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NEW ESTIMATES OF THE SINGULAR SERIES CORRESPONDING TO POSITIVE QUATERNARY QUADRATIC FORMS

GURAM GOGISHVILI

Abstract. Let $m \in \mathbb{N}$, f be a positive definite, integral, primitive, quaternary quadratic form of the determinant d and let $\rho(f,m)$ be the corresponding singular series.

When studying the best estimates for $\rho(f,m)$ with respect to d and m we proved in [4] that

$$\rho(f,m) = O(d^{-\frac{1}{3}} m \ln\ln b(dm)),$$

where $b(k)$ is the product of distinct prime factors of $16k$ if $k \neq 1$ and $b(k) = 3$ if $k = 1$.

The present paper proves a more precise estimate

$$\rho(f,m) = O(d_0^{-\frac{1}{3}} d_1^{-\frac{1}{2}} m \ln b(d_1) \ln\ln b(m)),$$

where $d = d_0 d_1$, $d = \prod\limits_{p|2^5 d} p^{h(p)}$, $d_0 = \prod\limits_{\substack{p|2^5 d \\ p|2m}} p^{h(p)}$, $d_1 = \prod\limits_{\substack{p|2^4 d \\ p\nmid m,\ p>2}} p^{h(p)}$, $h(p) \geqslant 0$ if $p > 2$; $h(2) \geqslant -4$.

The last estimate for $\rho(f,m)$ as a general result for quaternary quadratic forms of the above-mentioned type is unimprovable in a certain sense.

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1. Introduction

Let

$$f = \sum_{\alpha,\beta=1}^{4} a_{\alpha\beta} x_\alpha x_\beta \tag{1}$$

be any positive definite, integral, primitive, quaternary quadratic form of the determinant $d = d(f)$, so the $\gcd(a_{11}, a_{22}, a_{33}, a_{44}, 2a_{12}, \ldots, 2a_{34}) = 1$.

We consider the main term of formulas for the number of representations $r(f,m)$ of $m \in \mathbb{N}$ by f. The main term expressed by the so-called singular series $\rho(f,m)$ can be represented as an infinite product over all primes p

$$\rho(f,m) = \frac{\pi^2 m}{d^{\frac{1}{2}}} \prod_{p \geqslant 2} \chi(p). \tag{2}$$

The formulas for the $\chi(p)$ (even under more general assumptions) are obtained by Malyshev [6]. These formulas are simplified in some cases and represented in the convenient form in [1].

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The estimates of $\rho(f,m)$ with respect to d and m are important for the investigation of the asymptotic behavior of $r(f,m)$, determination of one-class genera of the forms (1), the existence of the so-called Gauss type formulas for $r(f,m)$ $(r(f,m)=\rho(f,m))$ and in other applications.

In the paper [6] some estimates of $\chi(p)$, $p\geqslant 2$, are given. They yield

$$\rho(f,m)=O(d^{\frac{1}{2}}m^{1+\varepsilon}) \tag{3}$$

for any $\varepsilon>0$.

Studying the representation of numbers by sums of squares, Rankin [7] estimated the corresponding $\rho(f,m)$. Some analogous results for a quaternary form of special type are obtained by Kiming [5]. In [2] we essentially improved the existing results and obtained

$$\rho(f,m)=O(d^{-\frac{1}{3}+\varepsilon_1}m^{1+\varepsilon_2}) \tag{4}$$

for any $\varepsilon_1>0,\varepsilon_2>0$ and calculated the constant in the "O-term". This constant depends only on ε_1 and ε_2.

The papers [3] and [4] give more precise estimates

$$\rho(f,m)=O(d^{-\frac{1}{3}}m\ln\ln d\ln\ln m) \tag{5}$$

and

$$\rho(f,m)=O(d^{-\frac{1}{3}}m\ln\ln b(dm)), \tag{6}$$

where $b(k)$ is the product of distinct prime factors of the number $16k$ if $k\neq 1$, and $b(k)=3$ if $k=1$.

