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QUADRATIC ENVELOPE OF THE CATEGORY OF CLASS TWO NILPOTENT GROUPS

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Dedicated to the blessed memory of Professor George Lomadze

Abstract. Using quadratic maps we define a category Niq with objects class two nilpotent groups. Hom's in this category are groups and we investigate functoriality properties of this group. The classification problem in Niq is reduced to a certain linear problem.

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Introduction

One of the distinguishing features of the theory of abelian groups is the fact that the set of homomorphisms between two abelian groups again forms an abelian group. This property is lost as soon as one moves out of the class. One can always consider the set of all maps, which is of course a group again, but the corresponding category becomes equivalent to the category of (nonempty) sets and therefore is not very interesting.

For nilpotent groups there is a less trivial possibility — to consider polynomial mappings [5]. These are closed under composition and valuewise addition of maps, but still form a very large class of maps compared to homomorphisms, and it is difficult to work with them.

In this paper we restrict attention to the class two nilpotent groups and show that there is a subclass of quadratic (i.e. degree two polynomial) maps called q-maps which turns out to be also closed under composition and valuewise addition. By definition, a map $f : G \to H$ is called a q-map if the expression $(x \mid y)_f = -(f(x) + f(y)) + f(x + y)$ lies in the commutator subgroup of H and is linear in x and y. In this way one obtains a category Niq whose objects are all class two nilpotent groups while the morphisms are all q-maps between them. Two nonisomorphic groups might become isomorphic when considered as objects of Niq, so that the classification problems (say, of finite groups) in Niq are easier (but still highly nontrivial) than the corresponding problems in the category of groups and homomorphisms.

We also indicate an approach to such classification questions using the notion of linear extension of categories from [3]. Namely, we construct several linear extensions connecting the category Niq to some simpler categories, among them

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some additive ones which might be susceptible to representation-theoretic classification methods. The point is that a linear extension of categories induces bijection on isomorphism classes of objects.

Here is a short description of the contents of the paper. The first section contains preliminaries on some standard facts about class two nilpotent groups. In Section 2 we present two variants of the notion of quadratic map for non-abelian groups and investigate some basic properties of such maps. Then in Section 3 we introduce the new class of maps that we call q-maps. Then in the central Section 4 using the q-maps we introduce the category Niq and give some of its features.

We then continue the study of Niq using linear extensions of categories. In the next Section 5 we recall this notion and exhibit the category Nil of class two nilpotent groups and homomorphisms as a linear extension of a simpler category $\mathsf{Nil}^{\sim}$ that up to equivalence can be described in terms of 2-cohomology classes of abelian groups. Then in Section 6 we do a similar thing with Niq in place of Nil; this time the simpler category $\mathsf{Niq}^{\sim}$ is even additive, unlike $\mathsf{Nil}^{\sim}$, and moreover is itself a linear extension of an even smaller additive category $\mathsf{Niq}^{\approx}$.

In Section 7 we introduce a particular class of nil_2-groups which we call q-split. This class seems to be the simplest nontrivial one admitting classification modulo isomorphism in Niq in terms of abelian groups.

In Section 8 we exhibit an analog of the notion of q-map and of the category Niq for Lie algebras and prove that in the uniquely 2-divisible situation the classical Maltsev correspondence between nil_2 groups and Lie algebras extends to q-maps. This fact has some consequences for the classification questions in view of further linear extensions on the Lie algebra side.

1. The Category Nil

The material in this section is well known and included for convenience of the reader and to compare with what follows next.

We fix some notation. Groups will be written additively. In particular, for elements $a, b \in G$ of a group G their commutator will be denoted by $[a, b] = -a - b + a + b$. We denote by $\mathsf{Z}(G)$ the center of G.

For any group G we denote by G^{ab} the abelianization of G, that is, the quotient

$$G^{\mathrm{ab}} := G/[G, G].$$

For an element $x \in G$ we let $\hat{x}$ denote the class of x in G^{ab}. For any abelian group A one denotes by $\Lambda^2(A)$ the *second exterior power* of A, which is the quotient of $A \otimes A$ by the subgroup generated by elements of the form $a \otimes a$, $a \in A$.

Recall that a group G is of *nilpotence class two*, or is a *nil*$_2$*-group*, if all triple commutators of G vanish, $[[G, G], G] = 0$.

We denote by Nil the category of groups of nilpotence class two and their homomorphisms.

We next recall some standard facts that we will need.

Lemma 1. *For any* $G \in \mathsf{Nil}$*:*

i) *There is a well-defined homomorphism* $\Lambda^2(G^{\mathrm{ab}}) \to G$ *given by* $\hat{a} \wedge \hat{b} \mapsto [a, b]$.
ii) *For any* $a, b \in G$ *one has* $[a, b] = a + b - a - b$.
iii) *One has the inclusion* $[G, G] \subset \mathsf{Z}(G)$.
iv) *For any* $a, b \in G$ *and any* $n \in \mathbf{Z}$ *one has*

$$na + nb = n(a + b) + \frac{n(n-1)}{2}[a, b].$$

The inclusion functor $\mathsf{Nil} \subset \mathsf{Groups}$ has a left adjoint given by

$$G \mapsto G^{\mathrm{nil}} := G/[[G, G], G].$$

Since left adjoints preserve all existing colimits, one can obtain coproducts in Nil as $(-)^{\mathrm{nil}}$ of coproducts in Groups. But in fact coproducts in Nil are much easier to construct directly than those in Groups. Namely, one easily checks

Proposition 1. *For two groups* G, H *let* $G \vee H$ *denote the set* $G^{\mathrm{ab}} \otimes H^{\mathrm{ab}} \times G \times H$. *The equalities*

$$\begin{aligned}(\xi, g, h) + (\xi', g', h') &= (\xi + \xi' - \hat{g}' \otimes \hat{h}, g + g', h + h'),\\ -(\xi, g, h) &= (-\xi - \hat{g} \otimes \hat{h}, -g, -h)\end{aligned}$$

equip this set with a group structure such that there is a central extension

$$0 \to G^{\mathrm{ab}} \otimes H^{\mathrm{ab}} \to G \vee H \to G \times H \to 0.$$

Moreover, if G *and* H *are nil*$_2$*-groups, then* $G \vee H$ *is also a nil*$_2$*-group and the maps* $i_G : G \to G \vee H$, $i_H : H \to G \vee H$ *given by* $i_G(g) = (0, g, 0)$ *and* $i_H(h) = (0, 0, h)$ *form a coproduct diagram in* Nil.

The forgetful functor $\mathsf{Nil} \to \mathsf{Sets}$ has a left adjoint, whose value on a set S is known as *the free nilpotent group of class two generated by* S and is denoted by $\mathbf{Z}_{\mathsf{Nil}}[S]$. If F_S is the free group spanned by S, then $\mathbf{Z}_{\mathsf{Nil}}[S] = (F_S)^{\mathrm{nil}}$. Moreover, since left adjoints preserve coproducts, and S is the coproduct of S copies of a singleton in Sets, one has

$$\mathbf{Z}_{\mathsf{Nil}}[S] = \bigvee_S \mathbf{Z}$$

in Nil.

As a consequence one can derive the simplest particular case of the famous result of Witt, which asserts that the graded Lie ring obtained by the lower central series of a free group is a free Lie ring:

Corollary 1. *For a free nil*$_2$*-group* G *one has the following central extension*

$$0 \to \Lambda^2(G^{\mathrm{ab}}) \to G \to G^{\mathrm{ab}} \to 0$$

2. Quadratic Maps between Nonabelian Groups

Our treatment of quadratic maps between groups is slightly different from that of [5] and is based on earlier work [1, 2]. We will begin with some facts concerning arbitrary groups; then starting from Lemma 6 all groups will be assumed nilpotent of class two.

Let G and H be arbitrary groups. Call a map $f : G \to H$ *weakly quadratic* if for any $a, b \in G$ the *cross-effect*

$$(a \mid b)_f := -(f(a) + f(b)) + f(a + b)$$

commutes with $f(c)$ for all $c \in G$ and is linear in a and b. Thus we have

$$f(a + b) = f(a) + f(b) + (a \mid b)_f,$$

and the equalities

$$\begin{aligned}(a_1 + a_2 \mid b)_f &= (a_1 \mid b)_f + (a_2 \mid b)_f,\\ (a \mid b_1 + b_2)_f &= (a \mid b_1)_f + (a \mid b_2)_f,\\ (a \mid b)_f + f(c) &= f(c) + (a \mid b)_f\end{aligned}$$

hold for any $a, a_1, a_2, b_1, b_2, b, c \in G$.

A weakly quadratic map $f : G \to H$ is *quadratic* [1] if in fact $(a \mid b)_f \in \mathsf{Z}(H)$ for all $a, b \in G$.

Obviously every weakly quadratic map to an abelian group is quadratic. We denote the set of all weakly quadratic maps from G to H by $\mathsf{wQuad}(G, H)$ and that of quadratic maps by $\mathsf{Quad}(G, H)$. It is clear that a map $f : G \to H$ is a homomorphism if and only if $(- \mid -)_f = 0$. Thus

$$\mathsf{Hom}(G, H) \subseteq \mathsf{Quad}(G, H) \subseteq \mathsf{wQuad}(G, H).$$

Lemma 2. *For $f \in \mathsf{wQuad}(G, H)$ the following assertions are true:*

i) *The cross-effect yields a well-defined homomorphism $(- \mid -)_f : G^{\mathrm{ab}} \otimes G^{\mathrm{ab}} \to H$.*

ii) *$f(0) = 0$.*

iii) *$f(-a) = -f(a) + (a \mid a)_f$.*

iv) *If $c \in [G, G]$, then for any $a \in G$ one has $f(a + c) = f(a) + f(c)$. In particular the restriction of f to the commutator subgroup is a homomorphism.*

v) *For any $a, b \in G$ one has*

$$f([a, b]) = -f(b + a) + f(a + b) = [f(a), f(b)] + (a \mid b)_f - (b \mid a)_f.$$

vi) *For any $a, b, c \in G$ one has $f([a, [b, c]]) = [f(a), [f(b), f(c)]]$.*

Proof. i) Since the elements $(a \mid b)_f$ commute with everything in the image of f, they centralize the subgroup generated by this image. But they belong to this subgroup themselves, so commute with each other. Thus for each $a \in G$ the map $(a \mid -)_f : G \to H$ is a homomorphism with abelian image, hence it factors trough G^{ab}. Similarly for $(- \mid b)_f$.

ii) $0 = (0 \mid 0)_f = -f(0) - f(0) + f(0) = f(0)$.

iii) By ii),

$$0 = f(0) = f(a - a) = f(a) + f(-a) + (a \mid -a)_f$$

and the statement follows.
iv) By i), $(a \mid c)_f = 0$.
v) We have

$$\begin{aligned} f([a,b]) &= f(-(b+a)+a+b) \\ &= f(-(b+a)) + f(a+b) + (-(b+a) \mid a+b)_f \\ &= -f(b+a) + (b+a \mid b+a)_f + f(a+b) + (-(b+a) \mid a+b)_f && \text{(by iii))} \\ &= -f(b+a) + f(a+b) && \text{(by i))} \\ &= -(f(b)+f(a)+(b \mid a)_f) + f(a) + f(b) + (a \mid b)_f \\ &= [f(a), f(b)] + (a \mid b)_f - (b \mid a)_f. \end{aligned}$$

vi) We have

$$\begin{aligned} f([a,[b,c]]) &= [f(a), f([b,c])] + (a \mid [b,c])_f - ([b,c] \mid a)_f && \text{(by v))} \\ &= [f(a), f([b,c])] && \text{(by i))} \\ &= [f(a), [f(b), f(c)] + (b \mid c)_f - (c \mid b)_f] && \text{(by v))} \\ &= [f(a), [f(b), f(c)]]. && \square \end{aligned}$$

Corollary 2. *Let $f : G \to H$ be a weakly quadratic map. If H is a nilpotent group of class two, then f factors through $G^{\mathrm{nil}} = G/[G,[G,G]]$. Thus*

$$\begin{aligned} \mathsf{wQuad}(G,H) &\cong \mathsf{wQuad}(G^{\mathrm{nil}}, H), \\ \mathsf{Quad}(G,H) &\cong \mathsf{Quad}(G^{\mathrm{nil}}, H). \end{aligned}$$

Proof. Indeed, if $c \in [G,[G,G]]$ then $f(c) = 0$ thanks to vi) of Lemma 2. Thus $f(a+c) = f(a)$ by iv) of Lemma 2. $\square$

The set of quadratic maps for nilpotent groups of class two has some remarkable properties. First of all, unlike $\mathsf{Hom}(G,H)$ or $\mathsf{wQuad}(G,H)$ the set $\mathsf{Quad}(G,H)$ is a group with respect to the valuewise addition of maps. This is the subject of the following Lemma.

