarrow{(F, \alpha, F')} & \downarrow d' \\
 * & \xrightarrow{\quad} & B \vee B & \xrightarrow{(g, g')} & B' \xrightarrow{H'_0} 0 \\
 \downarrow & \xrightarrow{w'} & \downarrow & \xrightarrow{W} & \downarrow \\
 tA & \xrightarrow{\quad} & C_{\partial_1} & \xrightarrow{\quad} & C_{d'} \\
 & & \uparrow i & \nearrow (\psi, \psi') & \\
 & & C_d \vee C_d & &
 \end{array}$$

Here w' , W and i are maps representing $\psi(\partial_{12}^{\square})$, $\psi(F, \alpha, F')$ and $d \vee d \subset \partial_1$ respectively and ψ and ψ' are maps representing $\psi(F)$, resp. $\psi(F')$. Since ψ is a functor we have $Wi = (\psi, \psi')$ in **A**.

Now assume $F \simeq F'$ in **HP** so that a, α, β are given. Then we can choose W as above and we observe that Ww' in **A** is represented by pasting tracks ∂_{12}^{\square} , (F, α, F') and H'_0 . Compare the proof of (2.4). This pasting yields the same element as the pasting of ∂_{12}^{\square} , (F, α, F') and β which by the definition of $F \simeq F'$ in (4.2) represents 0. Hence $Ww' = 0$ and therefore (5.1) and (5.2) show $\psi \simeq \psi'$ since v and v' are isomorphisms in **A**.

On the other hand, assume now $\psi \simeq \psi'$. Then by (5.1) and (5.2) there exists W with $Wi = (\psi, \psi')$ in **A** and $Ww' = 0$ in **A**. Since the functor ψ in (2.4) is full, we find a, α as above with $\psi(F, \alpha, F') = W$ and $\psi(F) = \psi$, $\psi(F') = \psi'$ in **A**. Moreover, there exists $\beta' : (f, a, f')\partial_2 \Rightarrow 0$, since $Ww' = 0$ induces $t((f, a, f')\partial_2) = 0$. Now pasting ∂_{12}^{\square} , (F, α, F') and β yields an element

$\xi \in \text{Hom}_{\mathbf{A}}(tA, B')$. This element maps to 0 in $\text{Hom}_{\mathbf{A}}(tA, C_{d'})$, since $Ww' = 0$. Hence there exists $\xi' \in \text{Hom}(tA, A')$ which maps to ξ , i.e., $d'\xi' = \xi$. Now we replace β' by ξ' and get β such that the pasting of ∂_{12}^{\square} , (F, α, F') and β represents the zero element. This shows $F \simeq F'$ in **HP**. \square

6. THE CATEGORY OF 2-STAGE SPECTRA

Let k be a commutative ring with unit and let **Mod** be the category of k -modules and k -linear maps. This is a symmetric monoidal category via the tensor product $A \otimes B$ over k . A pair of modules is a morphism

$$X = \left(X_1 \xrightarrow{\partial} X_0 \right)$$

in **Mod**. A morphism $f : X \rightarrow Y$ of pairs is a commutative diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ \partial \downarrow & & \downarrow \partial \\ X_0 & \xrightarrow{f_0} & Y_0 \end{array}$$

Clearly, a pair in **Mod** coincides with a chain complex concentrated in degrees 0 and 1. For two pairs X and Y the tensor product of the chain complexes corresponding to them is concentrated in degrees 0, 1 and 2 and is given by

$$X_1 \otimes Y_1 \xrightarrow{\partial_1} X_1 \otimes Y_0 \oplus X_0 \otimes Y_1 \xrightarrow{\partial_0} X_0 \otimes Y_0$$

with $\partial_0 = (\partial \otimes 1, 1 \otimes \partial)$ and $\partial_1 = (-1 \otimes \partial, \partial \otimes 1)$. Truncating $X \otimes Y$ we get the pair

$$X \bar{\otimes} Y = \left((X \bar{\otimes} Y)_1 = \text{coker}(\partial_1) \xrightarrow{\partial} X_0 \otimes Y_0 = (X \bar{\otimes} Y)_0 \right)$$

with ∂ induced by ∂_0 .

