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ON THE LACUNARITY OF TWO-ETA-PRODUCTS

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Dedicated to the memory of Prof. G. Lomadze

Abstract. We classify all lacunary modular forms corresponding to the two-eta-products $\eta^r(z)\eta^s(mz)$ for $m = 3, 4, 5$, where $r + s$ is even and $rs \neq 0$. We show that there are no lacunary non-cusp forms corresponding to the eta-product $\eta^r(z)\eta^s(mz)$, $m \geq 4$.

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1. Introduction

Consider the Dedekind η-function

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty}(1 - q^n), \quad q = e^{2\pi i z}, \quad z \in \mathbb{C}, \quad \operatorname{Im} z > 0$$

and consider the eta-product $f(z)$ defined by

$$f(z) := \prod_{\delta | N} \eta^{r_\delta}(\delta z), \quad \text{where} \quad N \geq 1 \quad \text{and} \quad r_\delta \in \mathbb{Z}. \tag{1}$$

This eta-product is a meromorphic modular form (holomorphic in the upper half-plane, but not necessarily at the cusps) of weight $k = \frac{1}{2} \sum\limits_{\delta|N} r_\delta$ with a multiplier system. A formal power series $x^\nu \sum\limits_{n=0}^{\infty} a(n)x^n$ is called *lacunary* if the arithmetic density of its non-zero coefficients is zero. In [12], Serre classified all lacunary even powers of the eta-function. Following this work, Gordon and Robins [4] considered the problem of classifying all lacunary eta-products of the form $\eta(z)^r\eta(mz)^s$ when $m = 2$.

In this paper, we classify all lacunary modular forms corresponding to the eta-products $\eta^r(z)\eta^s(mz)$ for $m = 3, 4, 5$, where $r + s$ is even and $rs \neq 0$. Further, for the non-cusp form part, we use the t-core partition theorem proved by Granville and Ono [6] to show the non-existence of lacunary non-cusp forms corresponding to the cases $m \geq 4$.

In this connection, it is relevant to quote the following comment of Serre [12]: "it seems to be enough numerical evidence to conjecture that no odd η-power except for η and η^3 is lacunary."

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In the light of Theorem 1.3 (as well as of Theorem 1.2), we observe that an affirmative answer to the following question of Gordon mentioned in [9] will immediately prove Serre's conjectural statement: "Is a half-integral weight modular form lacunary if and only if it is superlacunary?"

Before we state our main theorems, for the sake of completeness, we give below the earlier results.

Theorem 1.1 (Serre [12]). *Suppose that $r > 0$ is an even integer. Then $\eta^r(z)$ is lacunary if and only if $r = 2, 4, 6, 8, 10, 14$ or 26.*

Theorem 1.2 (Gordon and Robins [4]). *Suppose that $r + s$ is even and $rs \neq 0$. Then $\eta^r(z)\eta^s(2z)$ is lacunary if and only if (r, s) is one of the following 45 pairs:*

$k = 1 :$ $(1, 1)$, $(3, -1)$, $(-1, 3)$, $(4, -2)$, $(-2, 4)$;
$k = 2 :$ $(2, 2)$, $(3, 1)$, $(1, 3)$, $(5, -1)$, $(-1, 5)$, $(6, -2)$, $(-2, 6)$, $(7, -3)$, $(-3, 7)$;
$k = 3 :$ $(3, 3)$, $(4, 2)$, $(2, 4)$, $(5, 1)$, $(1, 5)$, $(7, -1)$, $(-1, 7)$, $(8, -2)$, $(-2, 8)$, $(9, -3)$, $(-3, 9)$, $(10, -4)$, $(-4, 10)$, $(11, -5)$, $(-5, 11)$;
$k = 5 :$ $(5, 5)$, $(7, 3)$, $(3, 7)$, $(14, -4)$, $(-4, 14)$, $(15, -5)$, $(-5, 15)$, $(16, -6)$, $(-6, 16)$, $(17, -7)$, $(-7, 17)$, $(18, -8)$, $(-8, 18)$, $(19, -9)$, $(-9, 19)$;
$k = 9 :$ $(9, 9)$.

We now state below the main theorems of this paper.

Theorem 1.3. *Let m be one of the integers $3, 4, 5$. Suppose that $r+s$ is even and $rs \neq 0$. Then $\eta^r(z)\eta^s(mz)$ is a lacunary cusp form if and only if (r, s) is one of the following pairs:*

m	(r, s)
3	$(1, 1)$, $(1, 3)$, $(3, 1)$, $(3, 3)$, $(-1, 5)$, $(5, -1)$, $(2, 2)$, $(-1, 9)$, $(9, -1)$, $(-2, 10)$, $(10, -2)$, $(-3, 11)$, $(11, -3)$, $(3, 5)$, $(5, 3)$, $(7, 7)$.
4	$(1, 1)$, $(1, 3)$, $(3, 1)$, $(3, 3)$, $(-1, 5)$, $(5, -1)$, $(2, 2)$, $(-1, 7)$, $(7, -1)$, $(5, 5)$.
5	$(1, 1)$, $(1, 3)$, $(3, 1)$, $(3, 3)$, $(-1, 7)$, $(7, -1)$, $(1, 5)$, $(5, 1)$, $(-1, 11)$, $(11, -1)$, $(5, 5)$.

Theorem 1.4. *Let $m \geq 3$ and $r, s \in \mathbb{Z}$ be integers with $rs \neq 0$. If the modular form corresponding to the eta-product $\eta^r(z)\eta^s(mz)$ is lacunary, it is a cusp form except when $m = 3$ and $(r, s) = (-1, 3)$ or $(3, -1)$.*

2. Preliminaries

For natural numbers k and N, let $M_k(\Gamma_0(N), \chi)$ denote the $\mathbb{C}$-vector space of modular forms of weight k and Nebentypus character χ on the congruence subgroup $\Gamma_0(N)$, holomorphic in the upper half-plane and at the cusps. If f belongs to $M_k(\Gamma_0(N),\chi)$, then it has a Fourier expansion at infinity given by

$$f(z)=\sum_{n=0}^{\infty}a_f(n)q^n.$$

The subspace of $M_k(\Gamma_0(N),\chi)$ consisting of cusp forms is denoted by $S_k(\Gamma_0(N),\chi)$. For a prime p, the p-th Hecke operator on $M_k(\Gamma_0(N),\chi)$ acts on f as follows:

$$f(z)|T(p):=\sum_{n=0}^{\infty}a_f(pn)\ q^n+\chi(p)p^{k-1}\sum_{n=0}^{\infty}a_f(n)\ q^{pn}. \tag{2}$$

We recall the following theorem by Gordon and Sinor.