Lemma 3. *Let G be a group and let H be a nilpotent group of class two. If the maps $f, g : G \to H$ are quadratic, then $f+g$ and $-f$ are also quadratic and*

$$\begin{aligned} (a \mid b)_{f+g} &= (a \mid b)_f + (a \mid b)_g + [f(b), g(a)], \\ (a \mid b)_{-f} &= [f(b), f(a)] - (a \mid b)_f. \end{aligned}$$

Proof. The above formulæ for $(- \mid -)_{f+g}$ and $(- \mid -)_{-f}$ can be easily checked. Since the commutators are central, it remains to show that $[f(b), g(a)]$ is linear in a and b for any quadratic f and g. But this is clear because $[-,-]$ is central, bilinear and vanishes on central elements. $\square$

Example 1 ([2])**.** For any group G there exists a *universal weakly quadratic map* $p_2 : G \to \mathrm{P}_2G$ such that for any other weakly quadratic map $q : G \to H$ there is a unique homomorphism $f_q : \mathrm{P}_2G \to H$ with $q = f_q p_2$. Thus

$$\mathsf{wQuad}(G, H) \cong \mathsf{Hom}(\mathrm{P}_2G, H).$$

One defines P_2G by the pullback square

$$\begin{tikzcd}
\mathrm{P}_2G \arrow[r, two heads] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & G \arrow[d, tail, "\text{diagonal}"] \\
G \vee G \arrow[r, two heads] & G \times G.
\end{tikzcd}$$

Thus by Proposition 1 there is a central extension

$$0 \to G^{\mathrm{ab}} \otimes G^{\mathrm{ab}} \xrightarrow{\iota} \mathrm{P}_2G \xrightarrow{\pi} G \to 0. \tag{1}$$

and P_2G is isomorphic to the set $G^{\mathrm{ab}} \otimes G^{\mathrm{ab}} \times G$ with the group structure given by

$$(\xi, g) + (\xi', g') = (\xi + \xi' - \hat{g} \otimes \hat{g}', g + g').$$

The universal weakly quadratic map $p_2 : G \to \mathrm{P}_2G$ is given by $p_2(g) = (0, g)$. Indeed, in P_2G one then has

$$(\hat{x} \otimes \hat{y}, 0) = -((0, x) + (0, y)) + (0, x + y) = (x \mid y)_{p_2}$$

and

$$(\xi, g) = (\xi, 0) + p_2(g),$$

so to factor a weakly quadratic map $q : G \to H$ through p_2 via a homomorphism $f_q : \mathrm{P}_2G \to H$ one is forced to put

$$f_q\,(\xi, g) = (- \mid -)_q(\xi) + q(g).$$

One then checks easily that this indeed gives the required factorization.

Note that the universal weakly quadratic map $p_2 : G \to \mathrm{P}_2G$ is not only weakly quadratic but actually also quadratic.

Lemma 4. *For any* $G \in \mathsf{Nil}$ *and* $A \in \mathsf{Ab}$ *one has an exact sequence*

$$0 \to \mathsf{Hom}(G, A) \to \mathsf{Quad}(G, A) \to \mathsf{Hom}(G^{\mathrm{ab}} \otimes G^{\mathrm{ab}}, A) \to H^2(G, A)$$

where the last homomorphism is given by $f \mapsto f_*([G])$*. Here* $f : G^{\mathrm{ab}} \otimes G^{\mathrm{ab}} \to A$ *is a homomorphism and* $[G] \in H^2(G, G^{\mathrm{ab}} \otimes G^{\mathrm{ab}})$ *is the class represented by the central extension* (1)*.*

Proof. The result follows immediately from the 5-term exact sequence in group cohomology (see for example [6], p. 15) applied to the central extension (1) and from the fact that for the abelian A one has $\mathsf{Hom}(\mathrm{P}_2G, A) \cong \mathsf{Quad}(G, A)$. ☐

Lemma 5. *For any groups* $(G_i)_{i \in I}$ *and* H *one has natural bijections*

$$\mathsf{Quad}(H, \prod_i G_i) \approx \prod_i \mathsf{Quad}(H, G_i).$$

If moreover $H \in \mathsf{Nil}$*, then there is a central extension*

$$\begin{aligned}0 \to \mathsf{Hom}(G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}}, \mathsf{Z}(H)) \xrightarrow{\alpha} \mathsf{Quad}(G_1 \times G_2, H)\\ \to \mathsf{Quad}(G_1, H) \times \mathsf{Quad}(G_2, H) \to 0\end{aligned}$$

where $(\alpha(\xi))(g_1, g_2) = \xi(\hat{g}_1, \hat{g}_2)$ *for* $\xi \in \mathsf{Hom}(G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}}, \mathsf{Z}(H))$ *and* $g_k \in G_k$*,* $k = 1, 2$*.*

Proof. The first assertion is clear. For the second one, take any $f_k \in \mathsf{Quad}(G_k, H)$, $k = 1, 2$. Then the composite maps $f_k p_k$ are again quadratic, where $p_k : G_1 \times G_2 \to G_k$ are projections. Thus $f = f_1 p_1 + f_2 p_2 : G_1 \times G_2 \to H$ is a quadratic map. It is clear that $f i_k = f_k$, where $i_k : G_k \to G_1 \times G_2$ are standard inclusions. This shows that the map $\mathsf{Quad}(G_1 \times G_2, H) \to \mathsf{Quad}(G_1, H) \times \mathsf{Quad}(G_2, H)$ is surjective. Let us compute the kernel of the latter homomorphism. Take any f from the kernel. Then $f : G_1 \times G_2 \to H$ is a quadratic map such that $f(g_1, 0) = 0 = f(0, g_2)$ for all $g_k \in G_k$. Define $\xi : G_1^{\mathrm{ab}} \times G_2^{\mathrm{ab}} \to \mathsf{Z}(H)$ by $\xi(\hat{g_1}, \hat{g_2}) := ((g_1, 0) \mid (0, g_2))_f$. Then one has

$$f(g_1, g_2) = f((g_1, 0) + (0, g_2)) = ((g_1, 0) \mid (0, g_2))_f = \xi(\hat{g}_1, \hat{g}_2)$$

and the lemma follows. ☐

As mentioned above, from now on we will assume that all groups under consideration are nilpotent of class two.

Lemma 6. *Let* $f : G \to H$ *be a weakly quadratic map. For any homomorphism* $h : G_1 \to G$ *the composite* $fh : G_1 \to H$ *is also weakly quadratic and*

$$(a \mid b)_{fh} = (h(a) \mid h(b))_f, \quad a, b \in G_1;$$

moreover, if f *is quadratic, then so is* fh*.*

For any homomorphism $g : H \to H_1$ *the composite* $gf : G \to H_1$ *is also weakly quadratic and*

$$(a \mid b)_{gf} = g((a \mid b)_f).$$

If moreover f *is quadratic, then* gf *will be quadratic provided* g *carries the central elements to the central elements.*

Thus for any $N \in \mathsf{Nil}$, one obtains functors

$$\begin{aligned}\mathsf{wQuad}(-, N) &: \mathsf{Nil}^{\mathrm{op}} \to \mathsf{Sets},\\ \mathsf{Quad}(-, N) &: \mathsf{Nil}^{\mathrm{op}} \to \mathsf{Nil},\\ \mathsf{wQuad}(N, -) &: \mathsf{Nil} \to \mathsf{Sets}.\end{aligned}$$

Indeed, by Example 1 the last functor is representable, i.e. one has

$$\mathsf{wQuad}(N, -) \approx \mathsf{Hom}_{\mathsf{Nil}}(\mathrm{P}_2 N, -).$$

However the mapping $\mathsf{Quad}(N, -)$ is NOT functorial.

Examples 1. i) For a fixed $n \in \mathbf{Z}$, consider the map $n : G \to G$ given by $a \mapsto na$. Then

$$(a \mid b)_n = -\frac{n(n-1)}{2}[a,b].$$

Thus $n \in \mathsf{Quad}(G,G)$.

ii) Let $+ : G \times G \to G$ be the map given by $(a,b) \mapsto a+b$. Then

$$((a,b) \mid (c,d))_+ = [c,b].$$

In particular $+ \in \mathsf{Quad}(G \times G, G)$.

iii) For any elements $a \in G$ and $b \in \mathsf{Z}(G)$ we put

$$f_{a,b}(n) = na + \frac{n(n-1)}{2}b.$$

The map $f_{a,b} : \mathbf{Z} \to G$ is a quadratic map with $(n \mid m)_{f_{a,b}} = nmb$ for any $n, m \in \mathbf{Z}$. We claim that any quadratic map $f : \mathbf{Z} \to G$ is of this form. Indeed, one puts $a = f(1)$, $b = (1 \mid 1)_f$ and considers $g = f - f_{a,b}$. Then one has $g(1) = 0 = (1 \mid 1)_g$. Since $(- \mid -)_g$ is bilinear, it follows that $(n \mid m)_g = nm(1 \mid 1)_g = 0$. Hence g is a homomorphism and the condition $g(1) = 0$ shows that $g = 0$ and the claim is proved. One easily computes that

$$f_{a,b} + f_{a',b'} = f_{a+a',b+b'+[a,a']}.$$

Thus valuewise sum of quadratic maps $\mathbf{Z} \to G$ is quadratic so that $\mathsf{Quad}(\mathbf{Z},G)$ has a group structure and one has the central extension

$$0 \to \mathsf{Z}(G) \to \mathsf{Quad}(\mathbf{Z},G) \xrightarrow{\mathsf{ev}(1)} G \to 0$$

where $\mathsf{ev}(1)(f) = f(1)$. A 2-cocycle $G \times G \to \mathsf{Z}(G)$ corresponding to this central extension is given by the commutator map.

Let us next investigate quadratic maps of the form $f : G_1 \vee G_2 \to H$. For such a map, denote

$$f_i = f|_{G_i} : G_i \to H,$$

$i = 1, 2$, and

$$f_\otimes = f|_{G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}}} : G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}} \to H,$$

where the inclusion $G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}} \subset G_1 \vee G_2$ is as in Proposition 1. Since $G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}}$ is contained in the commutator subgroup of $G_1 \vee G_2$, the map $f_\otimes$ is a homomorphism, and its image lies in the center of H (by v) of Lemma 2). As for f_i, they are quadratic maps. Since every element of $G_1 \vee G_2$ has the form $(\xi, g_1, g_2) = (\xi, 0, 0) + (0, g_1, 0) + (0, 0, g_2)$ with $\xi \in G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}}$ and

$$f(\xi, g_1, g_2) = f_\otimes(\xi) + f_1(g_1) + f_2(g_2) + (\hat{g_1} \mid \hat{g_2})_f,$$

it follows that f is uniquely reconstructed from the maps $f_\otimes$, f_1, f_2 and the homomorphism

$$(- \mid -)_f|_{G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}}} : G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}} \to \mathsf{Z}(H),$$

which we denote by $\hat{f}$.

Conversely, for any given maps

$$f_i \in \mathsf{Quad}(G_i, H), \quad i = 1, 2, \quad f_\otimes, \hat{f} \in \mathsf{Hom}(G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}}, \mathsf{Z}(H))$$

define the map $f : G_1 \vee G_2 \to H$ by

$$f(\xi, g_1, g_2) = f_\otimes(\xi) + f_1(g_1) + f_2(g_2) + \hat{f}(\hat{g}_1 \otimes \hat{g}_2).$$

Then

$$\begin{aligned}
&f((\xi, x_1, x_2) + (\eta, y_1, y_2)) = f(\xi + \eta - \hat{y}_1 \otimes \hat{x}_2, x_1 + y_1, x_2 + y_2)\\
&= f_\otimes(\xi + \eta - \hat{y}_1 \otimes \hat{x}_2) + f_1(x_1 + y_1) + f_2(x_2 + y_2) + \hat{f}((\hat{x}_1 + \hat{y}_1) \otimes (\hat{x}_2 + \hat{y}_2))\\
&= f_\otimes(\xi) + f_\otimes(\eta) - f_\otimes(\hat{y}_1 \otimes \hat{x}_2)\\
&\qquad + f_1(x_1) + f_1(y_1) + (x_1 \mid y_1)_{f_1} + f_2(x_2) + f_2(y_2) + (x_2 \mid y_2)_{f_2}\\
&\qquad + \hat{f}(\hat{x}_1 \otimes \hat{x}_2) + \hat{f}(\hat{x}_1 \otimes \hat{y}_2) + \hat{f}(\hat{y}_1 \otimes \hat{x}_2) + \hat{f}(\hat{y}_1 \otimes \hat{y}_2)\\
&= f(\xi, x_1, x_2) + f(\eta, y_1, y_2) - f_\otimes(\hat{y}_1 \otimes \hat{x}_2)\\
&\qquad + [f_1(y_1), f_2(x_2)] + (x_1 \mid y_1)_{f_1} + (x_2 \mid y_2)_{f_2} + \hat{f}(\hat{x}_1 \otimes \hat{y}_2) + \hat{f}(\hat{y}_1 \otimes \hat{x}_2).
\end{aligned}$$

It follows that f is a quadratic map, so that any choice of $f_\otimes$, f_1, f_2 and $\hat{f}$ as above is indeed valid.