We define the tensor product $A \otimes B$ of two graded modules in the usual way, i.e. by

$$(A \otimes B)^n = \bigoplus_{i+j=n} A^i \otimes B^j.$$

A (graded) pair module is a sequence $X^n = (\partial : X_1^n \rightarrow X_0^n)$ with $n \in \mathbb{Z}$ of pairs in **Mod**. We identify such a graded pair module X with the underlying morphism ∂ of degree 0 between graded modules

$$X = \left(X_1 \xrightarrow{\partial} X_0 \right).$$

Now the tensor product $X \bar{\otimes} Y$ of graded pair modules X, Y is defined by

$$(X \bar{\otimes} Y)^n = \bigoplus_{i+j=n} X^i \bar{\otimes} Y^j.$$

This defines a monoidal structure on the category of graded pair modules. Morphisms in this category are of degree 0. For two morphisms $f, g : X \rightarrow Y$

between graded pairs modules, a *homotopy* $H : f \implies g$ is a morphism $H : X_0 \longrightarrow Y_1$ of degree 0 as in the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ \partial \downarrow & \nearrow H & \downarrow \partial \\ X_0 & \xrightarrow{f_0} & Y_0 \\ & & \xrightarrow{g_0} \end{array}$$

satisfying $f_0 - g_0 = \partial H$ and $f_1 - g_1 = H\partial$. A *pair algebra* B is a monoid in the monoidal category of graded pair modules, with multiplication

$$\mu : B \bar{\otimes} B \longrightarrow B.$$

We assume that B is concentrated in nonnegative degrees, that is $B^n = 0$ for $n < 0$. A *left B -Module* is a graded pair module M together with a left action

$$\mu : B \bar{\otimes} M \longrightarrow M$$

of the monoid B on M . More explicitly pair algebras and modules over them can be described as follows.

Definition 6.1. A *pair algebra* B is a graded pair

$$\partial : B_1 \longrightarrow B_0$$

in **Mod** with $B_1^n = B_0^n = 0$ for $n < 0$ such that B_0 is a graded algebra in **Mod**, B_1 is a graded B_0 - B_0 -bimodule, and ∂ is a bimodule homomorphism. Moreover for $x, y \in B_1$ the equality

$$\partial(x)y = x\partial(y)$$

holds in B_1 .

Definition 6.2. A *(left) module* over a pair algebra B is a graded pair $M = (\partial : M_1 \longrightarrow M_0)$ in **Mod** such that M_1 and M_0 are left B_0 -modules and ∂ is B_0 -linear. Moreover a B_0 -linear map

$$\bar{\mu} : B_1 \otimes_{B_0} M_0 \longrightarrow M_1$$

is given, fitting in the commutative diagram

$$\begin{array}{ccc} B_1 \otimes_{B_0} M_1 & \xrightarrow{1 \otimes \partial} & B_1 \otimes_{B_0} M_0 \\ \mu \downarrow & \nearrow \bar{\mu} & \downarrow \mu \\ M_1 & \xrightarrow{\partial} & M_0 \end{array}$$

where $\mu(b \otimes m) = \partial(b)m$ for $b \in B_1$ and $m \in M_1 \cup M_0$.

For an indeterminate element x of degree $n = |x|$ let $B[x]$ denote the B -module with $B[x]_i$ consisting of expressions bx with $b \in B_i, i = 0, 1$, with bx having degree $|b| + n$, and structure maps given by $\partial(bx) = \partial(b)x, \mu(b' \otimes bx) = (b'b)x$ and $\bar{\mu}(b' \otimes bx) = (b'b)x$. A *free B -module* is a direct sum of several copies

of modules of the form $B[x]$, with $x \in I$ for some set I of indeterminates of possibly different degrees. It will be denoted by

$$B[I] = \bigoplus_{x \in I} B[x].$$

Let $B\text{-Mod}$ be the category of left modules over the pair algebra B . Morphisms $f = (f_0, f_1) : M \rightarrow N$ are pair morphisms which are B -equivariant, i.e., f_0 and f_1 are B_0 -equivariant and compatible with $\bar{\mu}$ above, i.e. the diagram

$$\begin{array}{ccc} B_1 \otimes_{B_0} M_0 & \xrightarrow{\bar{\mu}} & M_1 \\ 1 \otimes f_0 \downarrow & & \downarrow f_1 \\ B_1 \otimes_{B_0} N_0 & \xrightarrow{\bar{\mu}} & N_1 \end{array}$$

commutes. For two such maps $f, g : M \rightarrow N$ a track $H : f \Rightarrow g$ is a degree zero map

$$H : M_0 \rightarrow N_1$$

satisfying $f_0 - g_0 = \partial H$ and $f_1 - g_1 = H\partial$ such that H is B_0 -equivariant. For tracks $H : f \Rightarrow g, K : g \Rightarrow h$ their composition $K \square H : f \Rightarrow h$ is $K + H$.