Theorem 2.1 ([5])**.** *Let M be a natural number. Suppose that $f(z)$ is an eta-product defined by*

$$f(z)=\prod_{\delta|M}\eta^{r_\delta}(\delta z),\quad \text{where}\quad r_\delta\in\mathbb{Z}.$$

Assume $k:=\frac{1}{2}\sum_{\delta|M}r_\delta$ to be a positive integer and let

$$\frac{1}{24}\sum_{\delta|M}\delta\ r_\delta=\frac{c}{e},\qquad \frac{1}{24}\sum_{\delta|M}\frac{M}{\delta}r_\delta=\frac{c_0}{e_0},$$

where $\frac{c}{e}$ and $\frac{c_0}{e_0}$ are in lowest terms. Put $N=Mee_0$ and $F(z)=f(ez)$. Let χ be the character defined by

$$\chi(d):=\left(\frac{(-1)^k t}{d}\right),\quad t:=\prod_{\delta|M}\delta^{r_\delta}.$$

If for all positive divisors μ of N we have

$$\sum_{\delta|M}\frac{(\mu,\delta)^2}{\delta}r_\delta\ \geq 0, \tag{3}$$

then F belongs to $M_k(\Gamma_0(N),\chi)$. Further, if the above sum is strictly positive for all positive divisors μ of N, then $F(z)$ belongs to $S_k(\Gamma_0(N),\chi)$.

Let $f^{(m)}_{r,s}(z)=\eta^r(z)\eta^s(mz)$. We denote the corresponding modular form by $F^{(m)}_{r,s}(z)=f^{(m)}_{r,s}(ez)$. Also for a square-free integer d, we denote by χ_d the Kronecker character for $\mathbb{Q}(\sqrt{d})$. We list below the modular forms corresponding to the eta-products mentioned in Theorem 1.3 except for the pairs $(r,s)=(1,1)$, $(1,3)$, $(3,1)$, $(3,3)$.

m	$F_{r,s}^{(m)}$	Space
3	$\eta^{-1}(12z)\eta^5(36z),\ \eta^5(12z)\eta^{-1}(36z)$	$S_2(\Gamma_0(2^4\cdot 3^3),\chi_3)$
	$\eta^2(3z)\eta^2(9z)$	$S_2(\Gamma_0(3^3))$
	$\eta^{-1}(12z)\eta^9(36z),\ \eta^9(4z)\eta^{-1}(12z)$	$S_4(\Gamma_0(2^4\cdot 3^2),\chi_3)$
	$\eta^{-2}(6z)\eta^{10}(18z),\ \eta^{10}(6z)\eta^{-2}(18z)$	$S_4(\Gamma_0(2^2\cdot 3^3))$
	$\eta^{-3}(4z)\eta^{11}(12z),\ \eta^{11}(12z)\eta^{-3}(36z)$	$S_4(\Gamma_0(2^4\cdot 3^2),\chi_3)$
	$\eta^3(4z)\eta^5(12z),\ \eta^5(12z)\eta^3(36z)$	$S_4(\Gamma_0(2^4\cdot 3^2),\chi_3)$
	$\eta^7(6z)\eta^7(18z)$	$S_7(\Gamma_0(2^2\cdot 3^3),\chi_{-3})$
4	$\eta^{-1}(24z)\eta^5(96z),\ \eta^5(24z)\eta^{-1}(96z)$	$S_2(\Gamma_0(2^8\cdot 3^2))$
	$\eta^2(12z)\eta^2(48z)$	$S_2(\Gamma_0(2^6\cdot 3^2))$
	$\eta^{-1}(8z)\eta^7(32z),\ \eta^7(8z)\eta^{-1}(32z)$	$S_3(\Gamma_0(2^8),\chi_{-1})$
	$\eta^5(24z)\eta^5(96z)$	$S_5(\Gamma_0(2^8\cdot 3^2),\chi_{-1})$
5	$\eta^{-1}(12z)\eta^7(60z),\ \eta^7(12z)\eta^{-1}(60z)$	$S_3(\Gamma_0(2^4\cdot 3^2\cdot 5),\chi_{-5})$
	$\eta(12z)\eta^5(60z),\ \eta^5(12z)\eta(60z)$	$S_3(\Gamma_0(2^4\cdot 3^2\cdot 5),\chi_{-5})$
	$\eta^{-1}(4z)\eta^{11}(20z),\ \eta^{11}(4z)\eta^{-1}(20z)$	$S_5(\Gamma_0(2^4\cdot 5),\chi_{-5})$
	$\eta^5(4z)\eta^5(20z)$	$S_5(\Gamma_0(2^4\cdot 5),\chi_{-5})$

Hecke forms. Let K be a number field and $\mathcal{O}_K$ be its ring of integers. Also, let $\mathfrak{m}(\neq 0)$ be an ideal of $\mathcal{O}_K$. A Hecke character c of the exponent $(k-1)$ is a homomorphism from $I(\mathfrak{m})$, the group of fractional ideals prime to $\mathfrak{m}$ to $\mathbb{C}^\times$, such that $c(\mathfrak{a}) = \alpha^{k-1}$, where $\mathfrak{a} = \alpha\mathcal{O}_K$, α is totally positive, $\alpha \equiv 1 \pmod{\mathfrak{m}^\times}$ and $k \geq 1$. Now for an imaginary quadratic field K of the discriminant d, we define the Hecke form as follows:

$$\Phi_{K,c}(z) = \sum_{\substack{(\mathfrak{a},m)=\ 1,\\ \mathfrak{a}\ \text{integral}}} c(\mathfrak{a})q^{N(\mathfrak{a})}, \tag{4}$$

where $N(\mathfrak{a})$ is the norm of $\mathfrak{a}$.

