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ON THE BASIS OF THE SPACE OF CUSP FORMS OF THE LEVEL 12

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Dedicated to the memory of Professor G. Lomadze

Abstract. The basis of the space of cusp forms $S_{k/2}(\widetilde{\Gamma}_0(12),\chi)$ is constructed for any integer k and character mod 12.

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1. We will mostly use the notions and notation from [1] and [2].

Let

$$Q(X)=Q(x_1,\dots,x_f)=\sum_{1\le r\le s\le f} b_{rs}x_rx_s$$

be a positive quadratic form with integral coefficients b_{rs}; D denote the determinant of the quadratic form

$$2Q(X)=\sum_{r,s=1}^{f} a_{rs}x_rx_s \quad (a_{rr}=2b_{rr};\quad a_{sr}=a_{rs}=b_{rs},\quad r<s).$$

N be the level of $Q(X)$ (see [2], p. 207); $\chi(d)$ be the character of $Q(X)$, i.e., if $2\mid f$, then $\chi(d)=(\operatorname{sgn} d)^{f/2}\big(\frac{(-1)^{f/2}D}{|d|}\big)$ and if $2\nmid f$, then $\chi(d)=\big(\frac{2D}{|d|}\big)$ $\big((\frac{\cdot}{\cdot})$ is the Kronecker symbol). A positive quadratic form with f variables of level N with the character χ is called a quadratic form of type $(f/2,N,\chi)$. Further, suppose that $z=\exp(2\pi i\tau)$, $z_N=\exp(2\pi i\tau/N)$, $\operatorname{Im}\tau>0$, p is an odd prime number, $N,N_1\in\mathbb{N}$, $4\mid N$.

If $F(\tau)\in M_{k/2}(\widetilde{\Gamma}_0(N),\chi)$, $F(\tau)\ne 0$, then in the neighbourhood of the cusp $\zeta=\infty$ (see [1], Ch. IV, §1)

$$F(\tau)=\sum_{m=m_0\ge 0}^{\infty} a_{m_0}z^m,\quad a_{m_0}\ne 0.$$

The number

$$\operatorname{ord}(F(\tau),\infty,\Gamma_0(N))=m_0 \tag{1}$$

is called the order of $F(\tau)\ne 0$ at the cusp $\zeta=\infty$ with respect to $\Gamma_0(N)$.

In what follows let $\tau^{k/2}=(\sqrt{\tau})^k=|\tau|^{k/2}\exp(\pi i\varphi k)$, where $\sqrt{|\tau|}>0$, $-\frac{\pi}{2}<\varphi\le\frac{\pi}{2}$, $k\in\mathbb{Z}$, $\big(\frac{c}{d}\big)$ is the Jacobi symbol when $2\nmid d$, $d\in\mathbb{N}$, and if d is odd

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negative, then $\left(\frac{c}{d}\right) = \operatorname{sgn} c\left(\frac{c}{|d|}\right)$; also $\left(\frac{0}{\pm 1}\right) = 1$;

$$\varepsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ i & \text{if } d \equiv -1 \pmod 4. \end{cases}$$

Let $F(\tau)$ be any function on $\mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im} \tau > 0\}$. Then for any $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and $k \in \mathbb{Z}$ let

$$F(\tau)|_k L = (c\tau + d)^{-k} F(L\tau);$$
$$F(\tau)|_{k/2} \widetilde{L} = \left(\varepsilon_d \left(\frac{c}{d}\right) (c\tau + d)^{1/2}\right)^{-k} F(L\tau).$$

Let $Q(X)$ be a quadratic form of type $(f/2, N, \chi)$, A be the matrix of $2Q(X)$, $h \in \mathbb{Z}^f$, $hA \equiv 0 \pmod N$, $P_\nu(X)$ be a spherical function of order ν with respect to $Q(X)$ (see [2], p. 211) and

$$\vartheta(\tau; Q(X), P_\nu(X), h) = \sum_{\substack{g \in \mathbb{Z}^f \\ g \equiv h \pmod N}} P_\nu(g) z_N^{\frac{1}{N} Q(g)}.$$

Note ([2], p. 210) that

$$\vartheta(\tau; Q(X), P_\nu(X), h_1) = \vartheta(\tau; Q(X), P_\nu(X), h_2) \quad \text{if } h_1 \equiv h_2 \pmod N, \tag{2}$$
$$\vartheta(\tau; Q(X), P_\nu(X), -h) = (-1)^\nu \vartheta(\tau; Q(X), P_\nu(X), h). \tag{3}$$

Lemma 1 ([2], pp. 217, 218)**.** *If $2 \mid f$, then*
a) $\vartheta(\tau; Q(X), P_\nu(X), h) \in M_{f/2+\nu}(\Gamma(N))$;
b) *if $\nu > 0$, then* $\vartheta(\tau; Q(X), P_\nu(X), h) \in S_{f/2+\nu}(\Gamma(N))$;
c) *for any* $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$

$$\vartheta(\tau; Q(X), P_\nu(X), h) \mid_{f/2+\nu} L = \exp\left(\frac{2\pi i}{N^2} abQ(h)\right) \chi(d) \vartheta(\tau; Q(X), P_\nu(X), ah),$$

where χ is the character of $Q(X)$.

Lemma 2 ([3])**.** *If $2 \nmid f$, then*
a) $\vartheta(\tau; Q(X), P_\nu(X), h) \in M_{f/2+\nu}(\widetilde{\Gamma}(N))$;
b) *if $\nu > 0$, then* $\vartheta(\tau; Q(X), P_\nu(X), h) \in S_{f/2+\nu}(\widetilde{\Gamma}(N))$;
c) *for any* $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$

$$\vartheta(\tau; Q(X), P_\nu(X), h) \mid_{f/2+\nu} \widetilde{L} = \exp\left(\frac{2\pi i}{N^2} abQ(h)\right) (\operatorname{sgn} d)^\nu \left(\frac{-1}{|d|}\right)^\nu \chi(d) \vartheta(\tau; Q(X), P_\nu(X), ah),$$

where χ is the character of $Q(X)$.