Now suppose two quadratic maps $f, f' : G_1 \vee G_2 \to H$ are given. Then for their sum one clearly has $(f + f')_i = f_i + f'_i$, $i = 1, 2$, and $(f + f')_\otimes = f_\otimes + f'_\otimes$. Moreover, one calculates

$$\begin{aligned}
&\widehat{f + f'}(\hat{g}_1 \otimes \hat{g}_2) = ((0, g_1, 0) \mid (0, 0, g_2))_{f+f'}\\
&\quad = ((0, g_1, 0) \mid (0, 0, g_2))_f + ((0, g_1, 0) \mid (0, 0, g_2))_{f'} + [f(0, 0, g_2), f'(0, g_1, 0)]\\
&\qquad\qquad = \hat{f}(\hat{g}_1 \otimes \hat{g}_2) + \hat{f}'(\hat{g}_1 \otimes \hat{g}_2) + [f_2(g_2), f'_1(g_1)].
\end{aligned}$$

Thus identifying f with the quadruple $(f_1, f_2, f_\otimes, \hat{f})$ as above one has

$$(f_1, f_2, f_\otimes, \hat{f}) + (f'_1, f'_2, f'_\otimes, \hat{f}') = (f_1 + f'_1, f_2 + f'_2, f_\otimes + f'_\otimes, \hat{f} + \hat{f}' - [f'_1, f_2]),$$

where

$$[f'_1, f_2](\hat{g}_1 \otimes \hat{g}_2) = [f'_1(g_1), f_2(g_2)].$$

We thus have proved

Lemma 7. *For any nil$_2$-groups G_1, G_2, H there is a central extension*

$$\begin{aligned}
0 \to \mathsf{Hom}(G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}}, \mathsf{Z}(H)) \to \mathsf{Quad}(G_1 \vee G_2, H) \to \mathsf{Hom}(G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}}, \mathsf{Z}(H))&\\
\times \mathsf{Quad}(G_1, H) \times \mathsf{Quad}(G_2, H) \to 0.&
\end{aligned}$$

A cocycle defining this extension is given by

$$\begin{aligned}
((f_\otimes, f_1, f_2), (f'_\otimes, f'_1, f'_2)) \mapsto \alpha((f_\otimes, f_1, f_2), (f'_\otimes, f'_1, f'_2)) : G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}} \to \mathsf{Z}(H),&\\
\hat{g}_1 \otimes \hat{g}_2 \mapsto [f_2(g_2), f'_1(g_1)].&
\end{aligned}$$

Corollary 3. *Let G be a free nil$_2$-group on $x_1, \dots, x_n$. Then for any nil$_2$-group H and any elements $a_1, \dots, a_n \in H$, $a_{ij} \in \mathsf{Z}(H)$, $i < j$ and $b_{ij} \in \mathsf{Z}(H)$, $i \leqslant j$, there exists a unique quadratic map $f : G \to H$ such that*

$$\begin{aligned} f(x_i) &= a_i, \quad 1 \leqslant i \leqslant n, \\ f([x_i, x_j]) &= a_{ij}, \quad i < j, \\ (x_i \mid x_j)_f &= b_{ij}, \quad i \leqslant j. \end{aligned}$$

3. q-MAPS

The last identity of Lemma 1 suggests

Definition 1. A weakly quadratic map $f : G \to H$ between nil$_2$-groups is a *q-map* if one has $(a \mid b)_f \in [H, H]$ for all $a, b \in G$.

We denote by $\mathsf{Q}(G, H)$ the collection of all q-maps from G to H, so that

$$\mathsf{Hom}(G, H) \subseteq \mathsf{Q}(G, H) \subseteq \mathsf{Quad}(G, H).$$

Lemma 8. *The set $\mathsf{Q}(G, H)$ is a normal subgroup of $\mathsf{Quad}(G, H)$. In particular any linear combination of homomorphisms is a q-map.*

Proof. The first identity of Lemma 3 shows that $\mathsf{Q}(G, H) \subseteq \mathsf{Quad}(G, H)$ is a subgroup. By the same Lemma for any $f \in \mathsf{Quad}(G, H)$ and $g \in \mathsf{Q}(G, H)$ we have

$$(a \mid b)_{f+g-f} = (a \mid b)_g + [fb, ga] - [gb, fa]$$

and the result follows. □

Lemma 9. *A weakly quadratic map is in $\mathsf{Q}(G, H)$ if and only if its composite with $H \twoheadrightarrow H^{\mathrm{ab}}$ is a homomorphism. In particular any q-map $f : G \to H$ yields a well-defined homomorphism $f^{\mathrm{ab}} : G^{\mathrm{ab}} \to H^{\mathrm{ab}}$ such that the diagram*

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, two heads] & H \arrow[d, two heads] \\ G^{\mathrm{ab}} \arrow[r, "f^{\mathrm{ab}}"] & H^{\mathrm{ab}} \end{tikzcd}$$

commutes.

Proof. Indeed, $(a \mid b)_f \in [H, H]$ if and only if the image of $(a \mid b)_f$ vanishes in H^{ab}. □

Obviously, one has an embedding

$$\mathsf{Quad}(G, [H, H]) \subset \mathsf{Q}(G, H)$$

as a central subgroup.

Lemma 10. *For an abelian group H one has*

$$\mathsf{Q}(G, H) = \mathsf{Hom}(G, H)$$

for any $G \in \mathsf{Nil}$.

Proof. Since $[H,H]=0$, a map $f: G \to H$ is a q-map if and only if $(- \mid -)_f = 0$. □

Examples 2. The first two quadratic maps considered in Examples 1 are actually q-maps. Also, the map

$$\delta = i_1 + i_2 : G \to G \vee G$$

is a q-map, with $(x \mid y)_\delta = [i_1(y), i_2(x)]$.

On the other hand, the quadratic map $f_{a,b} : \mathbf{Z} \to G$ associated with elements $a \in G$ and $b \in \mathsf{Z}(G)$ as in iii) of Examples 1 is a q-map if and only if $b \in [G,G]$. Thus for any $G \in \mathsf{Nil}$ one has the central extension

$$0 \to [G,G] \to \mathsf{Q}(\mathbf{Z}, G) \xrightarrow{\mathsf{ev}(1)} G \to 0.$$

A 2-cocycle $G \times G \to [G,G]$ corresponding to this central extension is given by the commutator map.

It is also easy to calculate q-maps from the cyclic group of order two:

Lemma 11. *One has a natural bijection*

$$\mathsf{Q}(\mathbf{Z}/2\mathbf{Z}, G) \approx \{\ a \in G \mid 4a = 0\ \&\ 2a \in [G,G]\ \}.$$

Proof. If a is an element as above, then the map $f: \mathbf{Z}/2\mathbf{Z} \to G$ given by $f(0) = 0$ and $f(1) = a$ is a q-map. Conversely, if $f: \mathbf{Z}/2\mathbf{Z} \to G$ is a q-map then

$$0 = f(1+1) = a + a + (1 \mid 1)_f.$$

Thus $(1 \mid 1)_f = -2a$, which implies $2a \in [G,G]$. Since $(- \mid -)_f$ is bilinear we have

$$0 = (1+1 \mid 1)_f = 2(1 \mid 1)_f,$$

hence $4a = 0$. □

Exactly as for Lemma 5 one has

Lemma 12. *For any groups $(G_i)_{i \in I}$, H one has natural bijections*

$$\mathsf{Q}(H, \prod_i G_i) \cong \prod_i \mathsf{Q}(H, G_i).$$

If moreover $H \in \mathsf{Nil}$, then there is a central extension

$$0 \to \mathsf{Q}(G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}}, [H,H]) \to \mathsf{Q}(G_1 \times G_2, H) \to \mathsf{Q}(G_1, H) \times \mathsf{Q}(G_2, H) \to 0.$$

Moreover, one has exactly as in the case of Lemma 7

Lemma 13. *For any nil$_2$-groups G_1, G_2, H there is a central extension*

$$\begin{aligned} 0 \to \mathsf{Hom}(G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}}, [H,H]) \to \mathsf{Q}(G_1 \vee G_2, H) \to \mathsf{Hom}(G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}}, [H,H]) \\ \times \mathsf{Q}(G_1, H) \times \mathsf{Q}(G_2, H) \to 0. \end{aligned}$$

A cocycle defining this extension is given by

$$\begin{aligned} ((f_\otimes, f_1, f_2), (f'_\otimes, f'_1, f'_2)) \mapsto \alpha((f_\otimes, f_1, f_2), (f'_\otimes, f'_1, f'_2)) : G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}} \to [H,H], \\ \hat{g}_1 \otimes \hat{g}_2 \mapsto [f_2(g_2), f'_1(g_1)]. \end{aligned}$$

In particular, if G is a free nil$_2$-group on $x_1, \cdots, x_n$ then for any nil$_2$-group H and any elements $a_1, \cdots, a_n \in H$, $a_{ij} \in [H,H]$, $i < j$ and $b_{ij} \in [H,H]$, $i \leqslant j$ there exists a unique q-map $f : G \to H$ such that

$$\begin{aligned} f(x_i) &= a_i, \quad 1 \leqslant i \leqslant n, \\ f([x_i, x_j]) &= a_{ij}, \quad i < j, \\ (x_i \mid x_j)_f &= b_{ij}, \quad i \leqslant j. \end{aligned}$$

By Lemma 9 any q-map $f : G \to H$ yields a homomorphism $f^{\mathrm{ab}} : G^{\mathrm{ab}} \to H^{\mathrm{ab}}$. We now associate two more homomorphisms with any q-map.

Proposition 2. *Let $f : G \to H$ be a q-map. Then $f([G,G]) \subset [H,H]$ and the restriction of f to $[G,G]$ yields a homomorphism $[f,f] : [G,G] \to [H,H]$, which fits in the following commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & {[G,G]} \arrow[r] \arrow[d, "{[f,f]}"] & G \arrow[r] \arrow[d, "f"] & G^{\mathrm{ab}} \arrow[r] \arrow[d, "f^{\mathrm{ab}}"] & 0 \\ 0 \arrow[r] & {[H,H]} \arrow[r] & H \arrow[r] & H^{\mathrm{ab}} \arrow[r] & 0 \end{tikzcd}$$

Moreover, there exists a unique quadratic map

$$\beta(f) : \mathsf{Ker}(f^{\mathrm{ab}}) \to \mathsf{Coker}([f,f])$$

such that

$$\beta(f)(\hat{a}) = f(a) \mod f([G,G])$$

for any $a \in f^{-1}([H,H])$. Furthermore, if f is injective, then $[f,f]$ and $\beta(f)$ are monomorphisms and if f is surjective, then $\beta(f)$ and f^{ab} are epimorpisms.

Proof. If f is a q-map, then it follows from v) of Lemma 2 that $f[G,G] \subseteq [H,H]$. Hence by iv) of Lemma 2, $[f,f] : [G,G] \to [H,H]$ is a homomorphism and obviously the diagram commutes. We claim that $\beta(f)$ is well-defined. One observes that if $\hat{a}_1 = \hat{a}$, then $a_1 = a + b$ with $b \in [G,G]$. It follows by iv) of Lemma 2 that $f(a_1) = f(a) \mod f([G,G])$ and the claim is proved. The rest is just diagram chasing. ☐

4. THE CATEGORY Niq

In this section, our main character enters. This is the category Niq. The definition is based on the following result.

Proposition 3. *Any composite of q-maps is a q-map. More precisely, for q-maps $f : G \to H$ and $g : G_1 \to G$ the cross-effect of their composite is given by*

$$(a \mid b)_{fg} = f((a \mid b)_g) + (g(a) \mid g(b))_f, \quad a, b \in G_1.$$

Proof. One has

$$\begin{aligned} fg(a+b) &= f(g(a) + g(b) + (a \mid b)_g) \\ &= f(g(a) + g(b)) + f((a \mid b)_g) && \text{(by iv) of Lemma 2)} \\ &= f(g(a)) + f(g(b)) + (g(a) \mid g(b))_f + f((a \mid b)_g), \end{aligned}$$

which proves the equality above. $\square$

Hence there is a well-defined category Niq whose objects are nil$_2$-groups and morphisms are all q-maps between them. The hom-sets

$$\mathsf{Hom}_{\mathsf{Niq}}(G,H) = \mathsf{Q}(G,H)$$

are equipped with structures of nilpotent groups of class two. Nil is a subcategory of Niq, with the same objects. Moreover, the abelization and commutator subgroup functors $\mathsf{Nil} \to \mathsf{Ab}$ extend to Niq yielding well-defined functors $\mathsf{Niq} \to \mathsf{Ab}$ given respectively by $G \mapsto G^{\mathrm{ab}}$ and $G \mapsto [G,G]$.