Let N be a multiple of $|d|N(\mathfrak{m})$ and let $k \geq 2$. Then $\Phi_{K,c}(z)$ is a cusp form in $S_k(\Gamma_0(N), \chi)$, where $\chi_c(p) = \left(\frac{d}{p}\right)\frac{c(p\mathcal{O}_K)}{p^{k-1}}$ for all primes p not dividing N. For a given $k \geq 2$ and a Dirichlet character $\chi \pmod N$, the forms $\Phi_{K,c}(z)$ with $\chi_c = \chi$ span a subspace $S_{cm,k}(\Gamma_0(N), \chi)$ of $S_k(\Gamma_0(N), \chi)$. The elements of $S_{cm,k}(\Gamma_0(N), \chi)$ are called **CM** forms.

The following important theorems will be needed for our proof.

Theorem 2.2 (Serre [11])**.** *Suppose that* $F(z) = \sum_{n=1}^{\infty} a_F(n)q^n \in S_k(\Gamma_0(N), \chi)$ *with* $k \geq 2$ *and define*

$$M_F(t) := \#\left\{n \middle| \; 1 \leq n \leq t, \; a_F(n) \neq 0\right\}.$$

Then we have the following:

(i) *If* $F(z) \notin S_{cm,k}(\Gamma_0(N), \chi)$*, then*

$$M_F(t) \asymp t \quad \text{for } t \to \infty.$$

(ii) *If* $F(z) \in S_{cm,k}(\Gamma_0(N), \chi)$ *and* $F(z) \neq 0$*, then*

$$M_F(t) \asymp \frac{t}{\sqrt{\log t}} \quad \text{for } t \to \infty,$$

where $\phi(t) \asymp \psi(t)$ *means that* $\phi(t) = O(\psi(t))$ *and* $\psi(t) = O(\phi(t))$*.*

Remark 2.1. The above theorem implies that the cusp form $F(z)$ is lacunary if and only if it is a CM-form.

Theorem 2.3 (Ribet [10])**.** *Let* K *be an imaginary quadratic field and let* $\Phi_{K,c}(z)$ *be a Hecke form corresponding to the Hecke character* c *as in* (4)*. If a rational prime* p *is inert in* K*, then*

$$\Phi_{K,c}(z)\Big| T(p) = 0.$$

Next we define the concept of superlacunary series.

Definition (See [9])**.** A power series is called *superlacunary* if it has the form

$$G(z) = \sum_{n=-\infty}^{\infty} d(an^2 + bn + c)q^{an^2+bn+c},$$

where $a > 0$ and $a, b, c \in \mathbb{Z}$.

Remark 2.2. As observed by Ono and Robins [9, p. 1023], it is a fact that the product of two superlacunary series is a lacunary series.

3. Proof of Theorem 1.3

3.1. **Possible pairs of lacunary cusp forms.** It is known due to Serre [11] that all modular forms $F_{r,s}^{(m)}(z)$ belonging to $S_1(\Gamma_0(N),\chi)$ are lacunary, so we assume from now onwards that $k>1$. Using Theorem 2.1, we see that if $F_{r,s}^{(m)}(z)$ is a cusp form, then $r+ms>0$ and $s+mr>0$.

Let $a_{r,s}^m(n)$ be the Fourier coefficients of the eta-product

$$f_{r,s}^{(m)}(z)=q^{\frac{r+ms}{24}}\prod_{n=1}^{\infty}(1-q^n)^r(1-q^{mn})^s=q^{\frac{r+ms}{24}}\sum_{n=0}^{\infty}a_{r,s}^m(n)q^n.$$

As mentioned in Theorem 2.1, $F_{r,s}^{(m)}(z)=f_{r,s}^{(m)}(ez)$, where $\frac{r+ms}{24}=\frac{c}{e}$ and hence

$$F_{r,s}^{(m)}(z)=\sum_{n=0}^{\infty}a_{r,s}^m(n)q^{c+en}.$$

Now we consider some important lemmas which will help us to determine a possible list of lacunary cusp forms. These lemmas are similar to those considered in [4].

Lemma 3.1. *For $m=3,4,5$, let $F_{r,s}^{(m)}(z)$ be a lacunary cusp form of integer weight $k\geq 2$. Then*

$$F_{r,s}^{(m)}(z)\big|T(p)=0$$

for all primes $p\equiv p_m \pmod{N_m}$, where

$$p_m=\begin{cases}23 & \text{if } m=3,4,\\ 71 & \text{if } m=5\end{cases}\qquad \textit{and}\qquad N_m=\begin{cases}24 & \text{if } m=3,4,\\ 120 & \text{if } m=5.\end{cases}$$

Proof. Since $F_{r,s}^{(m)}$ is lacunary, then by Remark 2.1 $F_{r,s}^{(m)}$ is a CM form. Hence

$$F_{r,s}^{(m)}(z)=\sum_{v_m}a_{v_m}\Phi_{K_{v_m},c_{v_m}},$$

where $\Phi_{K_{v_m},c_{v_m}}$ are Hecke forms.