Lemma 3 ([1], Ch. IV, §1). *If $2 \mid f$ and $\phi(d) = \operatorname{sgn} d \left(\frac{-1}{|d|}\right)$, then*

$$M_{f/2}(\widetilde{\Gamma}_0(N), \chi) = M_{f/2}(\Gamma_0(N), \phi^{f/2}\chi),$$
$$S_{f/2}(\widetilde{\Gamma}_0(N), \chi) = S_{f/2}(\Gamma_0(N), \phi^{f/2}\chi).$$

Lemma 4 ([1], Ch. IV, §1). *If $F_r(\tau) \in M_{\frac{k_r}{2}}(\widetilde{\Gamma}_0(N), \chi_r)$ $(r = 1, 2)$, then $F_1(\tau)F_2(\tau) \in M_{(k_1+k_2)/2}(\widetilde{\Gamma}_0(N), \chi_1\chi_2)$; if F_1 or F_2 is a cusp form, then F_1F_2 is also a cusp form.*

Lemma 5 ([1], Ch. IV, §1). a) $M_{k/2}(\widetilde{\Gamma}_1(N)) = \oplus_\chi M_{k/2}(\widetilde{\Gamma}_0(N), \chi)$;
b) *if $\chi(-1) = -1$, then $M_{k/2}(\widetilde{\Gamma}_0(N), \chi) = 0$.*

Let $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$; $\Gamma' \subset SL_2(\mathbb{Z})$ be a subgroup of finite index, $\mu = [\overline{SL}_2(\mathbb{Z}) : \overline{\Gamma}']$, g be the genus of the surface $\Gamma' \setminus \mathbb{H}^*$; $e_1, e_2, \ldots, e_r$ be the orders of $\overline{\Gamma}'$-inequivalent elliptic points; ν_2 and ν_3 denote the numbers of $\overline{\Gamma}'$-inequivalent elliptic points of order 2 and 3, respectively; ν_∞ be the number of cusps with respect to Γ'; u be the number of regular cusps.

Lemma 6 ([4], Ch. 2, §6). a) $g = 1 + \frac{\mu}{12} - \frac{\nu_2}{4} - \frac{\nu_3}{3} - \frac{\nu_\infty}{2}$;
b) *if $2 \mid k$, then*

$$\dim S_k(\Gamma') = \begin{cases} (k-1)(g-1) + \left(\frac{k}{2} - 1\right)\nu_\infty + \sum\limits_{s=1}^{r} \left[\frac{k(e_s-1)}{2e_s}\right] & \text{if } k > 2, \\ g & \text{if } k = 2; \end{cases}$$

c) *if $2 \nmid k$, $k \geq 3$, and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \notin \Gamma'$, then*

$$\dim S_k(\Gamma') = (k-1)(g-1) + \frac{u(k-2)}{2} + \frac{(\nu_\infty - u)(k-1)}{2} + \sum_{s=1}^{r} \left[\frac{k(e_s-1)}{2e_s}\right];$$

d) *if $\Gamma' = \Gamma_0(N_1)$, then $\mu = N_1 \prod\limits_{p \mid N_1} \left(1 + \frac{1}{p}\right)$,*

$$\nu_2 = \begin{cases} 0 & \text{if } 4 \mid N_1, \\ \prod\limits_{p \mid N_1} \left(1 + \left(\frac{-1}{p}\right)\right) & \text{if } 4 \nmid N_1; \end{cases} \quad \nu_3 = \begin{cases} 0 & \text{if } 9 \mid N_1, \\ \prod\limits_{p \mid N_1} \left(1 + \left(\frac{-3}{p}\right)\right) & \text{if } 9 \nmid N_1, \end{cases}$$

where $\left(\frac{-3}{\cdot}\right)$ is the Kronecker symbol.

It is known (see [1], Ch. III, §1) that if $\Gamma' = \overline{\Gamma}_1(N_1)$, then

$$\mu = \frac{1}{2} N_1^2 \prod_{p \mid N_1} \left(1 - \frac{1}{p^2}\right). \tag{4}$$

Lemma 7 ([1], Ch. IV, §1; [5], p. 127). *Let $\rho = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$, $2 \nmid k$, $F(\tau)|_{k/2}\widetilde{\rho} = (N^{1/4}\sqrt{\tau})^{-k} F(\rho\tau)$, $\chi_1(d) = \left(\frac{N}{|d|}\right)$. Then the map*

$$|_{k/2}\widetilde{\rho} : S_{k/2}(\widetilde{\Gamma}_0(N), \chi) \to S_{k/2}(\widetilde{\Gamma}_0(N), \overline{\chi}\chi_1),$$

where $\overline{\chi}$ is the complex conjugate of χ, is the isomorphic imbedding.

2. Lemma 8. *The group $\Gamma_1(4p)$ has $5(p-1)$ cusps. When $2 \mid k$, they are k-regular.*

Proof. It is known (see [1], Ch. III, §1; [2], p. 102) that $[\overline{\Gamma}_0(4p) : \overline{\Gamma}_1(4p)] = p-1$ and

$$\nu_\infty = 6 \quad \text{for} \quad \Gamma_0(4p). \tag{5}$$

It is easy to verify that

$$B = \left\{ \begin{pmatrix} a & \frac{ab-1}{4p} \\ 4p & b \end{pmatrix} \,\middle|\, a = 3, 5, \dots, 2p-1;\ b \in \{3, 5, \dots, 4p-3\},\ ad \equiv 1 \ (\operatorname{mod} 4p) \right\} \cup \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is a complete system of representatives of right cosets of $\overline{\Gamma}_0(4p)$ by $\overline{\Gamma}_1(4p)$. One must seek the cusps of $\Gamma_1(4p)$ in the set

$$\left\{ L^{-1}\zeta \mid L \in B,\ \ \zeta \ \text{ is a cusp of } \ \Gamma_0(4p) \right\}.$$

$L_1\zeta$ and $L_2\zeta$ are $\Gamma_1(4p)$-equivalent iff there is $\sigma \in \Gamma_\zeta$ such that $L_1^{-1}\sigma L_2 \in \Gamma_1(4p)$. It is known (see [1], Ch. III, §3) that $\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \,\middle|\, n \in \mathbb{Z} \right\}$ and if $\zeta = \sigma^{-1}\infty$, then $\Gamma_\zeta = \sigma^{-1}\Gamma_\infty\sigma \cap \Gamma_0(4p)$. For $\zeta_1 = \infty$, $\zeta_2 = 0$, $\zeta_3 = -\frac{1}{4}$, $\zeta_4 = -\frac{1}{p}$, $\zeta_5 = -\frac{1}{2}$ and $\zeta_6 = -\frac{1}{2p}$, respectively, $\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$, $\sigma_4 = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}$, $\sigma_5 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ and $\sigma_6 = \begin{pmatrix} 1 & 0 \\ 2p & 1 \end{pmatrix}$.