The paper [4] gives an estimate for n-ary $(n\geqslant 5)$ quadratic forms too

$$\rho(f,m)=O(d^{-\frac{n-2}{2(n-1)}}m^{\frac{n}{2}-1}).$$

The present paper sharpens the result (6) and proves

$$\rho(f,m)=O(d_0^{-\frac{1}{3}}d_1^{-\frac{1}{2}}m\ln b(d_1)\ln\ln b(m)), \tag{7}$$

where $d_0d_1=d$, $d_0=\prod\limits_{\substack{p|2^5d\\ p|2m}}p^{h(p)}$, $d_1=\prod\limits_{\substack{p|2^4d\\ p\nmid m,\, p>2}}p^{h(p)}$, $h(p)\geqslant 0$ if $p>2$ and $h(2)\geqslant -4$.

The estimate (7) as a general result for quaternary quadratic forms of the above-mentioned type is unimprovable in a certain sense since the estimate $O(d_0^{-\frac{1}{3}}d_1^{-\frac{1}{2}}m)$ is not valid for any forms of such kind. An example of such extreme forms is constructed in [3].

2. Notation and Some Preliminary Results

It is known (cf., for example, [6]) that for any prime $p\geqslant 2$ and quadratic form (1) there exist integers e_α and quadratic forms ϕ_α, $\alpha=\overline{1,s}$, such that

$$f\equiv\sum_{\alpha=1}^{s}p^{e_\alpha}\phi_\alpha(\mod p^{e_s+3}),$$

where $-1 \leqslant e_1 < e_2 < \cdots < e_s$ (if $p = 2$, then any n_α-ary ϕ_α may be diagonal or of the type $\phi_\alpha = \sum_{\beta=1}^{n_\alpha/2} (2a'_{\alpha\beta}x^2_{\alpha\beta} + 2a''_{\alpha\beta}x_{\alpha\beta}y_{\alpha\beta} + 2a'''_{\alpha\beta}y^2_{\alpha\beta})$ and only then e_1 may be -1. If $p \neq 2$, then ϕ_α is diagonal and $p \nmid \det(\phi_\alpha)$, $\alpha = \overline{1, s}$).

Let p be any prime factor of d (more exactly, p be a factor of $2^4 d$, since d may be the number of type $2^{-4}d_*$ with d_* being an odd integer), $d = p^{h(p)}d_p$, $p \nmid d_p$, $m = p^w m_p$, $p \nmid m_p$, $w = w(p) \geqslant 0$. According to the formulas for $\chi(p)$ (cf., [6] or [1]) we obtain the estimates of $\chi(p)$, $p \mid 2^4 d$, $p > 2$. In all possible cases, for the representable m and the forms (1) we have

$$\chi(p) = 2 \quad \text{if} \quad w = 0, \quad n_1 = 1;$$

$$\chi(p) \leqslant 1 + \frac{1}{p} \quad \text{if} \quad w = 0, \quad n_1 > 1;$$

so

$$\chi(p) \leqslant 2 \quad \text{if} \quad w = 0. \tag{8}$$

An estimate for $w > 0$ is obtained in [2].

$$\chi(p) \leqslant p^{\frac{h(p)}{6}}(1 + p^{-2})(1 + p^{-1}) \quad \text{if} \quad w > 0. \tag{9}$$

From the formulas for $\chi(2)$ (cf., [2]) we obtain

$$\chi(2) \leqslant 2^{e_1} + \sum_{e_1 < t \leqslant w+2} 2^{t-2-\sum_{n=1}^{l(t)} n_\alpha(t-e_\alpha-1)/2+\nu(t)} + \begin{cases} 2^{w-B(w+2)/2+2.5-\nu(w+3)} & \text{if } 2 \nmid B(w+3), \\ 0 & \text{if } 2 \mid B(w+3), \end{cases}$$

where $\nu = \nu(t) = 1$ if $2 \mid \sum_{\alpha=1}^{l(t)} n_\alpha$; $\nu(t) = \frac{1}{2}$ if $2 \nmid \sum_{\alpha=1}^{l(t)} n_\alpha$,

$$B(t) = \sum_{\alpha=1}^{l(t)} n_\alpha(t - e_\alpha), \quad l(t) = \begin{cases} 0 & \text{if } t \leqslant e_1, \\ k & \text{if } e_k < t \leqslant e_{k+1}, \\ s & \text{if } t > e_s. \end{cases}$$

In a similar way as it was done in [2], from the last estimate we obtain

$$\chi(2) \leqslant 4 \cdot 2^{\frac{h(2)}{6}} \quad \text{if} \quad w \geqslant 0, \tag{10}$$

where $d = 2^{h(2)}d_2$, $h(2)$ and d_2 are integers, $h(2) \geqslant -4$, $2 \nmid d_2$.