Now by Theorem 2.1, $F_{r,s}^{(m)}(z)$ is in $S_k(\Gamma_0(N),\chi)$, where

$$N=\begin{cases}2^{\alpha_1}3^{\alpha_2} & \text{for some } \alpha_1,\alpha_2 & \text{if } m=3,4,\\ 2^{\beta_1}3^{\beta_2}5^{\beta_3} & \text{for some } \beta_1,\beta_2,\beta_3 & \text{if } m=5.\end{cases}$$

Since the discriminant d_{v_m} of K_{v_m} divides N, the only possibilities for d_{v_m} are

$$d_{v_m}=\begin{cases}-3,-4,-8 \text{ or } -24 & \text{if } m=3,4,\\ -3,-4,-8,-15,-20,-24,-40 \text{ or } -120 & \text{if } m=5.\end{cases}\tag{5}$$

Now the corresponding primes $p\equiv p_m \pmod{N_m}$ are inert in all the respective imaginary quadratic fields. Then the lemma follows from Theorem 2.3. □

Lemma 3.2. *Let $m=3,4,5$ and suppose that $F_{r,s}^{(m)}(z)\big|T(p_m)=0$, where p_m is defined as before.*

(i) *If* $r+ms \geq 3$, *then* $a^m_{r,s}(\ell) = a^m_{r,s}(\ell + p_m)$, *where* $0 \leq \ell \leq p_m - 1$ *and* $24\ell \equiv -(r+ms) \pmod{p_m}$.

(ii) *If* $r+ms = 2$, *then*

$$a^3_{r,s}(21) = 0 = a^4_{r,s}(21); \quad a^5_{r,s}(65) = 0.$$

(iii) *If* $r+ms = 1$, *then*

$$a^3_{r,s}(45) = 0 = a^4_{r,s}(45); \quad a^5_{r,s}(139) = 0.$$

Proof. Let $G(z) = F^{(m)}_{r,s}(24z)$. Then $G(z) \in S_k(\Gamma_0(24N), \chi)$ is lacunary if and only if $F^{(m)}_{r,s}(z)$ is lacunary. Since $F^{(m)}_{r,s}(z)|T(p_m) = 0$, it follows that $G(z)|T(p_m) = 0$. Hence by applying the Hecke operator $T(p_m) = U(p_m) + \chi(p_m)p_m^{k-1}B(p_m)$ to $G(z)$, we get

$$G(z)|T(p_m) = \sum_{e(r+ms+24n)\equiv 0 \pmod{p_m}} a^m_{r,s}(n) q^{e(r+ms+24n)/p_m}$$
$$+\chi(p_m){p_m}^{k-1} \sum_{n=0}^{\infty} a^m_{r,s}(n) q^{ep_m(r+ms+24n)}.$$

The first term in $G(z)|U(p_m)$ is $a^m_{r,s}(\ell) q^{[e(r+ms+24\ell)]/p_m}$, where ℓ is the least non-negative integer such that $e(r+ms+24\ell) \equiv 0 \pmod{p_m}$. Since $(e, p_m) = 1$, we have $24\ell \equiv -(r+ms) \pmod{p_m}$ and $0 \leq \ell \leq p_m - 1$.

(i) In this case $r+ms \geq 3$ and so

$$\ell + p_m \leq 2p_m - 1 < (p_m - 1)(r+ms).$$

Further, since $(p_m - 1)(p_m - 23) \geq 0$, we have

$$\frac{r+ms+24(\ell+p_m)}{p_m} < \frac{(r+ms)(1+24(p_m-1))}{p_m} \leq p_m(r+ms).$$

Thus the first two terms in $G(z)|U(p_m)$ appear before the first term in $G(z)|B(p_m)$, proving (i).

(ii) In this case $r+ms = 2$, hence

$$\ell = \begin{cases} 21 & \text{if } m = 3, 4, \\ 65 & \text{if } m = 5. \end{cases}$$

Since

$$\frac{(r+ms)+24\ell}{p_m} < p_m(r+ms),$$

the first term in $G(z)|U(p_m)$ appears before the first term in $G(z)|B(p_m)$, proving (ii).

(iii) In this case $r+ms = 1$, hence

$$\ell = \begin{cases} 22 & \text{if } m = 3, 4, \\ 68 & \text{if } m = 5. \end{cases}$$

Therefore

$$G(z)|T(p_m) = (a^m_{r,s}(\ell) + \chi(p_m){p_m}^{k-1})q^{ep_m} + a^m_{r,s}(\ell + p_m)q^{47e} + \cdots ,$$

which proves (iii). □

Now we define

$$b^3_{r,s}(l) = \begin{cases} a^3_{r,s}(l), & \text{if } l \equiv 0 \pmod 3, \\ a^3_{r,s}(l)/r, & \text{if } l \equiv 1 \pmod 3, \\ a^3_{r,s}(l)/r(r-3), & \text{if } l \equiv 2 \pmod 3, \end{cases}$$

$$b^4_{r,s}(l) = \begin{cases} a^4_{r,s}(l), & \text{if } l \equiv 0 \pmod 4, \\ a^4_{r,s}(l)/r, & \text{if } l \equiv 1, 2 \pmod 4, \\ a^4_{r,s}(l)/r(r-8), & \text{if } l \equiv 3 \pmod 4, \end{cases}$$

$$b^5_{r,s}(l) = \begin{cases} a^5_{r,s}(l), & \text{if } l \equiv 0 \pmod 5, \\ a^5_{r,s}(l)/r, & \text{if } l \equiv 1 \pmod 5, \\ a^5_{r,s}(l)/r(r-3), & \text{if } l \equiv 2 \pmod 5, \\ a^5_{r,s}(l)/r(r-1), & \text{if } l \equiv 3 \pmod 5, \\ a^5_{r,s}(l)/r(r-1)(r-3), & \text{if } l \equiv 4 \pmod 5, \end{cases}$$

except for $l = 9$, in which case we define

$$b^5_{r,s}(9) = a^5_{r,s}(9)/r(r-1)(r-3)(r-14)$$

Using MAPLE, we see that the polynomials $b^3_{r,s}(l)$, $b^4_{r,s}(l)$ for $0 \le l \le 45$ and $b^5_{r,s}(l)$ for $0 \le l \le 139$ are irreducible in $\mathbb{C}[r, s]$. Hence the curves C^m_l defined by

$$C^m_l = \{(r, s) \in \mathbb{C}^2 : b^m_{r,s}(l) = 0\}$$

are also irreducible.

Combining Lemma 3.1 and Lemma 3.2 and noting that $f^{(m)}_{r,s}(z)$ is lacunary if and only if $F^{(m)}_{r,s}(z)$ is lacunary, we get the following lemma.