Having performed suitable calculations, we get that the points $L\zeta$, $L \in B$, $\zeta = \infty, 0, -\frac{1}{4}$ or $-\frac{1}{p}$ are $\Gamma_1(4p)$-inequivalent. Their number is $4(p-1)$. Among the points $L\zeta$, $L \in B$, $\zeta = -\frac{1}{2}$ or $-\frac{1}{2p}$ there are only $p-1$ that are $\Gamma_1(4p)$-inequivalent. Hence, by the definition of cusp, the lemma is proved. □

Lemma 9. a) *If $4 \mid k$, then*

$$\dim S_{k/2}(\widetilde{\Gamma}_0(12)) = \begin{cases} k-5 & \text{if } k > 4, \\ 0 & \text{if } k = 4; \end{cases}$$

b) *if $2 \mid k$, then*

$$\dim S_{k/2}(\widetilde{\Gamma}_1(12)) = \begin{cases} 2k-9 & \text{if } k > 4, \\ 0 & \text{if } k = 4; \end{cases}$$

c) *if $4 \mid k$, $\chi(d) = \left(\frac{12}{d}\right)$, then*

$$\dim S_{k/2}(\widetilde{\Gamma}_0(12), \chi) = \begin{cases} k-4 & \text{if } k > 4, \\ 0 & \text{if } k = 4. \end{cases}$$

Proof. a) When $\Gamma' = \Gamma_0(12)$, by (5), Lemmas 3 and 6 we have

$$\nu_2 = \nu_3 = 0, \quad \nu_\infty = 6, \quad \mu = 24, \quad g = 0, \quad S_{k/2}(\widetilde{\Gamma}_0(12)) = S_{k/2}(\Gamma_0(12)).$$

Then the result follows from Lemma 6.

b) When $\Gamma' = \Gamma_1(12) \subset \Gamma_0(12)$, by (4), Lemmas 3, 5, 6 and 8 we get

$$\nu_2 = \nu_3 = 0, \quad \nu_\infty = 10, \quad \mu = 48, \quad g = 0, \quad S_{k/2}(\widetilde{\Gamma}_1(12)) = S_{k/2}(\Gamma_1(12)).$$

Then the result follows from Lemma 6.

c) Let φ be the Euler function. It is well known that the number of characters mod 12 is $\varphi(12) = 4$ and only two of them, the principal character and $\chi(d) = \left(\frac{12}{d}\right)$, are even. Therefore, by Lemma 5, we obtain

$$S_{k/2}(\widetilde{\Gamma}_1(12)) = S_{k/2}(\widetilde{\Gamma}_0(12)) \oplus S_{k/2}(\widetilde{\Gamma}_0(12), \chi).$$

Then the result follows from a) and b). □

Lemma 10. *Let $Q(X)$ be a quadratic form of type $(f/2, N, \chi)$, A be the matrix of $2Q(X)$, $P_\nu(X)$ be a spherical function of order ν with respect to $Q(X)$, $h \in \mathbb{Z}^f$, $hA \equiv 0 \pmod N$, $\Delta \in \mathbb{N}$. If $N^2 \mid Q(h)$ and for all a and d with $ad \equiv 1 \pmod N$ the equality*

$$\left(\frac{\det A}{|d|}\right) \vartheta(\tau; Q(X), P_\nu(X), ah)$$
$$= (\operatorname{sgn} d)^\nu \left(\frac{-1}{|d|}\right)^\nu \left(\frac{\Delta}{|d|}\right) \vartheta(\tau; Q(X), P_\nu(X), h) \qquad (6)$$

holds, then $\vartheta(\tau; Q(X), P_\nu(X), h) \in M_{f/2+\nu}(\widetilde{\Gamma}_0(N), \chi_1)$, and if $\nu > 0$, then $\vartheta(\tau; Q(X), P_\nu(X), h) \in S_{f/2+\nu}(\widetilde{\Gamma}_0(N), \chi_1)$, where $\chi_1(d) = (\operatorname{sgn} d)^{f/2} \left(\frac{(-1)^{f/2}\Delta}{|d|}\right)$ when $2 \mid f$ and $\chi_1(d) = \left(\frac{2\Delta}{|d|}\right)$ when $2 \nmid f$.

Proof directly follows from Lemmas 1–3, the definition of the character of a quadratic form and the definition of a modular form (see [5], p. 85). □

Corollary. *If $h = 0$ and $\nu = 0$, then $\vartheta(\tau, Q(X)) = \vartheta(\tau; Q(X), 1, 0) \in M_{f/2}(\widetilde{\Gamma}_0(N), \chi)$.*

3. Lemma 11. *Let $Q_1 = 3(x_1^2 + x_2^2 + x_3^2)$, $h = (4, 4, 4)$, $P_1(X) = x_1$. Then*

a) $$\psi_1(\tau) = \vartheta(\tau; Q_1, P_1(X), h) = 4 \sum_{n=1}^{\infty} \Bigg(\sum_{\substack{Q_1 = 9n \\ x_r \equiv 1 \pmod 3}} x_1 \Bigg) z^n$$
$$= 4(z - 3z^3 - 2z^4 + 6z^6 + 6z^7 - 3z^9 - 12z^{10} + \cdots) \in S_{5/2}(\widetilde{\Gamma}_0(12));$$

b) $\operatorname{ord}(\psi_1(\tau), \infty, \Gamma_0(12)) = 1.$

Proof. a) For Q_1 we have

$$A=\begin{pmatrix}6&0&0\\0&6&0\\0&0&6\end{pmatrix},\quad N=12,\quad hA=(24,24,24),$$
$$\det A=216,\quad Q_1(h)=144,\quad f=3.$$

By definition (see[2], p. 211), $P_1(X)$ is a spherical function of order $\nu=1$ with respect to Q_1.

From (2) it follows that one must verify condition (6) only for $a=d=5$, $a=d=7$ and $a=d=11$. It is easy to check that

$$5h\equiv -h\ (\operatorname{mod} 12),\quad \left(\frac{216}{5}\right)=1,\quad \left(\frac{18}{5}\right)=-1,\quad \left(\frac{-1}{5}\right)=1;$$
$$7h\equiv h\ (\operatorname{mod} 12),\quad \left(\frac{216}{7}\right)=-1,\quad \left(\frac{18}{7}\right)=1,\quad \left(\frac{-1}{7}\right)=-1;$$
$$11h\equiv -h\ (\operatorname{mod} 12),\quad \left(\frac{216}{11}\right)=-1,\quad \left(\frac{18}{11}\right)=-1,\quad \left(\frac{-1}{11}\right)=-1.$$

Bearing in mind (2) and (3) we obtain that for all a and d with $ad\equiv 1\ (\operatorname{mod} 12)$ the equality

$$\left(\frac{216}{|d|}\right)\vartheta(\tau;Q_1,P_1(X),ah)=\operatorname{sgn} d\left(\frac{-1}{|d|}\right)\left(\frac{18}{|d|}\right)\vartheta(\tau;Q_1,P_1(X),h)$$

holds. By Lemma 10, $\psi_1(\tau)\in S_{5/2}(\widetilde{\Gamma}_0(12))$, since $2\nmid f$ and $\chi_1(d)=\left(\frac{36}{|d|}\right)$ is the principal character mod 12.