The hom-functor of Niq (with values in sets) gives rise to a well-defined bifunctor

$$\mathsf{Q}(-,-) : \mathsf{Nil}^{\mathrm{op}} \times \mathsf{Nil} \to \mathsf{Niq}.$$

The authors are grateful to the referee for raising the natural question whether the above bifunctor also extends to a kind of internal hom bifunctor on Niq. The referee kindly provided the statement and proof of the following lemma showing that such an extension exists at least if one restricts the contravariant argument to the full subcategory of Niq on the free nil$_2$-groups:

Lemma 14. *Let F denote a free nil$_2$-group with generators $x_1, \dots, x_n$. Then*

1. *a q-map $f : F \to G$ lies in the commutator subgroup of $\mathsf{Q}(F,G)$ if and only if $f(x_i) \in [G,G]$, $(x_i \mid x_i)_f = 2f(x_i)$ and $f([x_i,x_j]) = 0$ for all i,j;*
2. *every q-map $g : G \to G'$ induces a q-map $g_* : \mathsf{Q}(F,G) \to \mathsf{Q}(F,G')$.*

Proof. Every element of F is of the form $\sum_i m_i x_i + \sum_{j<k} l_{jk}[x_j,x_k]$; hence every element of $\mathsf{Q}(F,G)$ has form $f = f_{(a_i),(a_{jk}),(b_{jk})}$ determined by

$$f(\sum_i m_i x_i + \sum_{j<k} l_{jk}[x_j,x_k]) = \sum_i m_i a_i + \sum_{j<k} l_{jk} a_{jk} + \sum_i \binom{m_i}{2} b_{ii} + \sum_{j<k} m_j m_k b_{jk}$$

for some $a_i \in G$, $a_{jk}, b_{jk} \in [G,G]$. Moreover, one has

$$\begin{aligned} f_{(a_i),(a_{jk}),(b_{jk})} + f_{(a'_i),(a'_{jk}),(b'_{jk})} &= f_{(a_i+a'_i),(a_{jk}+a'_{jk}),(b_{jk}+b'_{jk}+[a_k,a'_j])}, \\ [f_{(a_i),(a_{jk}),(b_{jk})}, f_{(a'_i),(a'_{jk}),(b'_{jk})}] &= f_{([a_i,a'_i]),(0),([a_j,a'_k]+[a_k,a'_j])} \end{aligned}$$

and so a q-map $f_{(a_i),(a_{jk}),(b_{jk})}$ lies in the commutator subgroup of $\mathsf{Q}(F,G)$ if and only if each a_i lies in $[G,G]$, all a_{jk} are zero and for all i one has $b_{ii} = 2a_i$.

Now consider the action of a q-map $g : G \to G'$ on $\mathsf{Q}(F,G)$. One calculates

$$\begin{aligned} g f_{(a_i),(a_{jk}),(b_{jk})} &= f_{(ga_i),(ga_{jk}),(gb_{jk}+(a_j|a_k)_g)}, \\ (f_{(a_i),(a_{jk}),(b_{jk})} \mid f_{(a'_i),(a'_{jk}),(b'_{jk})})_g &= f_{((a_i|a'_i)_g),(0),((a_j|a'_k)_g+(a_k|a'_j)_g)} \end{aligned}$$

and so the second part of the Lemma follows from the first. $\square$

It is however impossible to extend the hom bifunctor to the whole of Niq. Namely, Lemma 11 shows that one cannot define a functor $\mathsf{Q}(\mathbf{Z}/2\mathbf{Z},-) : \mathsf{Niq} \to$

Niq. To see this, let G be the nil$_2$-group generated by elements a, x, y subject to the relations

$$[x, y] = 2a, \ 4a = 0,$$
$$[a, x] = 0 = [a, y].$$

Thus one has the central extension

$$0 \to \mathbf{Z}/4\mathbf{Z} \to G \to \mathbf{Z} \times \mathbf{Z} \to 0.$$

It follows from Lemma 11 that $\mathsf{Q}(\mathbf{Z}/2\mathbf{Z}, G) \cong \mathbf{Z}/4\mathbf{Z}$. Now let $g : G \to G$ be the unique q-map satisfying

$$g(a) = a, \ (a \mid a)_g = 2a,$$
$$(x \mid y)_g = (y \mid x)_g = (a \mid x)_g = (a \mid y)_g = (y \mid a)_g = 0.$$

The action of g yields the quadratic map $g_* : \mathsf{Q}(\mathbf{Z}/2\mathbf{Z}, G) \to \mathsf{Q}(\mathbf{Z}/2\mathbf{Z}, G)$, which, upon the identification $\mathsf{Q}(\mathbf{Z}/2\mathbf{Z}, G) = \mathbf{Z}/4\mathbf{Z}$, is given by $n \mapsto n^2$ which is not a homomorphism and therefore not a q-map since, as we know, all q-maps between abelian groups are homomorphisms.

The composition in Niq is left distributive,

$$(f + f')g = fg + f'g,$$

but not right distributive; it is rather right quadratic in the following sense. First of all one has

$$f(g + g') = fg + fg' + (g \mid g')_f, \quad f \in \mathsf{Q}(G, H), \quad g, g' \in \mathsf{Q}(G_1, G), \tag{2}$$

where $(g \mid g')_f : G_1 \to H$ is given by

$$(g \mid g')_f(x) = (g(x) \mid g'(x))_f. \tag{3}$$

Secondly $(g \mid g')_f$ lies in the center of $\mathsf{Q}(G_1, H)$ and it is bilinear in g, g' and quadratic in f — more precisely, one has

$$(g \mid g')_{f+f'}(x) = (g \mid g')_f(x) + (g \mid g')_{f'}(x) + [fg'(x), f'g(x)].$$

One may thus view the embedding $\mathsf{Nil} \hookrightarrow \mathsf{Niq}$ as a sort of quadratic envelope—the result of closing Nil under the operation of valuewise addition of parallel morphisms.

The category Niq possesses all products and both the inclusion $\mathsf{Nil} \hookrightarrow \mathsf{Niq}$ and the forgetful functor $\mathsf{Niq} \to \mathsf{Sets}$ respect products.

Every object in Niq has a canonical internal group structure. However a morphism in Niq is compatible with the corresponding internal group structures if and only if it lies in Nil, i.e. is a homomorphism.

If $p_k : G_1 \times G_2 \to G_k$ are standard projections and $i_k : G_k \to G_1 \times G_2$ are standard inclusions, then one has $p_k i_k = \mathsf{Id}_{G_k}$, $i_1 p_1 + i_2 p_2 = \mathsf{Id}_{G_1 \times G_2}$ and $p_2 i_1 = 0$, $p_1 i_2 = 0$. Therefore Niq is a right quadratic category in the terminology of [1]. Trivial groups are zero objects in Niq.

Also note that it follows from Lemma 10 that

Proposition 4. *Any group isomorphic in* Niq *to an abelian group is itself abelian.*

Example 2. Let $f : \mathbf{Z}^3 \to \mathbf{Z} \vee \mathbf{Z}$ be the map given by

$$f(l,m,n) = l[x,y] + mx + ny,$$

where x and y are the generators of $\mathbf{Z} \vee \mathbf{Z}$. One then has

$$((l,m,n) \mid (l',m',n'))_f = m'n[x,y]$$

so that f is a q-map. It is obviously a bijection. However it cannot be an isomorphism in Niq because of Proposition 4.

Indeed,

$$(l[x,y] + mx + ny \mid l'[x,y] + m'x + n'y)_{f^{-1}} = (-m'n, 0, 0),$$

so that f^{-1} is quadratic, but not a q-map.

Let us point out that there exist nil$_2$-groups isomorphic in Niq but not in Nil. We will see such examples below (see Example 3).

Let us recall that a *weak coproduct* of objects X_1 and X_2 of some category is an object W together with morphisms $i_k : X_k \to W$ such that for any morphisms $f_k : X_k \to Z$ there exists a morphism (not necessarily unique) $f : W \to Z$ with $f_k = fi_k$, $k = 1,2$.

Lemma 15. *The category* Niq *possesses weak coproducts.*

Proof. We claim that $W = X_1 \times X_2$ does the job. Indeed, for any $f_k : X_k \to Z$ put $f = f_1p_1 + f_2p_2$. Then one has

$$fi_k = (f_1p_1 + f_2p_2)i_k = f_1p_1i_k + f_2p_2i_k = f_k. \qquad \square$$

5. The Category Nil as a Linear Extension

One of our main motivations for what follows is to reduce classification problems in Niq to those in other categories which are easier to handle. This is achieved using the notion of linear extension of a small category by a bifunctor [3].

Definition 2. A *linear extension* of a small category $\boldsymbol{C}$ by a bifunctor $D : \boldsymbol{C}^{\mathrm{op}} \times \boldsymbol{C} \to \mathsf{Ab}$

$$0 \to D \to \boldsymbol{E} \xrightarrow{P} \boldsymbol{C} \to 0$$

is a functor P with the following properties: $\boldsymbol{C}$ and $\boldsymbol{E}$ have the same objects and P is a full functor which is the identity on objects. For each pair of objects X and Y the abelian group $D(X,Y)$ acts transitively and effectively on the set $\mathsf{Hom}_{\boldsymbol{E}}(X,Y)$. We write $f + \alpha$ for the action of $\alpha \in D(X,Y)$ on $f \in \mathsf{Hom}_{\boldsymbol{E}}(X,Y)$. The action satisfies the linear distributivity law

$$(f + \alpha)(g + \beta) = fg + P(f)_*\beta + P(g)^*\alpha.$$

It is known and easy to prove that in any linear extension the functor q reflects isomorphisms and yields a bijection on isomorphism classes of objects.

Our aim is to obtain the category Nil as a linear extension. To do so we first recall some classical results on group (co)homology.

Proposition 5.

i) *For a central extension*

$$\boldsymbol{E} = \Big(0 \to A \xrightarrow{i} G \xrightarrow{p} Q \to 0\Big) \tag{4}$$

there is a well-defined class $\langle \boldsymbol{E} \rangle \in H^2(Q;A)$ *and in this way one obtains a one-to-one correspondence between the equivalence classes of central extensions of* Q *by* A *and elements of the group* $H^2(Q;A)$. *If* $\boldsymbol{E}'$ *is also a central extension of a group* Q' *by* A' *and* $f : Q \to Q'$, $g : A \to A'$ *are group homomorphisms, then* $g_* \langle \boldsymbol{E} \rangle$ *and* $f^* \langle \boldsymbol{E}' \rangle$ *are the same elements in* $H^2(Q;A')$ *if and only if there is a group homomorphism* $h : G \to G'$ *such that the diagram*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "g"'] & G \arrow[r] \arrow[d, "h"'] & Q \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & A' \arrow[r] & G' \arrow[r] & Q' \arrow[r] & 0
\end{tikzcd}$$

commutes.

ii) *Let* Q *be a group and* A *be an abelian group, considered as a* Q*-module via the trivial action of* Q *on* A. *Then one has the universal coefficient exact sequence*

$$0 \to \mathsf{Ext}(Q^{\mathrm{ab}}, A) \to H^2(Q;A) \xrightarrow{\mu} \mathsf{Hom}(H_2Q, A) \to 0.$$

iii) *For the central extension* (4) *one has the following Ganea exact sequence*

$$G^{\mathrm{ab}} \otimes A \to H_2G \to H_2Q \xrightarrow{\nu} A \to G^{\mathrm{ab}} \to Q^{\mathrm{ab}} \to 0,$$

where $\nu = \mu \langle \boldsymbol{E} \rangle$, *with* $\langle \boldsymbol{E} \rangle$ *as in i) above.*

iv) *If* B *is an abelian group then* $H_2(B) \cong \Lambda^2 B$ *and the homomorphism* $\mu \langle \boldsymbol{E} \rangle : \Lambda^2 B \to A$ *corresponding to a central extension*

$$\boldsymbol{E} = \Big(0 \to A \xrightarrow{i} G \xrightarrow{p} B \to 0\Big)$$

is determined by

$$i\left(\mu \langle \boldsymbol{E} \rangle \left(p(x) \wedge p(y)\right)\right) = [x,y], \quad x, y \in G.$$

Proof. These results are well known, see for example [6]. ☐

The class of the central extension

$$0 \to [G,G] \to G \to G^{\mathrm{ab}} \to 0 \tag{5}$$

in $H^2(G^{\mathrm{ab}};[G,G])$ is denoted by $\mathsf{e}(G)$.

Lemma 16. *The homomorphism* $\mu(\mathsf{e}(G)) : \Lambda^2(G^{\mathrm{ab}}) \to [G,G]$ *is surjective.*

Proof. This follows from iv) of Proposition 5 applied to the central extension (5). ☐

The exact sequence (5) is functorial in G, meaning that if $f : G \to H$ is a homomorphism, then one has the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & {[G,G]} \arrow[r, "i"] \arrow[d, "{[f,f]}"'] & G \arrow[r, "p"] \arrow[d, "f"'] & G^{\mathrm{ab}} \arrow[r] \arrow[d, "f^{\mathrm{ab}}"] & 0 \\
0 \arrow[r] & {[H,H]} \arrow[r, "j"] & H \arrow[r, "q"] & H^{\mathrm{ab}} \arrow[r] & 0
\end{tikzcd}$$

If $g : G \to H$ is another homomorphism, then we write $f \sim g$ provided $f^{\mathrm{ab}} = g^{\mathrm{ab}}$ and $[f,f] = [g,g]$. It is clear that $f \sim g$ if and only if there exists a homomorphism $k : G^{\mathrm{ab}} \to [H,H]$ such that $f - g = jkp$. We can consider the corresponding quotient category $\mathsf{Nil}^{\sim}$. Objects are the same as of Nil. Two homomorphisms $f, g : G \to H$ define the same morphism in $\mathsf{Nil}^{\sim}$ provided $f \sim g$. Comparing with the notion of linear extension of categories (see Definition 2) we obtain the following result.