Lemma 3.3. *Let $f^{(m)}_{r,s}(z)$ be lacunary.*

(i) *If $r + ms \ge 3$, then (r, s) is in the intersection of the curves C^m_l and C^m_{l+23} for some l with $0 \le l \le p_m - 1$.*

(ii) *If $r + ms = 2$, then (r, s) is on the curve*

$$b^3_{2-3s,s}(21) = 0 = b^4_{2-4s,s}(21); \quad b^5_{2-5s,s}(65) = 0.$$

(iii) *If $r + ms = 1$, then (r, s) is on the curve*

$$b^3_{1-3s,s}(45) = 0 = b^4_{1-4s,s}(45); \quad b^5_{1-5s,s}(139) = 0.$$

Since the curves C^m_l, $2 \le l \le 45$ for $m = 3, 4$ and $2 \le l \le 139$ for $m = 5$ are irreducible and distinct, by Bezout's theorem, there are only finitely many points in the intersection of two curves (see [7]). In fact, using the resultant one can explicitly find the points satisfying Lemma 3.3. This reduces to the possible pairs (r, s) given in Theorem 1.3.

3.2. **Lacunarity of the listed cases.** First we note that the lacunarity for a pair (r,s) implies the lacunarity of the pair (s,r). For, if χ is a quadratic character, then the Fricke involution

$$f(z) \to N^{-\frac{k}{2}} z^{-k} f\left(\frac{-1}{Nz}\right)$$

preserves the space $M_k(\Gamma_0(N),\chi)$. Further this involution commutes with the Hecke operators $T(p)$ for $(p,N) = 1$. Finally, using the functional equation

$$\eta\left(\frac{-1}{z}\right) = \sqrt{\frac{z}{i}}\eta(z)\ , \tag{6}$$

we see that $\eta^r(z)\eta^s(mz)$ is lacunary if and only if $\eta^s(z)\eta^r(mz)$ is lacunary.

Case (i): For any integer $m \geq 2$, the eta-products $\eta^r(z)\eta^s(mz)$ corresponding to the pairs $(1,1)$, $(1,3)$, $(3,1)$, $(3,3)$ are lacunary. Indeed, the identities:

(Euler) $$\eta(z) = q^{1/24}\sum_{-\infty}^{\infty}(-1)^n q^{(3n^2+n)/2},$$

(Jacobi) $$\eta^3(z) = q^{1/8}\sum_{0}^{\infty}(-1)^n(2n+1)q^{(n^2+n)/2}$$

show that η and η^3 are superlacunary and our observation follows from Remark 2.2.

We recall [1, p. 229] that two cusp forms belonging to $S_k(\Gamma_0(N),\chi)$ are equal if their Fourier expansions agree up to

$$1 + \frac{Nk}{12}\prod_{p|N}\left(1+\frac{1}{p}\right). \tag{7}$$

This would be relevant in proving the lacunarity of many of the listed pairs. Also since the lacunarity of $F_{r,s}^{(m)}(z)$ implies the lacunarity of $f_{r,s}^{(m)}(z)$, we prove the lacunarity of $F_{r,s}^{(m)}(z)$ by showing it is a CM-form.

Case (ii): $m = 3$**.** Gordon and Hughes proved that the eta-product for the pair $(-1,5)$ is lacunary in [3]. For the pair $(2,2)$, Dummit, Kisilevsky and McKay [2] proved that the modular form $\eta^2(3z)\eta^2(9z)$ is a CM-form and hence is lacunary. They obtained these forms as linear combinations of Hecke forms arising from the imaginary quadratic field $\mathbb{Q}(i\sqrt{3})$. Recently, S. Ahlgren [1] proved that the modular forms $F_{r,s}^{(3)}$ for the pairs $(-1,9)$, $(-2,10)$, $(-3,11)$ are lacunary by expressing them explicitly as linear combinations of Hecke forms arising from the fields $\mathbb{Q}(i)$ and $\mathbb{Q}(i\sqrt{3})$. The lacunarity of the case $(3,5)$ was dealt with Gordon and Hughes in [3]. So we omit the proof.

For the case $(7,7)$, we consider $F_{7,7}^{(3)}(z) = \eta^7(6z)\eta^7(18z) \in S_7(\Gamma_0(2^2.3^3),\chi_{-3})$. We show that $F_{7,7}^{(3)}(z)$ is a linear combination of Hecke forms arising from the field $\mathbb{Q}(i\sqrt{3})$. Consider the ideal $\mathfrak{m} = 6\mathcal{O}_K$ in $\mathcal{O}_K = \mathbb{Z}[\zeta]$, where $\zeta = \frac{1}{2} + \frac{i\sqrt{3}}{2}$.

Then $|(\mathcal{O}_K/6\mathcal{O}_K)^\times| = 18$; indeed, if $\alpha \in \mathcal{O}_K$ is prime to 6, then we have

$$\alpha \equiv \zeta^a \pmod{3\mathcal{O}_K}, \quad a \in \mathbb{Z}/6\mathbb{Z},$$
$$\alpha \equiv (\zeta^2)^b \pmod{2\mathcal{O}_K}, \quad b \in \mathbb{Z}/3\mathbb{Z}.$$

We can define Hecke characters c_1 and c_2 by

$$c_1(\alpha\mathcal{O}_K) := (-1)^a\zeta^{a+4b}\alpha^6; \qquad c_2(\alpha\mathcal{O}_K) := (-1)^a\zeta^{5a+2b}\alpha^6$$

since the left-hand sides are equal to 1 for $\alpha = \pm 1, \zeta, \zeta^2, \zeta^4, \zeta^5$.

Let

$$\begin{aligned}\varphi_{K,c_1}(z) &= q + 683q^7 - 4033q^{13} + 12851q^{19} + 15625q^{25} + \cdots \\ &\quad + 1058147q^{103} - 2172742q^{109} + 1771561q^{121} - 3952366q^{127} + \cdots, \\ \varphi_{K,c_2}(z) &= q - 397q^7 + 3527q^{13} - 2269q^{19} + 15625q^{25} + \cdots \\ &\quad - 2185093q^{103} - 2172742q^{109} + 1771561q^{121} - 3952366q^{127} + \cdots\end{aligned}$$

be the corresponding Hecke forms as defined in (4).