The expansion of $\psi_1(\tau)$ is obtained after easy calculations.

b) immediately follows from (1) and a). □

The following lemmas are proved exactly in the same manner as Lemma 11.

Lemma 12. *Let* $Q_2=3(x_1^2+x_2^2)+4(x_3^2+x_3x_4+x_4^2)$, $P_1(X)=x_1$, $h=(4,4,4,4)$. *Then*

a) $$\psi_2(\tau)=\vartheta(\tau;Q_2,P_1(X),h)=4\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{Q_2=9n\\ x_r\equiv 1\ (\operatorname{mod} 3)}}x_1\Bigg)z^n$$
$$=4(3z^2-3z^3-6z^4+3z^6+12z^7-6z^{10}+\cdots)\in S_{6/2}(\widetilde{\Gamma}_0(12));$$

b) $\operatorname{ord}(\psi_2(\tau),\infty,\Gamma_0(12))=2$.

Lemma 13. *Let* $Q_3=x_1^2+3(x_2^2+x_3^2)$, $h_1=(6,6,0)$, $P_2^{(1)}(X)=x_2^2-x_3^2$, $P_2^{(2)}(X)=x_1^2-x_2^2-2x_3^2$, $Q_4=2(x_1^2+x_1x_2+x_2^2)+3x_3^2$, $h_2=(4,4,4)$, $P_2(X)=x_1x_2+x_3^2$. *Then*

a) $$\psi_1^{(1)}(\tau)=\vartheta(\tau;Q_3,P_2^{(1)}(X),h_1)=36\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{Q_3=4n\\ 2\nmid x_1,\,2\nmid x_2,\,2\mid x_3}}x_2^2-x_3^2\Bigg)z^n$$

$$= 36(4z + 4z^3 - 24z^4 - 24z^6 + 40z^7 + 36z^9 + 16z^{10} + \cdots),$$

$$\psi_2^{(1)}(\tau) = \vartheta(\tau; Q_4, P_2(X), h_2) = 16 \sum_{n=1}^{\infty} \Bigg(\sum_{\substack{Q_4=9n \\ x_r \equiv 1 \,(\bmod\, 3)}} x_1x_2 + x_3^2 \Bigg) z^n$$

$$= 16(9z^2 - 9z^3 - 36z^5 + 27z^6 + 36z^8 + 0 \cdot z^{10} + \cdots),$$

$$\psi_3^{(1)}(\tau) = \vartheta(\tau; Q_3, P_2^{(2)}(X), h_1) = 36 \sum_{n=1}^{\infty} \Bigg(\sum_{\substack{Q_3=4n \\ 2\nmid x_1,\, 2\nmid x_2, 2|x_3}} x_1^2 - 2x_2^2 - 2x_3^2 \Bigg) z^n$$

$$= 36(32z^3 - 64z^4 + 64z^7 + 0 \cdot z^{10} + \cdots);$$

b) $\psi_s^{(1)}(\tau) \in S_{7/2}(\widetilde{\Gamma}_0(12))$;

c) $\operatorname{ord}\big(\psi_s^{(1)}(\tau), \infty, \Gamma_0(12)\big) = s \quad (s = 1, 2, 3)$.

Lemma 14. *Let* $P_3^{(1)}(X) = x_1x_2x_3$, $P_3^{(2)}(X) = x_1^3 - 3x_1x_2^2 + 2x_1x_2x_3$, $Q_5 = 4(x_1^2 + x_1x_2 + x_2^2) + 3x_3^2$, $P_3(X) = 11x_1^3 - 10x_2^3 - 33x_1x_2^2 - 3x_2x_3^2$, $h_1 = (3, -3, 6)$, $Q_6 = 3(x_1^2 + x_2^2 + x_3^2) + 4(x_4^2 + x_4x_5 + x_5^2)$, $P_2(X) = 6x_1^2 - 8x_4^2 + 3x_1x_2 - 4x_4x_5$, $h_2 = (4, 4, 4, 6, -6)$. *Then*

a) $$\psi_1^{(2)}(\tau) = \vartheta(\tau; Q_1, P_3^{(1)}(X), h) = 64 \sum_{n=1}^{\infty} \Bigg(\sum_{\substack{Q_1=9n \\ x_r \equiv 1 \,(\bmod\, 3)}} P_3^{(1)}(X) \Bigg) z^n$$

$$= 64(z - 6z^2 + 12z^3 - 8z^4 + 12z^6 - 48z^7 + 48z^8 - 15z^9 + 60z^{10} + \cdots),$$

$$\psi_2^{(2)}(\tau) = \vartheta(\tau; Q_1, P_3^{(2)}(X), h) = 64 \sum_{n=1}^{\infty} \Bigg(\sum_{\substack{Q_1=9n \\ x_r \equiv 1 \,(\bmod\, 3)}} P_3^{(2)}(X) \Bigg) z^n$$

$$= 64(-27z^2 + 27z^3 + 27z^6 + 216z^8 - 216z^9 + 0 \cdot z^{10} + \cdots),$$

$$\psi_3^{(2)}(\tau) = \vartheta(\tau; Q_5, P_3(X), h_1) = 27 \sum_{n=1}^{\infty} \Bigg(\sum_{\substack{Q_5=16n \\ x_1 \equiv 1 \,(\bmod\, 4) \\ x_2 \equiv -1 \,(\bmod\, 4) \\ x_3 \equiv 2 \,(\bmod\, 4)}} P_3(X) \Bigg) z^n$$

$$= 27(576z^3 - 1536z^4 + 3456z^6 - 2688z^7 - 3840z^{10} + \cdots),$$

$$\psi_4^{(2)}(\tau) = \vartheta(\tau; Q_6, P_2(X), h_2) = 4 \sum_{n=1}^{\infty} \Bigg(\sum_{\substack{Q_6=36n \\ x_1 \equiv x_2 \equiv x_3 \equiv 2 \,(\bmod\, 6) \\ x_4 \equiv x_5 \equiv 3 \,(\bmod\, 6)}} P_2(X) \Bigg) z^n$$

$$= 4(72z^4 - 216z^5 - 216z^6 + 936z^7 + 432z^8 - 1512z^9 - 1440z^{10} + \cdots),$$

where Q_1 *and* h *are defined by Lemma* 11*;*

b) $\psi_s^{(2)}(\tau) \in S_{9/2}(\widetilde{\Gamma}_0(12))$;

c) $\operatorname{ord}\left(\psi_s^{(2)}(\tau), \infty, \Gamma_0(12)\right) = s \quad (s = 1, \ldots, 4)$.