Theorem 1. *One has the following linear extension of categories*

$$0 \to D \to \mathsf{Nil} \to \mathsf{Nil}^{\sim} \to 0$$

where the bifunctor

$$D : (\mathsf{Nil}^{\sim})^{\mathrm{op}} \times \mathsf{Nil}^{\sim} \to \mathsf{Ab}$$

is given by

$$D(G,H) = \mathsf{Hom}(G^{\mathrm{ab}}, [H,H]).$$

Our next aim is to describe the quotient category $\mathsf{Nil}^{\sim}$ in cohomological terms. Define the category $\mathsf{Nil}^{\mathrm{ab}}$ as follows. The objects of $\mathsf{Nil}^{\mathrm{ab}}$ are triples (A,B,e), where A and B are abelian groups and $e \in H^2(A,B)$ is an element such that $\mu(e) : \Lambda^2(A) \to B$ is an epimorphism. A morphism from (A,B,e) to (A',B',e') is a pair (f,g), where $f : A \to A'$ and $g : B \to B'$ are homomorphisms such that the equation

$$f^*(e') = g_*(e)$$

holds in $H^2(A,B')$. Thus for any $G \in \mathsf{Nil}$ the triple

$$\mathsf{ch}(G) = (G^{\mathrm{ab}}, [G,G], \mathsf{e}(G))$$

is an object of $\mathsf{Nil}^{\mathrm{ab}}$. Moreover, if $f : G \to H$ is homomorphism of groups, then $(f^{\mathrm{ab}}, [f,f]) : \mathsf{ch}(G) \to \mathsf{ch}(H)$ is a morphism. In this way one obtains the functor

$$\mathsf{ch} : \mathsf{Nil}^{\sim} \to \mathsf{Nil}^{\mathrm{ab}}.$$

Theorem 2. *The functor* $\mathsf{ch} : \mathsf{Nil}^{\sim} \to \mathsf{Nil}^{\mathrm{ab}}$ *is an equivalence of categories.*

Proof. We claim that for any object $(A,B,e) \in \mathsf{Nil}^{\mathrm{ab}}$ there exist an object $G \in \mathsf{Nil}$ and an isomorphism $\mathsf{ch}(G) \to (A,B,e)$ in $\mathsf{Nil}^{\mathrm{ab}}$. Indeed, consider a central extension

$$0 \to B \to G \to A \to 0$$

corresponding to the element e. The exact sequence iii) of Proposition 5 in our case has the following form

$$H_2G \to H_2A \to B \to G^{\mathrm{ab}} \to A \to 0$$

Since $H_2A \to B$ is an epimorphism, it follows that $G^{\mathrm{ab}} \cong A$. Therefore $[G,G] \cong B$ and the claim is proved.

Now, we show that for any morphism $(f,g) : \mathsf{ch}(G) \to \mathsf{ch}(G_1)$ in $\mathsf{Nil}^{\mathrm{ab}}$, there is a unique homomorphism $h : G \cong G_1$ in $\mathsf{Nil}^{\sim}$ such that $\mathsf{ch}(h) = (f,g)$. Indeed, by the definition of morphisms in $\mathsf{Nil}^{\mathrm{ab}}$ and by part i) of Proposition 5 there exists an $h : G \to G_1$ in Nil such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & {[G,G]} \arrow[r, "i"] \arrow[d, "g"'] & G \arrow[r, "p"] \arrow[d, "h"'] & G^{\mathrm{ab}} \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & {[G_1,G_1]} \arrow[r, "i_1"] & G_1 \arrow[r, "p_1"] & G_1^{\mathrm{ab}} \arrow[r] & 0
\end{tikzcd}$$

If $h' : G \to G'$ is another such homomorphism, then, clearly, $h^{\mathrm{ab}} = (h')^{\mathrm{ab}}$ as well as $[h,h] = [h',h']$ and the result is proved. $\square$

6. The Category Niq as a Linear Extension

For $\alpha \in \mathsf{Hom}(G^{\mathrm{ab}} \otimes G^{\mathrm{ab}}, [H,H])$ and $f \in \mathsf{Q}(G,H)$ define $f + \alpha \in \mathsf{Q}(G,H)$ to be the map given by

$$(f+\alpha)(g) = f(g) + \alpha(\hat{g}, \hat{g}).$$

It is clear that for any $\alpha, \beta \in \mathsf{Hom}(G^{\mathrm{ab}} \otimes G^{\mathrm{ab}}, [H,H])$ and $f \in \mathsf{Q}(G,H)$ one has

$$f + (\alpha + \beta) = (f + \alpha) + \beta$$

and therefore the group $\mathsf{Hom}(G^{\mathrm{ab}} \otimes G^{\mathrm{ab}}, [H,H])$ acts on the set $\mathsf{Q}(G,H)$. In particular, this gives the following equivalence relation: for q-maps $f, g \in \mathsf{Q}(G,H)$ we put $f \sim g$ provided $g = f + \alpha$, for some homomorphism $\alpha : G^{\mathrm{ab}} \otimes G^{\mathrm{ab}} \to [H,H]$.

Lemma 17.

i) *Let $f_1, f_2, g_1, g_2 : G \to H$ be q-maps. If $f_1 \sim g_1$ and $f_2 \sim g_2$, then*

$$f_1 + f_2 \sim g_1 + g_2.$$

ii) *Let $f, g : G \to H$ be q-maps. Then*

$$f + g \sim g + f.$$

iii) *Let $f : G \to H$ and $g_1, g_2 : G_1 \to G$ be q-maps. Then*

$$f(g_1 + g_2) \sim fg_1 + fg_2.$$

iv) *Let $f : G \to H$ and $g : G_1 \to G$ be q-maps. Then for any homomorphisms $\alpha : G^{\mathrm{ab}} \otimes G^{\mathrm{ab}} \to [H,H]$ and $\beta : G_1^{\mathrm{ab}} \otimes G_1^{\mathrm{ab}} \to [G,G]$ one has*

$$(f+\alpha)(g+\beta) = fg + f_*\beta + g^*\alpha$$

where $f_*\beta$ and $g^*\alpha$ are homomorphisms $G_1^{\mathrm{ab}} \otimes G_1^{\mathrm{ab}} \to [H,H]$ given by $f_*\beta(x,y) = f(\beta(x,y))$ and $g^*\alpha(\hat{x}, \hat{y}) = \alpha(\widehat{g(x)}, \widehat{g(y)})$.

v) *Let $f_1, f_2 : G \to H$ and $g_1, g_2 : G_1 \to G$ be q-maps. If $f_1 \sim f_2$ and $g_1 \sim g_2$, then*

$$f_1 g_1 \sim f_2 g_2.$$

vi) *Let $f, g : G \to H$ be q-maps. If $f \sim g$, then they induce the same homomorphisms $G^{\mathrm{ab}} \to H^{\mathrm{ab}}$ and $[G, G] \to [H, H]$.*

Proof. i) We have $g_i = f_i + \alpha_i$, where $\alpha_i : G^{\mathrm{ab}} \otimes G^{\mathrm{ab}} \to [H, H]$ is a homomorphism $i = 1, 2$. Since the values of α_i are in the center, we obtain $g_1 + g_2 = f_1 + f_2 + (\alpha_1 + \alpha_2)$.

ii) It suffices to observe that $f + g = g + f + \alpha$, where $\alpha(\hat{x}, \hat{y}) = [f(x), g(y)]$.

iii) Thanks to equation (3) one has $f(g_1 + g_2) = fg_1 + fg_2 + \alpha$, where $\alpha(\hat{x}_1, \hat{x}_2) = (g_1(x_1) \mid g_2(x_2))_f$.

iv) We have

$$(f + \alpha)(g + \beta)(x) = f(g(x) + \beta(\hat{x}, \hat{x})) + \alpha(\widehat{g(x) + \beta(\hat{x}, \hat{x})}, \widehat{g(x) + \beta(\hat{x}, \hat{x})}).$$

Since the values of β lie in the commutator subgroup of H and α is defined on the abelization, we get

$$\alpha(\widehat{g(x) + \beta(\hat{x}, \hat{x})}, \widehat{g(x) + \beta(\hat{x}, \hat{x})}) = \alpha(\widehat{g(x)}, \widehat{g(x)}).$$

Thus the result follows from iv) of Lemma 2.

v) This property is an immediate consequence of iv).

vi) By assumption, $g = f + \alpha$, for some homomorphism $G^{\mathrm{ab}} \otimes G^{\mathrm{ab}} \to [H, H]$. If $c \in [G, G]$, then $\hat{c} = 0$ in G^{ab}, thus $\alpha(\hat{c}, \hat{c}) = 0$ and hence $g(c) = f(c)$. On the other hand, for any $x \in G$ the class of $\alpha(\hat{x}, \hat{x})$ in H^{ab} vanishes, hence $f^{\mathrm{ab}} = g^{\mathrm{ab}}$. □

Corollary 4. *There is a well-defined category $\mathsf{Niq}^{\sim}$ with objects nil$_2$-groups, and morphisms $\sim$-equivalence classes of q-maps. The category $\mathsf{Niq}^{\sim}$ is an additive category.*

Proof. By v), $\mathsf{Niq}^{\sim}$ is a well-defined category. By i) and ii) hom's in $\mathsf{Niq}^{\sim}$ are abelian groups. Since Niq was left distributive, it follows from iii) that the composition in $\mathsf{Niq}^{\sim}$ is distributive. One easily sees that the product in Niq remains also a product in $\mathsf{Niq}^{\sim}$ and therefore Niq is an additive category with products. □

For q-maps $f, g \in \mathsf{Q}(G, H)$ we put $f \approx g$ provided both of them yield the same homomorphisms $G^{\mathrm{ab}} \to H^{\mathrm{ab}}$ and $[G, G] \to [H, H]$. The corresponding quotient category is denoted by $\mathsf{Niq}^{\approx}$. By iv) in Lemma 17 the quotient functor $\mathsf{Niq} \to \mathsf{Niq}^{\approx}$ factors through $\mathsf{Niq}^{\sim}$.

For nil$_2$-groups G and H let $D^{\sim}(G, H)$ be the quotient of

$$\mathsf{Hom}(G^{\mathrm{ab}} \otimes G^{\mathrm{ab}}, [H, H])$$

by the subgroup spanned by such $\alpha \in \mathsf{Hom}(G^{\mathrm{ab}} \otimes G^{\mathrm{ab}}, [H, H])$ that $\alpha(\hat{x}, \hat{x}) = 0$ for all $x \in G$. In this way one obtains a bifunctor $D^{\sim} : (\mathsf{Niq}^{\approx})^{\mathrm{op}} \times \mathsf{Niq}^{\approx} \to \mathsf{Ab}$. We also need another bifunctor $D^{\approx} : (\mathsf{Niq}^{\approx})^{\mathrm{op}} \times \mathsf{Niq}^{\approx} \to \mathsf{Ab}$ given by $D^{\approx}(G, H) = \mathsf{Quad}(G^{\mathrm{ab}}, [H, H])$. There is a natural transformation $\rho : D^{\sim} \to D^{\approx}$, which takes $\alpha : G^{\mathrm{ab}} \otimes G^{\mathrm{ab}} \to [H, H]$ to the quadratic map $\rho(\alpha) : G^{\mathrm{ab}} \to [H, H]$ given by $\rho(\alpha)(\hat{x}) = \alpha(\hat{x}, \hat{x})$. It follows from the definition of $D^{\sim}$, that ρ is a monomorphism. We define $\tilde{D} := \mathsf{Coker}(\rho)$. Using the quotient functors $\mathsf{Niq} \twoheadrightarrow \mathsf{Niq}^{\sim} \twoheadrightarrow \mathsf{Niq}^{\approx}$ one considers $D^{\sim}$, $D^{\approx}$ also as bifunctors on $\mathsf{Niq}^{\sim}$, or Niq.