3. Estimates of $\chi(p)$, $p \nmid 2^4 d$, $p > 2$

Let $m = p^w m_p$, $p \nmid m_p$ and

$$\delta = \left(\frac{d}{p}\right)$$

be the Jacobi symbol. The paper gives the formulas for the corresponding $\chi(p)$ in the above-mentioned case.

$$\chi(p) = (1-\delta p^{-2}) \sum_{0\leqslant t\leqslant w} \delta^t p^t.$$

It follows from the last formula that

$$\chi(p) \leqslant 1+p^{-2} \text{ if } p\nmid 2^5 dm, \tag{11}$$

$$\begin{aligned}\chi(p) &< (1-p^{-2})\sum_{t\geqslant 0} p^{-t} = (1-p^{-2})(1-p^{-1})^{-1} \\ &= 1+p^{-1} \quad \text{if} \quad p\nmid 2^5 d, \quad p\mid m.\end{aligned} \tag{12}$$

4. Estimate of $\rho(f,m)$

Now using (10), (8), (9), (11) and (12) we obtain

$$\begin{aligned}\prod_{p\geqslant 2}\chi(p) &= \chi_2 \prod_{\substack{p\mid 2^4 d\\ p\nmid 2m}}\chi(p) \prod_{\substack{p\mid 2^4 d\\ p\mid m,\, p>2}}\chi(p) \prod_{p\nmid 2^5 dm}\chi(p) \prod_{\substack{p\nmid 2^5 d\\ p\mid m}}\chi(p) \\ \leqslant 4\cdot 2^{\frac{h(2)}{6}} \prod_{\substack{p\mid 2^4 d\\ p\nmid 2m}} 2 \prod_{\substack{p\mid 2^4 d\\ p\mid m,\, p>2}} & p^{\frac{h(p)}{6}}(1+p^{-2})(1+p^{-1}) \prod_{p\nmid 2^5 dm}(1+p^{-2}) \prod_{\substack{p\nmid 2^5 d\\ p\mid m}}(1+p^{-1}) \\ &\leqslant 4\cdot d_0^{\frac{1}{6}} 2^{\sigma(d_1)} \prod_{p\geqslant 2}(1+p^{-2}) \prod_{p\mid m}(1+p^{-1}),\end{aligned} \tag{13}$$

where $\sigma(d_1)$ is the number of prime divisors of d_1.

It is obvious that $2^{\sigma(d_1)} < b(d_1)$, so $\sigma(d_1) = O(\ln b(d_1))$.

Let $m>1$ and $p_1,\dots,p_{\sigma(m)}$ be the first $\sigma(m)$ prime numbers, then using the well-known estimates (cf., for example, [8]) we obtain

$$\begin{aligned}\prod_{p\mid m}(1+p^{-1}) &\leqslant \prod_{2\leqslant p\leqslant p_{\sigma(m)}} (1+p^{-1}) = O(\ln p_{\sigma(m)}) \\ &= O(\ln(\sigma(m)\ln\sigma(m))) = O(\ln\sigma(m)) = O(\ln\ln b(m)).\end{aligned} \tag{14}$$

This result together with the estimate

$$\prod_{p\geqslant 2}(1+p^{-2}) \leqslant \sum_{a=1}^{\infty} a^{-2} = \frac{\pi^2}{6}, \tag{15}$$

lead us to the final result for the product

$$\prod_{p\geqslant 2}\chi(p) = O(d_0^{\frac{1}{6}} \ln b(d_1) \ln\ln b(m)). \tag{16}$$

Clearly, (2) and (16) give the desired result (7).

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Author's address:

Faculty of Exact and Natural Sciences
I. Javakhishvili Tbilisi State University
1, Chavchavadze Ave., Tbilisi 0128
Georgia
E-mail: guram@mzera.com