Since $F^{(3)}_{7,7}(z), \varphi_{K,c_1}(z),\ \varphi_{K,c_2}(z) \in S_7(\Gamma_0(2^2.3^3), \chi_{-3})$, using condition (7), we get the identity

$$\eta^7(6z)\eta^7(18z) = \frac{1}{1080}\left(\varphi_{K,c_1}(z) - \varphi_{K,c_2}(z)\right).$$

This shows that $F^{(3)}_{7,7}(z)$ is lacunary.

Case (iii): $m = 4$**.** The pairs $(5, -1)$, $(2, 2)$ can be written as follows:

$$\eta^5(z)\eta^{-1}(4z) = (\eta^5(z)\eta^{-2}(2z))(\eta^2(2z)\eta^{-1}(4z)),$$
$$\eta^2(z)\eta^2(4z) = \eta(2z)(\eta^2(z)\eta^{-1}(2z)\eta^2(4z)).$$

Since $\eta(z)$, $\eta^5(z)\eta^{-2}(2z)$, $\eta^2(z)\eta^{-1}(2z)$, $\eta^2(z)\eta^{-1}(2z)\eta^2(4z)$ are superlacunary forms [9], the forms $\eta^5(z)\eta^{-1}(4z)$ and $\eta^2(z)\eta^2(4z)$ are lacunary. The lacunarity of the eta-products for the pairs $(-1, 7)$, $(5, 5)$ was proved by Ahlgren [1].

Case (iv): $m = 5$**.** To prove the lacunarity of the eta-products for the pairs $(7, -1)$, $(1, 5)$, we express the modular forms $F^{(5)}_{7,-1}(z) = \eta^7(12z)\eta^{-1}(60z) \in S_3(\Gamma_0(2^4 \cdot 3^2 \cdot 5), \chi_{-5})$ and $F^{(5)}_{1,5}(z) = \eta(12z)\eta^5(60z) \in S_3(\Gamma_0(2^4 \cdot 3^2 \cdot 5), \chi_{-5})$ as linear combinations of Hecke forms arising from the field $\mathbb{Q}(i)$.

Let $K = \mathbb{Q}(i)$. Consider the ideals $\mathfrak{m}_1 = 6\mathcal{P}_1\mathcal{O}_K$ and $\mathfrak{m}_2 = 6\mathcal{P}_2\mathcal{O}_K$ in $\mathcal{O}_K = \mathbb{Z}[i]$, where $\mathcal{P}_1$ is generated by $(2+i)$ and $\mathcal{P}_2$ is generated by $(2-i)$. Then $|(\mathcal{O}_K/2\mathcal{P}_j)^\times| = 64$, $j = 1, 2$; indeed, if $\mathfrak{a}$ is an ideal prime to $6\mathcal{P}_j\mathcal{O}_K$ and generated by α, then we have

$$\alpha \equiv i^a \pmod{2\mathcal{O}_K}, \quad a \in \mathbb{Z}/2\mathbb{Z},$$
$$\alpha \equiv (1 \mp i)^{b+c} \pmod{3\mathcal{P}_j\mathcal{O}_K}, \quad b \in \mathbb{Z}/8\mathbb{Z}, \quad c \in \mathbb{Z}/4\mathbb{Z}, \quad j = 1, 2.$$

We can define Hecke characters $c_1^{\pm}$ and $c_2^{\pm}$ by

$$c_j^{\pm}(\mathfrak{a}) := (-1)^{a+c}(\pm i)^b\alpha^2, \quad j = 1, 2$$

since the left-hand side is equal to 1 for $\alpha = \pm 1, \pm i$.

Let

$$\begin{aligned}\varphi_{K,c_1^+}(z) &= q + (4+3i)q^5 - 24i\ q^{13} - 30i\ q^{17} + (7+24i)q^{25} + 40q^{29} + \cdots \\ &\quad - 782q^{409} + 58q^{421} + (720-210i)q^{425} - 816i\ q^{433} + \cdots\,, \\ \varphi_{K,c_1^-}(z) &= q - (4+3i)q^5 - 24i\ q^{13} + 30i\ q^{17} + (7+24i)q^{25} - 40q^{29} + \cdots \\ &\quad - 782q^{409} + 58q^{421} - (720-210i)q^{425} - 816i\ q^{433} + \cdots\,, \\ \varphi_{K,c_2^+}(z) &= q - (4-3i)q^5 + 24i\ q^{13} - 30i\ q^{17} + (7-24i)q^{25} - 40q^{29} + \cdots \\ &\quad - 782q^{409} + 58q^{421} - (720+210i)q^{425} + 816i\ q^{433} + \cdots\,, \\ \varphi_{K,c_2^-}(z) &= q + (4-3i)q^5 + 24i\ q^{13} + 30i\ q^{17} + (7-24i)q^{25} + 40q^{29} + \cdots \\ &\quad - 782q^{409} + 58q^{421} + (720+210i)q^{425} + 816i\ q^{433} + \cdots\end{aligned}$$

be the corresponding Hecke forms $\in S_3(\Gamma_0(2^4 \cdot 3^2 \cdot 5), \chi_{-5})$.

By checking a sufficient number of coefficients, we get the following identities:

$$\begin{aligned}\eta(12z)^7\eta^{-1}(60z) &= \frac{1}{4}\left\{\Big(\varphi_{K,c_1^+}(z) + \varphi_{K,c_1^-}(z)\Big) + \Big(\varphi_{K,c_2^+}(z) + \varphi_{K,c_2^-}(z)\Big)\right\} \\ &\quad + \frac{7}{96i}\left\{\Big(\varphi_{K,c_1^+}(z) + \varphi_{K,c_1^-}(z)\Big) - \Big(\varphi_{K,c_2^+}(z) + \varphi_{K,c_2^-}(z)\Big)\right\}, \\ \eta(12z)\eta^5(60z) &= \frac{-1}{96i}\left\{\Big(\varphi_{K,c_1^+}(z) + \varphi_{K,c_1^-}(z)\Big) - \Big(\varphi_{K,c_2^+}(z) + \varphi_{K,c_2^-}(z)\Big)\right\}.\end{aligned}$$

Hence the forms $F^{(5)}_{7,-1}(z)$ and $F^{(5)}_{1,5}(z)$ are lacunary.