Proposition 6. *One has the following commutative diagram of linear extensions:*

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & D^{\sim} \arrow[r, "\rho"] \arrow[d, equal] & D^{\approx} \arrow[r] \arrow[d] & \tilde{D} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D^{\sim} \arrow[r] & \mathsf{Niq} \arrow[r] \arrow[d] & \mathsf{Niq}^{\sim} \arrow[r] \arrow[d] & 0 \\
 & & \mathsf{Niq}^{\approx} \arrow[r, equal] \arrow[d] & \mathsf{Niq}^{\approx} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

In particular $\mathsf{Niq}^{\approx}$ *is also an additive category and the quotient functors* $\mathsf{Niq} \to \mathsf{Niq}^{\sim} \to \mathsf{Niq}^{\approx}$ *reflect isomorphisms and yield bijections on isomorphism classes of objects.*

Proof. The operation $\mathsf{Q}(G,H) \times \mathsf{Hom}(G^{\mathrm{ab}} \otimes G^{\mathrm{ab}}, [H,H]) \to \mathsf{Q}(G,H)$ given by $(f,\alpha) \mapsto f + \alpha$ yields the action of $D^{\sim}$ on the category Niq and by the property iv) one obtains a linear extension of categories

$$0 \to D^{\sim} \to \mathsf{Niq} \to \mathsf{Niq}^{\sim} \to 0.$$

By Proposition 2 for q-maps $f, g : G \to H$ one has $f \approx g$ if and only if there is a quadratic map $h : G^{\mathrm{ab}} \to [H,H]$ such that $f - g$ factors through h. This shows that

$$0 \to D^{\approx} \to \mathsf{Niq} \to \mathsf{Niq}^{\approx} \to 0$$

is a linear extension of categories. The rest follows from the properties of linear extensions. ☐

Remark 1. For an abelian group A the group $A \otimes A$ has a canonical involution $(a \otimes b)^{\sigma} = b \otimes a$. We put $\tilde{\Gamma}^2(A) := \{x \in A \otimes A \mid x^{\sigma} = x\}$. Then one has an exact sequence

$$0 \to \tilde{\Gamma}^2(A) \to A \otimes A \to \Lambda^2(A) \to 0$$

One easily sees that the class of abelian groups for which this sequence splits is closed under direct sums and contains all cyclic groups (and all uniquely 2-divisible groups). In particular the sequence splits provided A is a direct sum of cyclic groups. The exact sequence yields the following exact sequence

$$0 \to \mathsf{Hom}(\Lambda^2(A), B) \to \mathsf{Hom}(A \otimes A, B) \xrightarrow{\xi_{A,B}} \mathsf{Hom}(\tilde{\Gamma}^2(A), B)$$

for all abelian group B. It follows from the definition that there is an isomorphism $D^{\sim}(G,H) \cong \mathsf{Im}(\xi_{G^{\mathrm{ab}},[H,H]})$. In particular, if G^{ab} is a direct sum of cyclic groups, then $D^{\sim}(G,H) = \mathsf{Hom}(\tilde{\Gamma}^2(G^{\mathrm{ab}}), [H,H])$.

Definition 3. For abelian groups A and B we denote by $H_b^2(A,B)$ the subgroup of $H^2(A,B)$ generated by bilinear 2-cocycles. Thus by definition one has the following exact sequence

$$\mathsf{Quad}(A,B) \xrightarrow{(-|-)_?} \mathsf{Hom}(A\otimes A,B) \to H_b^2(A,B) \to 0$$

where the first map assigns to a quadratic map f its cross-effect $(-\mid -)_f$.

We now define the category $\mathsf{Niq}^{\mathrm{ab}}$, which has the same objects as the category $\mathsf{Nil}^{\mathrm{ab}}$. Thus objects are triples (A,B,e), where A and B are abelian groups, and $e\in H^2(A,B)$ is an element such that $\mu(e):\Lambda^2(A)\to B$ is an epimorphism. A morphism from (A,B,e) to (A',B',e') in $\mathsf{Niq}^{\mathrm{ab}}$ is a pair (f,g), where $f:A\to A'$ and $g:B\to B'$ are homomorphisms such that

$$f^*(e')-g_*(e)\in H_b^2(A,B').$$

Theorem 3. *The functor* $\mathsf{ch}:\mathsf{Nil}^{\sim}\to\mathsf{Nil}^{\mathrm{ab}}$ *has a canonical extension*

$$\mathsf{ch}:\mathsf{Niq}^{\approx}\to\mathsf{Niq}^{\mathrm{ab}}$$

which is an equivalence of categories.

Proof. On the objects one puts

$$\mathsf{ch}(G)=(G^{\mathrm{ab}},[G,G],\mathsf{e}(G)).$$

If $f:G\to H$ is a q-map, then one puts

$$\mathsf{ch}(G\xrightarrow{f}H)=(f^{\mathrm{ab}},[f,f]).$$

We claim that one has

$$\left(f^{\mathrm{ab}}\right)^*(\mathsf{e}(H))-[f,f]_*(\mathsf{e}(G))\in H_b^2(G^{\mathrm{ab}},[H,H]).$$

Let α (resp. β) be a 2-cocycle representing the class $\mathsf{e}(G)$ (resp. $\mathsf{e}(H)$). Thus $G=G^{\mathrm{ab}}\times[G,G]$ (resp. $H=H^{\mathrm{ab}}\times[H,H]$) as a set, with the group structure $(a,u)+(b,v)=(a+b,\alpha(a,b)+u+v)$, where $a,b\in G^{\mathrm{ab}}$ and $u,v\in[G,G]$ (resp. $(c,x)+(d,y)=(c+d,\beta(c,d)+x+y)$, $c,d\in H^{\mathrm{ab}}$, $x,y\in[H,H]$). Any q-map $f:G\to H$ has the form $f(a,u)=(f^{\mathrm{ab}}(a),[f,f](u)+\gamma(a))$, where $f^{\mathrm{ab}}:G^{\mathrm{ab}}\to H^{\mathrm{ab}}$ and $[f,f]:[G,G]\to[H,H]$ are homomorphisms, while $\gamma:G^{\mathrm{ab}}\to[H,H]$ is a map. One has

$$\begin{aligned}f((a,u)+(b,v))&=f(a+b,\alpha(a,b)+u+v)=\\&(f^{\mathrm{ab}}(a)+f^{\mathrm{ab}}(b),\gamma(a+b)+[f,f](\alpha(a,b))+[f,f](u)+[f,f](v)).\end{aligned}$$

On the other hand, we have

$$f((a,u)+(b,v))=f((a,u))+f((b,v))+((a,u),(b,v))_f.$$

Since the cross-effect of f factors through $\delta:G^{\mathrm{ab}}\otimes G^{\mathrm{ab}}\to[H,H]$, we obtain

$$\begin{aligned}&f((a,u)+(b,v))=\\&\quad(f^{\mathrm{ab}}(a),[f,f](u)+\gamma(a))+(f^{\mathrm{ab}}(b),[f,f](v)+\gamma(b))+(0,\delta(a,b))=\\&(f^{\mathrm{ab}}(a)+f^{\mathrm{ab}}(b),\beta(f^{\mathrm{ab}}(a),f^{\mathrm{ab}}(b))+\gamma(a)+\gamma(b)+\delta(a,b)+[f,f](u)+[f,f](v)).\end{aligned}$$

Comparing these expressions we obtain

$$\gamma(a+b)+[f,f]\left(\alpha(a,b)\right)=\beta(f^{\mathrm{ab}}(a),f^{\mathrm{ab}}(b))+\gamma(a)+\gamma(b)+\delta(a,b).$$

Thus the class $\left(f^{\mathrm{ab}}\right)^*(\mathsf{e}(H))-[f,f]_*(\mathsf{e}(G))$ in the group $H^2_b(G^{\mathrm{ab}},[H,H])$ coincides with the class of $-\delta$ and the claim is proved. It follows that ch is a well-defined functor $\mathsf{Niq}\to\mathsf{Niq}^{\mathrm{ab}}$, which obviously factors trough the category $\mathsf{Niq}^{\approx}$. By our construction and by the definition of $\mathsf{Niq}^{\approx}$ the induced map

$$\mathsf{Hom}_{\mathsf{Niq}^{\approx}}(G,H)\to\mathsf{Hom}_{\mathsf{Niq}^{\mathrm{ab}}}(\mathsf{ch}(G),\mathsf{ch}(H))$$

is an injection. Let us show that this map is surjective as well. Take any morphism $(g,h):\mathsf{ch}(G)\to\mathsf{ch}(H)$ in $\mathsf{Niq}^{\mathrm{ab}}$. Then $g:G^{\mathrm{ab}}\to H^{\mathrm{ab}}$ and $h:[G,G]\to[H,H]$ are homomorphisms such that

$$h(\alpha(a,b))-\beta(g(a),g(b))=-\gamma(a+b)+\gamma(a)+\gamma(b)+\delta(a,b)$$

where $\delta:G^{\mathrm{ab}}\otimes G^{\mathrm{ab}}\to[H,H]$ is a homomorphism, $\gamma:G^{\mathrm{ab}}\to[H,H]$ is a map, while α and β are as above. Define the map $f:G\to H$ via $f(a,u)=(g(a),\gamma(a)+h(u))$. Then one has

$$((a,u),(b,v))_f=\delta(a,b).$$

Thus f is a q-map with $f^{\mathrm{ab}}=g$ and $[f,f]=h$. Therefore ch is full and faithful. By Theorem 2 the functor ch is surjective on isomorphism classes of objects and the result follows. ☐

7. q-Split Groups

We start with the following definitions.

Definition 4. Call nil$_2$-groups *similar* if they have isomorphic abelianizations and isomorphic commutator subgroups.

Definition 5. Call a nil$_2$-group G *q-split* if the quotient map $G\twoheadrightarrow G^{\mathrm{ab}}$ has a quadratic section. It is easy to see that this section is then a q-map.

Lemma 18. *The class of q-split groups contains all abelian groups and is closed under products and coproducts.*

Proof. For products and abelian groups this is obvious. For coproducts, note that the central extension

$$0\to G_1^{\mathrm{ab}}\otimes G_2^{\mathrm{ab}}\to G_1\vee G_2\to G_1\times G_2\to 0$$

has a quadratic section s given by $s(g_1,g_2)=(0,g_1,g_2)$. One easily checks that

$$((x_1,x_2)\mid(y_1,y_2))_s=(y_1\otimes x_2,0,0)=[(0,y_1,0),(0,0,x_2)].$$

Thus s is a q-map. Since $(G_1\vee G_2)^{\mathrm{ab}}=(G_1\times G_2)^{\mathrm{ab}}=G_1^{\mathrm{ab}}\times G_2^{\mathrm{ab}}$ we see that for any quadratic sections $s_i:G_i^{\mathrm{ab}}\to G_i$ of the natural projections $G_i\twoheadrightarrow G_i^{\mathrm{ab}}$, $i=1,2$, the composite $s\circ(s_1\times s_2):(G_1\vee G_2)^{\mathrm{ab}}\to G_1\vee G_2$ is a section. Since s, s_1, s_2 are q-maps, $s\circ(s_1\times s_2)$ is also a q-map and the result follows. ☐

Example 3. It follows that the dihedral group $D_4 \cong \mathbf{Z}/2\mathbf{Z} \vee \mathbf{Z}/2\mathbf{Z}$ of order 8 is q-split. Let us show that the quaternion group

$$Q_8 = \langle \tau, \omega \mid 2\tau = 2\omega, \omega + \tau - \omega = -\tau \rangle$$

of order 8 is also q-split. Observe that τ and ω are of order 4 and $[\omega, \tau] = 2\tau$. So one has $Q_8^{\mathrm{ab}} \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \cong D_4^{\mathrm{ab}}$ and $[Q_8, Q_8] \cong \mathbf{Z}/2\mathbf{Z} \cong [D_4, D_4]$. One easily checks that the map $s : Q_8^{\mathrm{ab}} \to Q_8$ given by $s(0) = 0$, $s(\hat{\omega}) = \omega$, $s(\hat{\tau}) = \tau$, $s(\hat{\omega} + \hat{\tau}) = \omega + \tau$ is a quadratic section of $Q_8 \to Q_8^{\mathrm{ab}}$. It follows from Corollary 4 below that Q_8 and D_4 are isomorphic in Niq.

Lemma 19. *A nil$_2$-group G is q-split if and only if the class $\mathsf{e}(G) \in H^2(G^{\mathrm{ab}}, [G, G])$ belongs to the subgroup $H_b^2(G^{\mathrm{ab}}, [G, G])$.*

Proof. Let $u : G^{\mathrm{ab}} \to G$ be a quadratic section. Then the class $\mathsf{e}(G)$ can be represented by the cocycle $(a_1, a_2) \mapsto u(a_1) + u(a_2) - u(a_1 + a_2)$ which is bilinear and therefore lies in $H_b^2(G^{\mathrm{ab}}, [G, G])$. Conversely, if the class $\mathsf{e}(G) \in H^2(G^{\mathrm{ab}}, [G, G])$ is represented by a bilinear map $f : G^{\mathrm{ab}} \times G^{\mathrm{ab}} \to [G, G]$, then G is isomorphic to the set $G^{\mathrm{ab}} \times [G, G]$ with the group structure defined by $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2 + f(a_1, a_2))$, and the projection of the latter to G^{ab} has a quadratic section given by $a \mapsto (a, 0)$. □

We denote, respectively, by $\mathsf{Spl}(\mathsf{Niq})$ and $\mathsf{Spl}(\mathsf{Niq}^{\approx})$ the full subcategories of Niq and $\mathsf{Niq}^{\approx}$ with objects all q-split groups. They are related via the linear extension

$$0 \to D^{\approx} \to \mathsf{Spl}(\mathsf{Niq}) \to \mathsf{Spl}(\mathsf{Niq}^{\approx}) \to 0$$

and in particular they have the same isoclasses of objects. According to Theorem 3 and Lemma 19 the category $\mathsf{Spl}(\mathsf{Niq}^{\approx})$ is equivalent to the category $\mathsf{Spl}^{\mathrm{ab}}$, which is the full subcategory of the category $\mathsf{Nil}^{\mathrm{ab}}$ on those objects (A, B, e) of $\mathsf{Nil}^{\mathrm{ab}}$ satisfying $e \in H_b^2(A, B)$. Let us observe that

$$\mathsf{Hom}_{\mathsf{Spl}^{\mathrm{ab}}}((A, B, e), (A', B', e')) = \mathsf{Hom}(A, A') \times \mathsf{Hom}(B, B')$$

because the compatibility condition required in the definition of morphisms in $\mathsf{Nil}^{\mathrm{ab}}$ holds automatically in $\mathsf{Spl}^{\mathrm{ab}}$.