Lemma 15. *Let* $P_3(X) = 4x_1^3 - 7x_3^3 - 6x_1x_2^2 + 21x_1^2x_3 + 12x_1x_3x_4$, $Q_7 = 3(x_1^2+x_2^2)+4(x_3^2+x_3x_4+x_4^2+x_5^2+x_5x_6+x_6^2)$, $P_2^{(1)}(X) = 2x_1^2-6x_3^2+4x_5^2+x_1x_2$, $P_2^{(2)}(X) = 2x_1^2 + 3x_3^2 - 6x_5^2 + x_1x_2 - 2x_5x_6$, $h_1 = (4,4,4,4,6,-6)$. *Then*

a)
$$\psi_4^{(3)}(\tau) = \vartheta(\tau; Q_2, P_3(X), h) = 64\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{Q_2=9n\\ x_r\equiv 1\,(\bmod\ 3)}} P_3(X)\Bigg)z^n$$
$$= 64(162z^4 - 486z^6 - 324z^7 + 972z^8 - 1296z^{10} + \cdots),$$
$$\psi_5^{(3)}(\tau) = \vartheta(\tau; Q_7, P_2^{(1)}(X), h_1) = 4\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{Q_7=36n\\ x_1\equiv\cdots\equiv x_4\equiv 2\,(\bmod\ 6)\\ x_5\equiv x_6\equiv 3\,(\bmod\ 6)}} P_2^{(1)}(X)\Bigg)z^n$$
$$= 4(216z^5 - 648z^7 - 864z^8 + 648z^9 + 2592z^{10} + \cdots),$$
$$\psi_6^{(3)}(\tau) = \vartheta(\tau; Q_7, P_2^{(2)}(X), h_1) = 4\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{Q_7=36n\\ x_1\equiv\cdots\equiv x_4\equiv 2\,(\bmod\ 6)\\ x_5\equiv x_6\equiv 3\,(\bmod\ 6)}} P_2^{(2)}(X)\Bigg)z^n$$
$$= 4(-432z^6 + 432z^7 + 1728z^8 - 864z^9 - 3456z^{10} + \cdots),$$

where Q_2 *and* h *are defined by Lemma* 12*;*

b) $\psi_s^{(3)}(\tau) \in S_{10/2}(\widetilde{\Gamma}_0(12))$;

c) $\operatorname{ord}\left(\psi_s^{(3)}(\tau), \infty, \Gamma_0(12)\right) = s \quad (s = 4,5,6)$.

In what follows let $\chi(d) = \left(\frac{12}{|d|}\right)$.

Lemma 16. *Let* $P_1(X) = x_3$. *Then*

a)
$$\Omega_1(\tau) = \vartheta(\tau; Q_4, P_1(X), h_2) = 4\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{Q_4=9n\\ x_r\equiv 1\,(\bmod\ 3)}} x_3\Bigg)z^n = 4(3z - 6z^2$$
$$+ 3z^3 - 6z^4 + 6z^5 + 12z^8 - 9z^9 + 12z^{10} + \cdots) \in S_{5/2}(\widetilde{\Gamma}_0(12), \chi),$$

where Q_4 *and* h_2 *are defined by Lemma* 13*;*

b) $\operatorname{ord}(\Omega_1(\tau), \infty, \Gamma_0(12)) = 1$.

Lemma 17. *Let* $Q_8 = x_1^2 + 3x_2^2$, $P_2(X) = x_1^2 - 3x_2^2$, $h = (6,6)$. *Then*

a)
$$\Omega_1^{(1)}(\tau) = \vartheta(\tau; Q_8, P_2(X), h) = 36\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{Q_8=4n\\ 2\nmid x_1,\, 2\nmid x_2}} P_2(X)\Bigg)z^n$$
$$= 36(-8z + 24z^3 - 16z^7 - 72z^9 + 0\cdot z^{10} + \cdots) \in S_{6/2}(\widetilde{\Gamma}_0(12), \chi);$$

b) $\operatorname{ord}\left(\Omega_1^{(1)}(\tau), \infty, \Gamma_0(12)\right) = 1.$

Lemma 18. *Let* $P_2(X) = x_1x_2$, $Q_9 = x_1^2+x_2^2+3x_3^2$, $P_2^{(1)}(X) = 2x_1^2-3x_2^2+3x_3^2$, $P_2^{(2)}(X) = 7x_2^2 - 10x_1^2 + 9x_3^2$, $h_1 = (0,6,6)$. *Then*

a) $$\Omega_2^{(1)}(\tau) = \vartheta(\tau; Q_9, P_2^{(1)}(X), h_1) = 36\sum_{n=1}^{\infty}\left(\sum_{\substack{Q_9=4n\\ 2|x_1,\, 2\nmid x_2,\, 2\nmid x_3}} P_2^{(1)}(X)\right)z^n$$
$$= 36(64z^2 - 96z^3 - 128z^4 + 256z^5 - 128z^7 - 256z^8 + 640z^{10} + \cdots);$$
$$\Omega_3^{(1)}(\tau) = \vartheta(\tau; Q_9, P_2^{(2)}(X), h_1) - 144\vartheta(\tau; Q_1, P_2(X), h)$$
$$= 36\sum_{n=1}^{\infty}\left(\sum_{\substack{Q_9=4n\\ 2|x_1,\, 2\nmid x_2,\, 2\nmid x_3}} P_2^{(2)}(X)\right)z^n - 2304\sum_{n=1}^{\infty}\left(\sum_{\substack{Q_1=9n\\ x_r\equiv 1 \pmod 3}} P_2(X)\right)z^n$$
$$= 36(288z^3-1152z^5-576z^6+1152z^7+2304z^8+1152z^9-2304z^{10} + \cdots),$$

where Q_1 *and* h *are defined by Lemma* 11*;*

b) $\Omega_s^{(1)}(\tau) \in S_{7/2}(\widetilde{\Gamma}_0(12), \chi)$;

c) $\operatorname{ord}\left(\Omega_s^{(1)}(\tau), \infty, \Gamma_0(12)\right) = s \quad (s = 2, 3).$