To prove the lacunarity of the eta-product for the pair $(-1, 11)$, we consider the modular form $F^{(5)}_{-1,11}(z) = \eta^{-1}(4z)\eta^{11}(20z)$ in $S_5(\Gamma_0(2^4 \cdot 5), \chi_{-5})$. We show that $F^{(5)}_{-1,11}(z)$ is a linear combination of Hecke forms arising from the fields $\mathbb{Q}(i)$ and $\mathbb{Q}(i\sqrt{5})$.

Let $K = \mathbb{Q}(i)$. Consider the ideals $\mathfrak{m}_1 = 2\mathcal{P}_1\mathcal{O}_K$ and $\mathfrak{m}_2 = 2\mathcal{P}_2\mathcal{O}_K$ in $\mathcal{O}_K = \mathbb{Z}[i]$, where $\mathcal{P}_1$ is generated by $(2+i)$ and $\mathcal{P}_2$ is generated by $(2-i)$. Then $|(\mathcal{O}_K/2\mathcal{P}_j)^\times| = 8$, $j = 1, 2$; indeed, if $\mathfrak{a}$ is an ideal prime to $2\mathcal{P}_j\mathcal{O}_K$ and generated by α, then we have

$$\begin{aligned}&\alpha \equiv i^a \pmod{2\mathcal{O}_K}, \quad a \in \mathbb{Z}/2\mathbb{Z}, \\ &\alpha \equiv (1 \mp i)^b \pmod{\mathcal{P}_j O_K}, \quad b \in \mathbb{Z}/4\mathbb{Z}, \quad j = 1, 2.\end{aligned}$$

We can define Hecke characters c_1 and c_2 by

$$c_j(\mathfrak{a}) := (-1)^{a+b}\alpha^4, \quad j = 1, 2,$$

since the left-hand side is equal to 1 for $\alpha = \pm 1, \pm i$.

Let

$$\begin{aligned}\varphi_{K,c_1}(z) &= q + (7+24i)q^5 - 81q^9 + 240i\ q^{13} + 480i\ q^{17} + \cdots \\ &\quad + (-567-1944i)q^{45} - 2401q^{49} - 5040i\ q^{53} + 6958q^{61} + \cdots\,, \\ \varphi_{K,c_2}(z) &= q - (-7+24i)q^5 - 81q^9 - 240i\ q^{13} - 480i\ q^{17} + \cdots \\ &\quad + (-567+1944i)q^{45} - 2401q^{49} + 5040i\ q^{53} + 6958q^{61} + \cdots\end{aligned}$$

be the corresponding Hecke forms in $S_5(\Gamma_0(2^4\cdot 5),\chi_{-5})$.

Now, let $L=\mathbb{Q}(i\sqrt{5})$. Consider the ideal $\mathfrak{m}=2\mathcal{O}_L$ in $\mathcal{O}_L=\mathbb{Z}[i\sqrt{5}]$. In this case, every principal ideal $\mathfrak{a}$ prime to $\mathfrak{m}$ is generated by an element $a+bi\sqrt{5}$ with a and b of opposite parity. Define $c(\mathfrak{a})=(-1)^{a+1}(a+bi\sqrt{5})^4$. The class group of L is of order two and it is generated by the class of $\mathcal{I}$, a prime ideal dividing $3\mathcal{O}_L$. We extend c to a Hecke character of L by defining $c(\mathcal{I})$ to be a square root of the known value $c(\mathcal{I}^2)$. These choices would give two extensions $c^{\pm}$ to non-principal ideals. Note that if $\mathfrak{a}$ is non-principal, then $c^+(\mathfrak{a})+c^-(\mathfrak{a})=0$. If we denote the corresponding Hecke forms in $S_5(\Gamma_0(2^4\cdot 5),\chi_{-5})$ by $\varphi_{L,c^+},\varphi_{L,c^-}$, then we have

$$\begin{aligned}\varphi_{L,c^+}(z)+\varphi_{L,c^-}(z)&=2q-50q^5+478q^9+1920q^{21}+1250q^{25}-2396q^{29}\\&\quad-964q^{41}-11950q^{45}+958q^{49}+8156q^{61}+\cdots.\end{aligned}$$

By checking a sufficient number of coefficients, one has

$$\begin{aligned}\eta^{-1}(4z)\eta^{11}(20z)&=\frac{1}{640}\left\{(\varphi_{L,c^+}(z)+\varphi_{L,c^-}(z))-(\varphi_{K,c_1}(z)+\varphi_{K,c_2}(z))\right\}\\&\quad-\frac{i}{480}\left(\varphi_{K,c_1}(z)-\varphi_{K,c_2}(z)\right).\end{aligned}$$

This shows the lacunarity of the eta-product for the pair $(-1,11)$.

Next consider the pair $(5,5)$. To prove the lacunarity of the eta-product for the pair $(5,5)$, we express the modular form $F_{5,5}^{(5)}(z)=\eta^5(4z)\eta^5(20z)$ in $S_5(\Gamma_0(2^4\cdot 5),\chi_{-5})$ as a linear combination of Hecke forms arising from the fields $\mathbb{Q}(i)$ and $\mathbb{Q}(i\sqrt{5})$.

Define Hecke forms $\varphi_{K,c_1},\varphi_{K,c_1},\varphi_{L,c^\pm}$ as above. Then the following identity can be checked easily:

$$\begin{aligned}\eta^{5}(4z)\eta^{5}(20z)&=\frac{1}{128}\left\{(\varphi_{K,c_1}(z)+\varphi_{K,c_2}(z))-(\varphi_{L,c^+}(z)+\varphi_{L,c^-}(z))\right\}\\&\quad-\frac{i}{96}\left(\varphi_{K,c_1}(z)-\varphi_{K,c_2}(z)\right).\end{aligned}$$

This shows that $F_{5,5}^{(5)}(z)$ is lacunary.