We now consider another category Spl, which is a full subcategory of the product category $\mathsf{Ab} \times \mathsf{Ab}$. Objects of the category Spl are pairs of abelian groups (A, B) for which there exists a homomorphism $f : A \otimes A \to B$ such that $f^a : \Lambda^2(A) \to B$ is an epimorphism, where $f^a(x, y) := f(x, y) - f(y, x)$.

Theorem 4. *The categories Spl and $\mathsf{Nil}^{\mathrm{ab}}$ are equivalent. Thus, two q-split groups are isomorphic in Niq if and only if they are similar.*

Proof. Take any object (A, B) of Spl and choose $f : A \otimes A \to B$ for which f^a is an epimorphism. Let $e_f \in H_b^2(A, B)$ be the class corresponding to f. Then $(A, B, e_f) \in \mathsf{Nil}^{\mathrm{ab}}$. Then $(A, B, f) \mapsto (A, B, e_f)$ yields the expected equivalence of categories. □

Remark 2. One easily sees that the class S of abelian groups for which the natural short exact sequence $0 \to \Lambda^2(A) \to A \otimes A \to \mathrm{S}^2(A) \to 0$ splits contains all cyclic groups, all uniquely 2-divisible groups and is closed under direct sums. In particular any finitely generated abelian group lies in S. If $A \in \mathsf{S}$ then for any homomorphism $g : \Lambda^2(A) \to B$ there exists a homomorphism $f : A \otimes A \to B$ such that $f^a = g$. It follows that a pair of abelian groups (A, B) with $A \in \mathsf{S}$ belongs to Spl if and only if there exists an epimorphism $\Lambda^2(A) \to B$.

Proposition 7. *For abelian groups A, B there is a commutative diagram with exact rows*

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{Hom}(\mathrm{S}^2A, B) \arrow[r, tail] \arrow[d, dashed] & \mathsf{Hom}(A \otimes A, B) \arrow[r, tail] \arrow[d, "\alpha"] & \mathsf{Hom}(\Lambda^2A, B) \arrow[d, equal] & \\
0 \arrow[r] & \mathsf{Ext}(A, B) \arrow[r] & H^2(A; B) \arrow[r, "c"] & \mathsf{Hom}(\Lambda^2A, B) \arrow[r] & 0
\end{tikzcd} \tag{6}$$

where the image of the homomorphism α is equal to the subgroup $H^2_b(A, B)$ from Definition 3.

Proof. For any abelian group A one has a short exact sequence

$$0 \to \Lambda^2A \to A \otimes A \to \mathrm{S}^2A \to 0,$$

We place in the upper row of (6) the sequence induced by this short exact sequence. The lower row is the universal coefficient exact sequence, and the map α is given by considering a bilinear map as a 2-cocycle. The rest is obvious. □

Remark 3. The arrow on the upper right of (6) is surjective if A is either uniquely 2-divisible or is a direct sum of cyclic groups. Indeed, in these cases the afore-mentioned short exact sequence splits.

Proposition 8. *If* 2 *is invertible in B, then the above homomorphism*

$$\mathsf{Hom}(\mathrm{S}^2A, B) \to \mathsf{Ext}(A, B)$$

is zero.

Proof. For a homomorphism $f : \mathrm{S}^2A \to B$ the class $\alpha(f)$ is represented by the cocycle $(x, y) \mapsto f(xy)$. This cocycle is the coboundary of the cochain $g : A \to B$ given by $a \mapsto \frac{1}{2}f(a^2)$. □

Lemma 20. *Let A be any abelian group and let B be a uniquely 2-divisible group. Then for any $x \in H^2_b(A; B)$ and any $0 \neq a \in \mathsf{Ext}(A, B)$ one has $x + a \notin H^2_b(A, B)$.*

Proof. Otherwise one would have $a \in \mathsf{Im}\,\alpha$, which contradicts the preceding lemma. □

Remark 4. It follows that for any object (A, B, x) of $\mathsf{Niq}^{\mathrm{ab}}$ and any $a \in \mathsf{Ext}(A, B)$, $(A, B, x + a)$, too, gives an object of $\mathsf{Niq}^{\mathrm{ab}}$, since in the universal coefficient exact sequence in (6) one has $c(x + a) = c(x)$. In particular, if (A, B, x) with uniquely 2-divisible B lies in the subcategory $\mathsf{Spl}^{\mathrm{ab}}$ and $a \neq 0$, then $(A, B, x + a) \in \mathsf{Niq}^{\mathrm{ab}}$ cannot belong to $\mathsf{Spl}^{\mathrm{ab}}$, since by Lemma 20, $x + a \notin H^2_b(A; B)$. Thus not all nil$_2$-groups are q-split. Some explicit examples of non-q-split groups follow.

Example 4. For each prime p consider the semidirect product $\mathbf{Z}/p^2\mathbf{Z} \rtimes \mathbf{Z}/p\mathbf{Z}$, where the generator of $\mathbf{Z}/p\mathbf{Z}$ acts on $\mathbf{Z}/p^2\mathbf{Z}$ via multiplication by $p+1$. This group is similar in the sense of Definition 4 to $\mathbf{Z}/p\mathbf{Z} \vee \mathbf{Z}/p\mathbf{Z}$ (both have abelianizations isomorphic to $(\mathbf{Z}/p\mathbf{Z})^2$ and commutator subgroups isomorphic to $\mathbf{Z}/p\mathbf{Z}$). For $p = 2$ these groups are in fact isomorphic; however for odd p they are not, since the former has exponent p^2 and the latter has exponent p. Thus in the diagram (6) for $A = (\mathbf{Z}/p\mathbf{Z})^2$ and $B = \mathbf{Z}/p\mathbf{Z}$, the classes of $\mathbf{Z}/p\mathbf{Z} \vee \mathbf{Z}/p\mathbf{Z}$ and $\mathbf{Z}/p^2\mathbf{Z} \rtimes \mathbf{Z}/p\mathbf{Z}$ in $H^2((\mathbf{Z}/p\mathbf{Z})^2; \mathbf{Z}/p\mathbf{Z})$ are not equal. On the other hand, one can choose isomorphisms of their commutator subgroups with $\mathbf{Z}/p\mathbf{Z}$ in a way which makes obvious that these classes have the same image under the homomorphism c defined in (6) above, hence they differ by a nonzero element of $\mathsf{Ext}(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z})$. But $\mathbf{Z}/p\mathbf{Z} \vee \mathbf{Z}/p\mathbf{Z}$ is q-split by Lemma 18, hence its class is in the image of the homomorphism α from (6). Then by Lemma 20 we conclude that $\mathbf{Z}/p^2\mathbf{Z} \rtimes \mathbf{Z}/p\mathbf{Z}$ is not q-split. In particular, the above similar groups are also not isomorphic in Niq.

8. q-Maps for Uniquely 2-Divisible Groups

Let us recall the relevant part of the classical Maltsev correspondence between nilpotent groups and Lie algebras. In the nil_2 case it amounts to an isomorphism of categories from the category of nil_2 Lie algebras over $\mathbf{Z}[\frac{1}{2}]$, i.e. Lie algebras with $[L,[L,L]] = 0$, to the category of uniquely 2-divisible nil_2-groups. In what follows, all Lie algebras are understood to be of the above kind, i.e. class two nilpotent Lie $\mathbf{Z}[\frac{1}{2}]$-algebras. Let us denote by $\mathsf{Nil}(\mathbf{Z}[\frac{1}{2}])$ the category of these algebras and their homomorphisms. Moreover, we will denote by $\mathsf{Nil}^{\frac{1}{2}}$ the category of uniquely 2-divisible nil_2-groups.

One defines an isomorphism of categories

$$\exp : \mathsf{Nil}(\mathbf{Z}[\tfrac{1}{2}]) \to \mathsf{Nil}^{\frac{1}{2}}$$

by declaring, for an algebra $L \in \mathsf{Nil}(\mathbf{Z}[\frac{1}{2}])$, $\exp(L)$ to be the set L equipped with the operation

$$a \oplus b = a + b + \tfrac{1}{2}[a,b].$$

This is a group, with zero element 0 and the inverse of an element a given by $-a$. Moreover, the commutator with respect to this group structure coincides with the Lie bracket so that for any L one has $[\exp(L), \exp(L)] = [L,L]$ and $\exp(L)^{\mathrm{ab}} = L^{\mathrm{ab}}$, where $L^{\mathrm{ab}} = L/[L,L]$ is the abelianization of the Lie algebra L.

Now, clearly, any Lie algebra homomorphism is also a homomorphism with respect to $\oplus$. Moreover, we have $a \oplus c = a + c$ for $c \in [L,L]$ so that for any $a, b \in L$

$$a + b = a \oplus b \oplus \tfrac{1}{2}[b,a].$$

It follows that likewise conversely, a map which is a homomorphism with respect to $\oplus$ is a Lie algebra homomorphism so that exp is an isomorphism of categories, with the inverse isomorphism log defined as follows: for a uniquely 2-divisible nil_2-group G the Lie algebra $\log(G)$ is the set G equipped with the addition as above and with the bracket equal to the commutator map.

Our aim in this section is to prove

Theorem 5. *Two uniquely 2-divisible nil*$_2$ *groups* G, G' *are isomorphic as objects of* Niq *if and only if there exists an isomorphism of the abelian groups* $g : \log(G) \to \log(G')$ *such that* $g[G, G] = [G', G']$.

For the proof we must define an analog of the category Niq from Section 4 for Lie algebras. For this, we first formulate

Definition 6. A map $f : L \to L'$ between the Lie algebras is called a q-map if it is a quadratic map between the underlying abelian groups and moreover for any $a, b \in L$ and any $c \in [L, L]$ one has $(a \mid b)_f \in [L', L']$, $f(a+c) = f(a) + f(c)$ and $f(c) \in [L', L']$.

Moreover, we consider the following category $\mathsf{Niq}(\mathbf{Z}[\frac{1}{2}]) \supset \mathsf{Nil}(\mathbf{Z}[\frac{1}{2}])$ with the same objects as the latter. A morphism $L \to L'$ in $\mathsf{Niq}(\mathbf{Z}[\frac{1}{2}])$ is a q-map in the sense just defined.

The key observation is then

Theorem 6. *The functor* exp *extends to an isomorphism of categories*

$$\mathsf{Niq}(\mathbf{Z}[\tfrac{1}{2}]) \simeq \mathsf{Niq}^{\frac{1}{2}}.$$

This theorem follows immediately from

Proposition 9. *Let* $f : L \to L'$ *be a map between Lie algebras. Then the following assertions are equivalent:*

i) f *is a q-map in the sense of Definition* 6*;*
ii) f *is a q-map when considered as a map* $\exp(L) \to \exp(L')$*;*
iii) *there exists a linear map* $g : L \to L'$ *with* $g[L, L] \subseteq [L', L']$ *and a symmetric bilinear map* $h : L^{\mathrm{ab}} \times L^{\mathrm{ab}} \to [L', L']$ *such that one has*

$$f(a) = g(a) + \tfrac{1}{2}h(\hat{a}, \hat{a})$$

for all $a \in L$.