Lemma 19. *Let* $P_2(X) = 18x_3^2 - 7x_1^2 + 22x_3x_4 + 2x_1x_3 - 4x_1x_2$, $P_2^{(1)}(X) = 4x_3^2 + 3x_1x_2 - 12x_1x_3 + 8x_3x_4$, $h = (4,4,2,2)$. *Then*

a) $$\Omega_3^{(2)}(\tau) = \vartheta(\tau; Q_2, P_2^{(1)}(X), h) = 4\sum_{n=1}^{\infty}\left(\sum_{\substack{Q_2=36n\\ x_1\equiv x_2\equiv 2 \pmod 6\\ x_3\equiv x_4\equiv 1 \pmod 6}} P_2^{(1)}(X)\right)z^n = 4(252z^3$$
$$- 144z^4 - 360z^5 - 576z^6 - 144z^7 + 1440z^8 + 1080z^9 + 288z^{10} + \cdots),$$
$$\Omega_4^{(2)}(\tau) = \vartheta(\tau; Q_2, P_2(X), h) = 4\sum_{n=1}^{\infty}\left(\sum_{\substack{Q_2=36n\\ x_1\equiv x_2\equiv 2 \pmod 6\\ x_3\equiv x_4\equiv 1 \pmod 6}} P_2(X)\right)z^n$$
$$= 4(360z^4 - 360z^5 - 1080z^6 + 360z^7 + 1440z^8 + 1080z^9 - 720z^{10} + \cdots),$$

where Q_2 *is defined by Lemma* 12*;*

b) $\Omega_r^{(2)}(\tau) \in S_{8/2}(\widetilde{\Gamma}_0(12), \chi)$;

c) $\operatorname{ord}\left(\Omega_r^{(2)}(\tau), \infty, \Gamma_0(12)\right) = r \quad (r = 3, 4).$

Lemma 20. *Let* $P_3^{(1)}(X) = x_2^3-6x_2x_3^2$, $P_3^{(2)}(X) = 2x_2^3-12x_2x_3^2+3x_2^2x_3-2x_3^3$, $Q_{10} = 3x_1^2 + 4(x_2^2 + x_2x_3 + x_3^2 + x_4^2 + x_4x_5 + x_5^2)$, $P_2(X) = 3x_1^2 - 6x_2^2 + 10x_4^2 +$

$6x_1x_4 + 14x_4x_5$, $h_1 = (4,4,4,2,2)$. *Then*

a) $$\Omega_1^{(3)}(\tau) = \vartheta(\tau; Q_4, P_3^{(1)}(X), h_2) = 64\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{Q_4=9n\\ x_r\equiv 1 \,(\mathrm{mod}\ 3)}} P_3^{(1)}(X)\Bigg)z^n = 64(-6z - 6z^2 + 48z^3 + 48z^4 - 120z^5 - 126z^6 + 48z^8 + 414z^9 + 300z^{10} + \cdots),$$

$$\Omega_3^{(3)}(\tau) = \vartheta(\tau; Q_4, P_3^{(2)}(X), h_2) = 64\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{Q_4=9n\\ x_r\equiv 1 \,(\mathrm{mod}\ 3)}} P_3^{(2)}(X)\Bigg)z^n = 64(162z^3 - 972z^6 + 1944z^9 + 0\cdot z^{10} + \cdots),$$

$$\Omega_5^{(3)}(\tau) = \vartheta(\tau; Q_{10}, P_2(X), h_1) = 4\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{Q_{10}=36n\\ x_1\equiv x_2\equiv x_3\equiv 2 \,(\mathrm{mod}\ 6)\\ x_4\equiv x_5\equiv 1 \,(\mathrm{mod}\ 6)}} P_2(X)\Bigg)z^n = 4(648z^5 - 1296z^6 - 648z^7 + 1296z^8 + 1944z^9 - 1296z^{10} + \cdots),$$

where Q_4 and h_2 are defined by Lemma 13*;*

b) $\Omega_s^{(3)}(\tau) \in S_{9/2}(\widetilde{\Gamma}_0(12), \chi)$;

c) $\operatorname{ord}\big(\Omega_s^{(3)}(\tau), \infty, \Gamma_0(12)\big) = s \quad (s = 1,3,5)$.

Lemma 21. *Let* $Q_{11} = 2(x_1^2 + x_1x_2 + x_2^2) + 4(x_3^2 + x_3x_4 + x_4^2)$, $P_3^{(1)}(X) = x_3^2x_4 + x_3x_4^2$, $P_3^{(2)}(X) = x_1^2x_2 + x_1x_2^2 - x_3^2x_4 - x_3x_4^2$, $h = (4,4,4,4)$. *Then*

a) $$\Omega_2^{(3)}(\tau) = \vartheta(\tau; Q_{11}, P_3^{(1)}(X), h) = 64\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{Q_{11}=9n\\ x_r\equiv 1 \,(\mathrm{mod}\ 3)}} P_3^{(1)}(X)\Bigg)z^n = 64(18z^2 + 18z^4 - 108z^6 - 144z^8 + 108z^{10} + \cdots),$$

$$\Omega_4^{(3)}(\tau) = \vartheta(\tau; Q_{11}, P_3^{(2)}(X), h) = 64\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{Q_{11}=9n\\ x_r\equiv 1 \,(\mathrm{mod}\ 3)}} P_3^{(2)}(X)\Bigg)z^n = 64(-162z^4 + 486z^6 - 972z^{10} + \cdots);$$

b) $\Omega_s^{(3)}(\tau) \in S_{10/2}(\widetilde{\Gamma}_0(12), \chi)$;

c) $\operatorname{ord}\big(\Omega_s^{(3)}(\tau), \infty, \Gamma_0(12)\big) = s \quad (s = 2,4)$.