Thus we have completed proving the lacunarity of all the listed cases.

4. Proof of Theorem 1.4

Note that by Theorem 2.1, the eta-quotient $\eta^r(z)\eta^s(mz)$ is a modular form but not a cusp form if and only if $dr+s=0$ or $r+ds=0$ for some positive divisor $d\neq 1$ of m. If $dr+s=0$, then $(d-1)$ divides $r+s$ and we have $r=-(r+s)/(d-1)$ and $s=-dr$. Writing the eta-quotient $\eta^r(z)\eta^s(mz)$ by (r,s), we have

$$(r,s)=\left(-\frac{(r+s)}{d-1},d\frac{(r+s)}{d-1}\right)=(-1,d)^{\frac{(r+s)}{d-1}}. \qquad (8)$$

Otherwise $r + ds = 0$, so arguing as before, we get

$$(r, s) = \left(d\frac{(r+s)}{d-1}, -\frac{(r+s)}{d-1} \right) = (d, -1)^{\frac{(r+s)}{d-1}}. \tag{9}$$

Since the lacunarity of the pair $(-1, d)^{\frac{(r+s)}{d-1}}$ is equivalent to the lacunarity of the pair $(d, -1)^{\frac{(r+s)}{d-1}}$, we will consider only the pair $(-1, d)^{\frac{(r+s)}{d-1}}$. Now using the t-core partition theorem proved by Granville and Ono, we will show that for any integer $m \geq 4$, $f_{r,s}^{(m)}(z)$ can not be a lacunary non-cusp form.

Theorem 4.1 (Granville–Ono [6, Theorem 1])**.** *Let*

$$\prod_{n \geq 1} (1 - q^n)^{-1} (1 - q^{tn})^t = \sum_{n \geq 0} c_t(n) q^n. \tag{10}$$

If $t \geq 4$, then $c_t(n) > 0$ for all n.

By taking $t = m$ in Theorem 4.1, we see that the product $(r, s) = (-1, m)$ is non-lacunary with positive coefficients if $m \geq 4$. Then any positive power of $(-1, m)$ also has positive Fourier coefficients and hence is non-lacunary. Now let $m \geq 4$ and let $d \neq 1$ be any divisor of m. Then

$$(-1, d) = q^{\nu} \prod_{n \geq 1} (1 - q^n)^{-1} (1 - q^{mn})^d = (-1, m) \prod_{n \geq 1} (1 - q^{mn})^{d-m}. \tag{11}$$

Since $d - m$ is negative (if $d \neq m$), the series $\prod_{n \geq 1} (1 - q^{mn})^{d-m}$ has positive coefficients. So multiplying by a non-lacunary series with positive coefficients will give again a non-lacunary series. Hence $(-1, d)$ is non-lacunary. Then any positive power of $(-1, d)$ is also non-lacunary as it has positive Fourier coefficients. This completes the proof of the first part of Theorem 1.4. In order to complete the proof, we have to show that when $m = 3$, $f_{r,s}^{(m)}(z)$ is a lacunary non-cusp form if and only if $(r, s) = (-1, 3)$ or $(3, -1)$. This follows from the next lemma.

Lemma 4.2. *The eta-product $\left(f_{-1,3}^{(3)}(z) \right)^t$ is lacunary if and only if $t = 1$.*

Proof. In [3, p. 427], Gordon and Hughes proved that $F_{-1,3}^{(3)}(z)$ is a lacunary form and showed it as a Hecke form. Further they showed that $\left(F_{-1,3}^{(3)}(z) \right)^2$ is a non-lacunary form. Since $F_{-1,3}^{(3)}(z)$ is equal to the Hecke form corresponding to the trivial character, it has non-negative Fourier coefficients. Hence $f_{-1,3}^{(3)}(z)$ also has non-negative Fourier coefficients. Hence $\left(f_{-1,3}^{(3)}(z) \right)^t$ cannot be lacunary for $t > 1$. □

4.1. **Another proof of Theorem 1.4 for the case** $m = 4$**.** Using (8), (9), we see that the eta-product $\eta^r(z)\eta^s(4z)$ is a non-cusp form if and only if $(r, s) =$

$(-2,8)^{m_1}$ or $(r,s)=(8,-2)^{m_1}$, where $m_1 \geq 1$. We show that $\left(f^{(4)}_{-2,8}(z)\right)^{m_1}$ is never a lacunary non-cusp form.

Consider the product

$$\prod_{n\geq 1}(1-q^n)^{-2}(1-q^{4n})^8 = H(z)^2H(2z)^4, \tag{12}$$

where

$$H(z)=\prod_{n\geq 1}(1-q^n)^{-1}(1-q^{2n})^2=\sum_{n\geq 0}q^{(n^2+n)/2}. \tag{13}$$

From the definition, it is clear that $H(z)$ is a super-lacunary form with non-negative coefficients and hence $H(z)^2$ is a lacunary form with non-negative coefficients. Let

$$H(z)^k=\sum_{n\geq 0}t_k(n)q^n, \tag{14}$$

where $t_k(n)$ is the number of representations of n as a sum of k triangular numbers. It is well known that $t_k(n)>0$ for all n when $k\geq 3$. This shows that $H(z)^4$ is non-lacunary with positive coefficients and hence $(-2,8)=H(z)^2H(2z)^4$ is also non-lacunary form with positive Fourier coefficients. Hence $(-2,8)^{m_1}$ is non-lacunary for all $m_1 \geq 1$.

5. Conjecture

In Subsection 3.2, we have observed that the two-eta-products $\eta^r(z)\eta^s(mz)$ corresponding to the pairs $(r,s)=(1,1),(1,3),(3,1),(3,3)$ are lacunary. Based on numerical calculations, it appears that there are only finitely many lacunary modular forms corresponding to the eta-products $\eta^r(z)\eta^s(mz)$, $r,s\notin\{1,3\}$.

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