Proof. ii) $\iff$ iii):

Let $(a \mid b)^{+}_{f}$, $(a \mid b)^{\oplus}_{f}$ denote the cross-effect of f with respect to the corresponding operations. Thus f is a q-map when considered as a map $\exp(L) \to \exp(L')$ if and only if $(a \mid b)^{\oplus}_{f}$ is bilinear and lands in $[L', L']$. In that case we have

$$f(a+b) = f(a \oplus b \oplus \tfrac{1}{2}[b, a]) = f(a \oplus b) \oplus f(\tfrac{1}{2}[b, a]) = fa \oplus fb \oplus (a \mid b)^{\oplus}_{f} \oplus \tfrac{1}{2}f[b, a]$$

and

$$fa + fb = fa \oplus fb \oplus \tfrac{1}{2}[fb, fa],$$

hence

$$\begin{aligned}
&(a \mid b)^{+}_{f} = -(fa + fb) + f(a + b) = \\
&-(fa + fb) \oplus f(a + b) \oplus \tfrac{1}{2}[f(a + b), -(fa + fb)] = \\
&\tfrac{1}{2}[fa, fb] \oplus (a \mid b)^{\oplus}_{f} \oplus \tfrac{1}{2}f[b, a] \oplus \tfrac{1}{2}[fa \oplus fb \oplus (a \mid b)^{\oplus}_{f} \oplus \tfrac{1}{2}f[b, a], -(fa \oplus fb \oplus \tfrac{1}{2}[fb, fa])] \\
&= \tfrac{1}{2}[fa, fb] \oplus (a \mid b)^{\oplus}_{f} \oplus \tfrac{1}{2}f[b, a].
\end{aligned}$$

The latter expression is then symmetric since it is the cross-effect of some map with respect to the commutative operation $+$. It is bilinear with respect to $\oplus$ and satisfies

$$(a \oplus c \mid b)_f^+ = (a \mid b \oplus c)_f^+ = (a \mid b)_f^+$$

for any $c \in [L, L]$ and any $a, b \in L$. Hence it is also bilinear with respect to $+$ and defining

$$h(\hat{a}, \hat{b}) = (a \mid b)_f^+$$

gives a well-defined symmetric bilinear map $h : L^{\mathrm{ab}} \times L^{\mathrm{ab}} \longrightarrow [L', L']$. Then the map $g : L \longrightarrow L'$ given by

$$g(a) = f(a) - \tfrac{1}{2}h(\hat{a}, \hat{a}) = f(a) - \tfrac{1}{2}(a \mid a)_f^\oplus$$

carries $[L, L]$ to $[L', L']$. Moreover, this map is linear since $(a \mid a)_f^\oplus = (a \mid a)_f^+$ for any $a \in L$, so that

$$\begin{aligned} g(a+b) =& f(a+b) - \tfrac{1}{2}(a+b \mid a+b)_f^+ \\ =& fa + fb + (a \mid b)_f - \tfrac{1}{2}(a \mid a)_f^+ - \tfrac{1}{2}(b \mid b)_f^+ - \tfrac{1}{2}(a \mid b)_f^+ - \tfrac{1}{2}(b \mid a)_f^+ \\ =& g(a) + g(b). \end{aligned}$$

Conversely, given g and h as in iii), we compute

$$\begin{aligned} &(a \mid b)_f^\oplus = -(fa \oplus fb) \oplus f(a \oplus b) = \\ &-(fa + fb + \tfrac{1}{2}[fa, fb]) + f(a + b + \tfrac{1}{2}[a, b]) + \tfrac{1}{2}[fa + fb + \tfrac{1}{2}[fa, fb], f(a + b + \tfrac{1}{2}[a, b])] \\ &= -(ga + \tfrac{1}{2}h(\hat{a}, \hat{a}) + gb + \tfrac{1}{2}h(\hat{b}, \hat{b}) + \tfrac{1}{2}[ga, gb]) \\ &+ ga + gb + \tfrac{1}{2}g[a, b] + \tfrac{1}{2}h(\widehat{a+b}, \widehat{a+b}) + \tfrac{1}{2}[-(ga + gb), ga + gb] \\ &= -\tfrac{1}{2}[ga, gb] + \tfrac{1}{2}g[a, b] + h(\hat{a}, \hat{b}) \end{aligned}$$

which lies in $[L', L']$ and is bilinear, so f is a q-map.

i) $\iff$ iii):

Obviously, any f satisfying iii) is quadratic. Moreover, a map f between the $\mathbf{Z}[\frac{1}{2}]$-modules is quadratic if and only if it has the form

$$f(a) = g(a) + \tfrac{1}{2}h(a, a)$$

for a unique linear map g and a bilinear symmetric map h. One just takes $g(a) = 2f(a) - \frac{1}{2}f(2a)$ and $h(a, b) = f(a + b) - f(a) - f(b)$. Then it is easy to check that a quadratic map is a q-map of Lie algebras if and only if the corresponding g and h satisfy the conditions in iii). $\square$

This enables us to obtain an extension to the q-world of the above classical Maltsev equivalence by identifying the full subcategory $\mathsf{Niq}^{\frac{1}{2}} \subset \mathsf{Niq}$ on the uniquely 2-divisible nil$_2$-groups with the following category defined in terms of Lie $\mathbf{Z}[\frac{1}{2}]$-algebras.

Moreover, in this situation Theorem 3 admits a strengthening. To formulate it we will need some more categories.

Definition 7. Let $\mathsf{Niq}_0(\mathbf{Z}[\frac{1}{2}]) \subset \mathsf{Niq}(\mathbf{Z}[\frac{1}{2}])$ be the subcategory with the same objects and those morphisms which are actually linear. That is, a morphism from L to L' in $\mathsf{Niq}_0(\mathbf{Z}[\frac{1}{2}])$ is an abelian group homomorphism $g : L \to L'$ with $g[L, L] \subseteq [L', L']$.

Moreover, let $\widetilde{\mathsf{Niq}}_0(\mathbf{Z}[\frac{1}{2}])$ be the quotient category of $\mathsf{Niq}_0(\mathbf{Z}[\frac{1}{2}])$ obtained by identifying those $g_1, g_2 : L \to L'$ for which $g_1|_{[L,L]} = g_2|_{[L,L]}$ and $g_1^{\mathrm{ab}} = g_2^{\mathrm{ab}} : L^{\mathrm{ab}} \to L'^{\mathrm{ab}}$.

We then have

Proposition 10. *There are linear extensions*

$$0 \to \Phi \to \mathsf{Niq}(\mathbf{Z}[\tfrac{1}{2}]) \xrightarrow{q} \mathsf{Niq}_0(\mathbf{Z}[\tfrac{1}{2}]) \to 0$$

and

$$0 \to \tilde{\Phi} \to \mathsf{Niq}_0(\mathbf{Z}[\tfrac{1}{2}]) \xrightarrow{\tilde{q}} \widetilde{\mathsf{Niq}}_0(\mathbf{Z}[\tfrac{1}{2}]) \to 0$$

defined as follows. The functor $q : \mathsf{Niq}(\mathbf{Z}[\frac{1}{2}]) \to \mathsf{Niq}_0(\mathbf{Z}[\frac{1}{2}])$ *is identity on objects and given on the morphisms via*

$$q(f)(a) = 2f(a) - \tfrac{1}{2}f(2a).$$

The bifunctor $\Phi : \mathsf{Niq}_0(\mathbf{Z}[\frac{1}{2}])^{\mathrm{op}} \times \mathsf{Niq}_0(\mathbf{Z}[\frac{1}{2}]) \to \mathsf{Ab}$ *is given by*

$$\Phi(L, L') = \mathsf{Hom}(\mathrm{S}^2(L^{\mathrm{ab}}), [L', L']).$$

The functor $\tilde{q}$ *is a canonical quotient functor, and* $\tilde{\Phi}$ *is given by*

$$\tilde{\Phi}(L, L') = \mathsf{Hom}(L^{\mathrm{ab}}, [L', L']).$$

Moreover, the categories $\mathsf{Niq}_0(\mathbf{Z}[\frac{1}{2}])$ *and* $\widetilde{\mathsf{Niq}}_0(\mathbf{Z}[\frac{1}{2}])$ *are both additive, and the functor* q *has a section given by the embedding.*

Proof. The additivity of $\mathsf{Niq}_0(\mathbf{Z}[\frac{1}{2}])$ follows from the obvious fact that for any morphisms $g_1, g_2 : L \to L'$ in $\mathsf{Niq}_0(\mathbf{Z}[\frac{1}{2}])$ the maps $g_1 \pm g_2$ are morphisms of $\mathsf{Niq}_0(\mathbf{Z}[\frac{1}{2}])$ too.

The rest is clear in view of the above consideration. Indeed, we can replace a morphism $f : L \to L'$ in $\mathsf{Niq}(\mathbf{Z}[\frac{1}{2}])$ by a pair (g, h) as in iii) of Proposition 9. Under this identification the functor q becomes the projection sending (g, h) to g and the first linear extension becomes obvious. The second one is straightforward. □

Definition 8. Let $\mathsf{Niq}^{\mathrm{ab}}(\mathbf{Z}[\frac{1}{2}])$ denote the category whose objects are short exact sequences

$$0 \to B \to E \to A \to 0$$

of $\mathbf{Z}[\frac{1}{2}]$-modules such that there exists a surjective homomorphism $\pi : \Lambda^2(A) \twoheadrightarrow B$. A morphism from $0 \to B \to E \to A \to 0$ to $0 \to B' \to E' \to A' \to 0$ is a pair $(\alpha : A \to A', \beta : B \to B')$ of homomorphisms such that there exists

$\varepsilon : E \to E'$ making the diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, "\beta"'] & E \arrow[r] \arrow[d, dashed, "\varepsilon"'] & A \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & B' \arrow[r] & E' \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

commute. We do not make π or ε part of the structure, in particular π is not required to be compatible with α and β in any way.

Note that $\mathsf{Niq}^{\mathrm{ab}}(\mathbf{Z}[\frac{1}{2}])$ is an additive category, since for any A, A' there are surjective homomorphisms $\Lambda^2(A \oplus A') \twoheadrightarrow \Lambda^2(A) \oplus \Lambda^2(A')$ and, moreover, for any morphism (α, β) in $\mathsf{Niq}^{\mathrm{ab}}(\mathbf{Z}[\frac{1}{2}])$ the pair $(-\alpha, -\beta)$ is also a morphism.

There is a functor $r : \widetilde{\mathsf{Niq}}_0(\mathbf{Z}[\frac{1}{2}]) \to \mathsf{Niq}^{\mathrm{ab}}(\mathbf{Z}[\frac{1}{2}])$ sending a Lie algebra L to the short exact sequence

$$0 \to [L, L] \to L \to L^{\mathrm{ab}} \to 0$$

and the morphism $[g] : L \to L'$ to the pair $(g^{\mathrm{ab}}, g|_{[L,L]})$, where $[g]$ denotes the equivalence class of g and $g^{\mathrm{ab}} : L^{\mathrm{ab}} \to L'^{\mathrm{ab}}$ is the homomorphism induced by g which exists since $g[L, L] \subseteq [L', L']$.

Proposition 11. *The above functor* r *yields an equivalence of categories*

$$\widetilde{\mathsf{Niq}}_0(\mathbf{Z}[\tfrac{1}{2}]) \simeq \mathsf{Niq}^{\mathrm{ab}}(\mathbf{Z}[\tfrac{1}{2}]).$$

Proof. First, r is surjective on the objects, since for any object $0 \to B \to E \to A \to 0$ of $\mathsf{Niq}^{\mathrm{ab}}(\mathbf{Z}[\frac{1}{2}])$ any surjective homomorphism $\Lambda^2(A) \twoheadrightarrow B$ determines a bracket

$$[,] : \Lambda^2(E) \twoheadrightarrow \Lambda^2(A) \twoheadrightarrow B \rightarrowtail E$$

on E which turns it into a nil_2 Lie algebra with $[E, E] = B$ and $E^{\mathrm{ab}} = A$.

Next, r is full since a morphism from the object $0 \to [L, L] \to L \to L^{\mathrm{ab}} \to 0$ to the object $0 \to [L', L'] \to L' \to L'^{\mathrm{ab}} \to 0$ in $\mathsf{Niq}^{\mathrm{ab}}(\mathbf{Z}[\frac{1}{2}])$ is, by definition, a pair of linear maps $\beta : [L, L] \to [L', L']$, $\alpha : L^{\mathrm{ab}} \to L'^{\mathrm{ab}}$ for which there exists a linear map $g : L \to L'$ fitting in the appropriate diagram, which means that $\beta = g|_{[L,L]}$ and $\alpha = g^{\mathrm{ab}}$.

Finally, r is faithful since for $g_1, g_2 : L \to L'$ one has $r[g_1] = r[g_2]$ if and only if g_1 and g_2 are equivalent in the sense of Definition 7, i.e. if and only if $[g_1] = [g_2]$. ☐

We can now finish the proof of our theorem.

Proof of Theorem 5. There is a chain of functors

$$\mathsf{Niq}^{\frac{1}{2}} \xrightarrow{(6)} \mathsf{Niq}(\mathbf{Z}[\tfrac{1}{2}]) \xrightarrow{(10)} \mathsf{Niq}_0(\mathbf{Z}[\tfrac{1}{2}]) \xrightarrow{(10)} \widetilde{\mathsf{Niq}}_0(\mathbf{Z}[\tfrac{1}{2}]) \xrightarrow{(11)} \mathsf{Niq}^{\mathrm{ab}}(\mathbf{Z}[\tfrac{1}{2}])$$

each of which is either an equivalence or a linear extension. The statement of Theorem 5 is that the objects on the left are isomorphic if and only if their images under the composite functor are. This is clear since any linear extension reflects the isomorphy of objects. ☐

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