Lemma 22. *Let* $Q_{12} = 4(x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2)$, $h_1 = (6,-6,6,-6)$. *Then*

a) $$\vartheta(\tau; Q_{11}, 1, h) = \sum_{n=1}^{\infty}\Bigg(\sum_{\substack{Q_{11}=9n\\ x_r\equiv 1 \,(\mathrm{mod}\ 3)}} 1\Bigg)z^n = 9z^2 + 9z^4 + 27z^6 + 9z^8 + 54z^{10} + \cdots \in M_{4/2}(\widetilde{\Gamma}_0(12)),$$

where Q_{11} and h are defined by Lemma 21,

$$\vartheta(\tau; Q_{12}, 1, h_1) = \sum_{n=1}^{\infty} \Bigg(\sum_{\substack{Q_{12}=4n \\ 2\nmid x_r}} 1 \Bigg) z^n$$
$$= 4z^2 + 8z^4 + 4z^6 + 16z^8 + 24z^{10} + \cdots \in M_{4/2}(\widetilde{\Gamma}_0(12));$$

b) $$F(\tau) = \frac{1}{4}\,\vartheta(\tau; Q_{12}, 1, h_1) - \frac{1}{9}\,\vartheta(\tau; Q_{11}, 1, h)$$
$$= z^4 - 2z^6 + 3z^8 + 0 \cdot z^{10} + \cdots \in M_{4/2}(\widetilde{\Gamma}_0(12)),$$
$$\operatorname{ord}(F(\tau), \infty, \Gamma_0(12)) = 4.$$

Let $Q(X) = x^2$. $Q(X)$ is the quadratic form of type $\left(\frac{1}{2}, 4, 1\right)$. By the Corollary of Lemma 10 we have

$$\vartheta(\tau) = \vartheta(\tau, Q(X)) = 1 + 2z + 2z^4 + 2z^9 + 0 \cdot z^{10} + \cdots$$
$$\in M_{1/2}(\widetilde{\Gamma}_0(4)) \subset M_{1/2}(\widetilde{\Gamma}_0(12)), \tag{7}$$
$$\operatorname{ord}(\vartheta(\tau), \infty, \Gamma_0(12)) = 0. \tag{8}$$

Theorem. *Let*

$$\lambda_k = \begin{cases} k-4 & \text{if } 4 \nmid k, \\ k-5 & \text{if } 4 \mid k, \end{cases} \qquad \delta_k = \begin{cases} k-4 & \text{if } k \not\equiv 2 \pmod 4, \\ k-5 & \text{if } k \equiv 2 \pmod 4; \end{cases}$$

$B_k = (b_{sr})$ $(s = 1, 2, \ldots, 14;\ r = 1, 2, \ldots, \lambda_k)$ and $C_k = (c_{sr})$ $(s = 1, 2, \ldots, 13;\ r = 1, 2, \ldots, \delta_k)$ be the matrices whose elements are non-negative integers satisfying the conditions

$$\frac{5}{2}\,b_{1r} + 3b_{2r} + \frac{7}{2}\sum_{s=3}^{5} b_{sr} + \frac{9}{2}\sum_{s=6}^{9} b_{sr} + 5\sum_{s=10}^{12} b_{sr} + 2b_{13,r} + \frac{1}{2}\,b_{14,r} = \frac{k}{2}\,, \tag{9}$$

$$b_{1r} + b_{3r} + b_{6r} + 2(b_{2r} + b_{4r} + b_{7r}) + 3(b_{5r} + b_{8r}) + 4(b_{9r} + b_{10,r} + b_{13,r})$$
$$+ 5b_{11,r} + 6b_{12,r} = r, \qquad \sum_{s=1}^{11} b_{sr} > 0, \tag{10}$$

$$\frac{5}{2}\,c_{1r} + 3c_{2r} + \frac{7}{2}\,(c_{3r} + c_{4r}) + 4(c_{5r} + c_{6r}) + \frac{9}{2}\sum_{s=7}^{11} c_{sr} + \frac{1}{2}\,(c_{8r} + c_{10,r})$$
$$+ 2c_{12,r} + \frac{1}{2}\,c_{13,r} + \frac{t}{2} = \frac{k}{2}\,, \tag{11}$$

$$c_{1r} + c_{2r} + c_{7r} + 2(c_{3r} + c_{8r}) + 3(c_{4r} + c_{5r} + c_{9r}) + 4(c_{6r} + c_{10,r} + c_{12,r})$$
$$+ 5c_{11,r} + \ell = r, \quad 2 \nmid \sum_{s=1}^{11} c_{sr}, \quad \ell \geq 0, \quad t \geq 0, \tag{12}$$

$$t = 0 \quad \textit{iff} \quad \ell = 0; \tag{13}$$

$$G_{k/2,r}(\tau) = F^{b_{13,r}}(\tau)\vartheta^{b_{14,r}}(\tau)\prod_{s=1}^{2}\psi_s^{b_{sr}}(\tau)\prod_{s=3}^{5}(\psi_{s-2}^{(1)}(\tau))^{b_{sr}}$$

$$\times\prod_{s=6}^{9}(\psi_{s-5}^{(2)}(\tau))^{b_{sr}}\prod_{s=10}^{12}(\psi_{s-6}^{(3)}(\tau))^{b_{sr}}, \quad r = 1, 2, \dots, \lambda_k, \tag{14}$$

$$H_{k/2,r}(\tau) = G_{t/2,\ell}(\tau)F^{c_{12,r}}(\tau)\vartheta^{c_{13,r}}(\tau)\Omega_1^{c_{1r}}(\tau)\prod_{s=2}^{4}(\Omega_{s-1}^{(1)}(\tau))^{c_{sr}}\prod_{s=5}^{6}(\Omega_{s-2}^{(2)}(\tau))^{c_{sr}}$$

$$\times\prod_{s=7}^{11}(\Omega_{s-6}^{(3)}(\tau))^{c_{sr}}, \quad r = 1, 2, \dots, \delta_k \quad (G_{0,0}(\tau) \equiv 1). \tag{15}$$

Then

a) *for any integer* $k \geq 5$

$$\dim S_{k/2}(\widetilde{\Gamma}_0(12)) = \lambda_k, \quad \dim S_{k/2}(\widetilde{\Gamma}_0(12), \chi) = \delta_k \tag{16}$$

and the systems of functions (14) *and* (15) *are the bases of the spaces* $S_{k/2}(\widetilde{\Gamma}_0(12))$ *and* $S_{k/2}(\widetilde{\Gamma}_0(12), \chi)$*, respectively;*

b) *if* $F(\tau) \in S_{k/2}(\widetilde{\Gamma}_0(12))$ (*or* $S_{k/2}(\widetilde{\Gamma}_0(12), \chi)$) *and its first* λ_k (*or* δ_k) *Fourier coefficients vanish, then* $F(\tau)$ *is identically zero, i.e., this space has no "Weierstrass gaps" at* ∞*.*

Proof. a) First we prove that for any integer $k \geq 5$ there exist matrices B_k and C_k with the properties required by the theorem. By (7), (8) and Lemmas 4, 11–13, 16–19 and 22 we have