The category $B\text{-Mod}$ with these tracks is a well defined additive track category with strong direct sums $M \oplus N$. Let

$$B\text{-mod} \subset B\text{-Mod} \tag{6.3}$$

be the full subcategory consisting of finitely generated free B -modules. The category of homotopy pairs $\mathbf{HP}(B\text{-mod})$ is defined dually to (2.3). Morphisms $d' \rightarrow d$ are diagrams

$$\begin{array}{ccc} A & \xleftarrow{f} & A' \\ d \uparrow & \underline{\underline{F}} & \uparrow d' \\ B & \xleftarrow{g} & B' \end{array} \tag{6.4}$$

in $B\text{-mod}$. Two such morphisms $(f, g, F), (f', g', F') : d' \rightarrow d$ are *homotopic* if there exists (a, α, β) as in the diagram

$$\begin{array}{ccc} & & A \\ & \partial_2 \uparrow & \leftarrow \beta \\ A \oplus B \oplus A & \xleftarrow{(f, a, f')} & A' \\ \partial_1 \uparrow & \underline{\underline{(F, \alpha, F')}} & \uparrow d' \\ B \oplus B & \xleftarrow{(g, g')} & B' \end{array}$$

with $\partial_1 = (dp_1, -p_1 + p_2, dp_2)$ and $\partial_2 = p_1 + dp_2 - p_3$. Here p_1, p_2, p_3 denote the projections. This defines a natural equivalence relation \simeq on $\mathbf{HP}(B\text{-mod})$.

Theorem 6.5. *Let B be the pair algebra of secondary cohomology operations computed in [3]. Then one has an equivalence of categories*

$$(\mathbf{HP}(B\text{-mod})/\simeq)^{op} \xrightarrow{\sim} 2\text{-stage}.$$

Here **2-stage** is the category of 2-stage spectra and principal maps in (1.5).

Proof. It is proved in [3] (5.5.6) that the track category of finite direct sums of Eilenberg–MacLane spectra is equivalent to the track category $(B\text{-mod})^{op}$. This equivalence is compatible with strong coproducts and hence induces the equivalence in the theorem by the use of theorem (4.4). Since the composition in $B\text{-mod}$ is bilinear, Γ -tracks are trivial in the notion of homotopy in (6.4). \square

REFERENCES

1. H.-J. BAUES, Triangulated track categories. *Georgian Math. J.* **13**(2006), No. 4, 607–634.
2. H.-J. BAUES, On the cohomology of categories, universal Toda brackets and homotopy pairs. *K-Theory* **11**(1997), No. 3, 259–285.
3. H.-J. BAUES, The algebra of secondary cohomology operations. *Preprint Max-Planck-Institut für Mathematik*, 2004.
4. H.-J. BAUES and G. GAUDENS, Secondary chain complexes. *Preprint Max-Planck-Institut für Mathematik*, 2004.
5. H.-J. BAUES and T. PIRASHVILI, Third MacLane cohomology and strengthening of additive tracks theories. *Preprint Max-Planck-Institut für Mathematik*, 2004.
6. S. I. GELFAND and YU. I. MANIN, Methods of homological algebra. Second edition. *Springer Monographs in Mathematics. Springer-Verlag, Berlin*, 2003.
7. D. M. KAN, Semisimplicial spectra. *Illinois J. Math.* **7**(1963), 463–478.
8. L. KRISTENSEN and IB. MADSEN, On the structure of the operation algebra for certain cohomology theories. *Conf. on Algebraic Topology (Univ. of Illinois at Chicago Circle, Chicago, Ill., 1968)*, 134–160. *Univ. of Illinois at Chicago Circle, Chicago, Ill*, 1969 .
9. A. NEEMAN, Triangulated categories. *Princeton University Press, Annals of Math. Studies* 148.
10. C. A. WEIBEL, An introduction to homological algebra. *Cambridge Studies in Advanced Mathematics*, 38. *Cambridge University Press, Cambridge*, 1994.

(Received 5.02.2006)

Author’s address:

Max-Planck-Institut fuer Mathematik
 Vivatsgasse 7, D-53111 Bonn
 Germany
 E-mail: baues@mpim-bonn.mpg.de