1) if $k = 5$, then $\lambda_5 = \delta_5 = 1$, $G_{5/2,1}(\tau) = \psi_1(\tau) \in S_{5/2}(\widetilde{\Gamma}_0(12))$; $H_{5/2,1}(\tau) = \Omega_1(\tau) \in S_{5/2}(\widetilde{\Gamma}_0(12), \chi)$;

2) if $k = 6$, then $\lambda_6 = 2$, $\delta_6 = 1$, $G_{6/2,1}(\tau) = \psi_1(\tau)\vartheta(\tau)$, $G_{6/2,2}(\tau) = \psi_2(\tau)$, $G_{6/2,r}(\tau) \in S_{6/2}(\widetilde{\Gamma}_0(12))$, $\mathrm{ord}(G_{6/2,r}(\tau), \infty, \Gamma_0(12)) = r$ $(r = 1, 2)$; $H_{6/2,1}(\tau) = \Omega_1^{(1)}(\tau) \in S_{6/2}(\widetilde{\Gamma}_0(12), \chi)$;

3) if $k = 7$, then $\lambda_7 = \delta_7 = 3$, $G_{7/2,r}(\tau) = \psi_r^{(1)}(\tau) \in S_{7/2}(\widetilde{\Gamma}_0(12))$, $\mathrm{ord}(G_{7/2,r}(\tau), \infty, \Gamma_0(12)) = r$ $(r = 1, 2, 3)$; $H_{7/2,1}(\tau) = \Omega_1^{(1)}(\tau)\vartheta(\tau)$, $H_{7/2,s}(\tau) = \Omega_s^{(1)}(\tau)$ $(s = 2, 3)$, $\mathrm{ord}(H_{7/2,r}(\tau), \infty, \Gamma_0(12)) = r$, $H_{7/2,r}(\tau) \in S_{7/2}(\widetilde{\Gamma}_0(12), \chi)$ $(r = 1, 2, 3)$;

4) if $k = 8$, then $\lambda_8 = 3$, $\delta_8 = 4$, $G_{8/2,r}(\tau) = G_{7/2,r}(\tau)\vartheta(\tau) \in S_{8/2}(\widetilde{\Gamma}_0(12))$, $\mathrm{ord}(G_{8/2,r}(\tau), \infty, \Gamma_0(12)) = r$ $(r = 1, 2, 3)$; $H_{8/2,r}(\tau) = H_{7/2,r}(\tau)\vartheta(\tau)$ $(r = 1, 2, 3)$, $H_{8/2,4}(\tau) = \Omega_4^{(2)}(\tau)$, $H_{8/2,s}(\tau) \in S_{8/2}(\widetilde{\Gamma}_0(12), \chi)$, $\mathrm{ord}(H_{8/2,s}(\tau), \infty, \Gamma_0(12)) = s$ $(s = 1, \dots, 4)$.

Now we show that if for some $k \geq 9$ there exist matrices B_k and C_k, then there also are matrices B_{k+4} and C_{k+4}.

It follows from the definition that $\lambda_{k+4} = \lambda_k + 4$ and $\delta_{k+4} = \delta_k + 4$. By (7)–(15) and Lemmas 4, 11–22 we obtain

$$G_{k/2,r}(\tau) \in S_{k/2}(\widetilde{\Gamma}_0(12)), \tag{17}$$

$$\operatorname{ord}(G_{k/2,r}(\tau),\infty,\Gamma_0(12)) = r \quad (r=1,2,\dots,\lambda_k), \tag{18}$$

$$H_{k/2,r}(\tau) \in S_{k/2}(\widetilde{\Gamma}_0(12),\chi), \tag{19}$$

$$\operatorname{ord}(H_{k/2,r}(\tau),\infty,\Gamma_0(12)) = r \quad (r=1,2,\dots,\delta_k). \tag{20}$$

Consider the systems of functions

$$G_{\frac{k+4}{2},r}(\tau) = G_{k/2,r}(\tau)\vartheta^4(\tau) \quad (r=1,2,\dots,\lambda_k), \tag{21}$$

$$G_{\frac{k+4}{2},r}(\tau) = G_{k/2,r-4}(\tau)F(\tau) \quad (r=\lambda_k+1,\dots,\lambda_k+4), \tag{22}$$

$$H_{\frac{k+4}{2},r}(\tau) = H_{k/2,r}(\tau)\vartheta^4(\tau) \quad (r=1,2,\dots,\delta_k), \tag{23}$$

$$H_{\frac{k+4}{2},r}(\tau) = H_{k/2,r-4}(\tau)F(\tau) \quad (r=\delta_k+1,\dots,\delta_k+4). \tag{24}$$

(7), (8), (17)–(24) and Lemmas 4 and 22 imply that

$$G_{\frac{k+4}{2},r}(\tau) \in S_{\frac{k+4}{2}}(\widetilde{\Gamma}_0(12)),$$

$$\operatorname{ord}(G_{\frac{k+4}{2},r}(\tau),\infty,\Gamma_0(12)) = r \quad (r=1,2,\dots,\lambda_{k+4}),$$

$$H_{\frac{k+4}{2},r}(\tau) \in S_{\frac{k+4}{2}}(\widetilde{\Gamma}_0(12),\chi),$$

$$\operatorname{ord}(H_{\frac{k+4}{2},r}(\tau),\infty,\Gamma_0(12)) = r \quad (r=1,2,\dots,\delta_{k+4}).$$

Thus for any integer $k \geq 5$ there exist matrices B_k and C_k, whose elements are non-negative integers satisfying conditions (9)–(13). Furthermore, for functions (14) and (15), (17)–(20) are fulfilled.

It follows from (18) and (20) that the systems of functions (17) and (19) are linearly independent. Therefore

$$\dim S_{k/2}(\widetilde{\Gamma}_0(12)) \geq \lambda_k, \quad \dim S_{k/2}(\widetilde{\Gamma}_0(12),\chi) \geq \delta_k. \tag{25}$$

Consider following cases:

1°. Let $k=4m$. Then by Lemma 9

$$\dim S_{k/2}(\widetilde{\Gamma}_0(12)) = \lambda_k, \quad \dim S_{k/2}(\widetilde{\Gamma}_0(12),\chi) = \delta_k.$$

2°. Let $k=4m+2$. According to Lemma 9

$$\dim S_{k/2}(\widetilde{\Gamma}_0(12)) + \dim S_{k/2}(\widetilde{\Gamma}_0(12),\chi) = \dim S_{k/2}(\widetilde{\Gamma}_1(12)) = 2k-9; \tag{26}$$

but if $k=4m+2$, then

$$\lambda_k + \delta_k = 2k-9. \tag{27}$$

From (25)–(27) follows (16), since $\lambda_k \geq 0$ and $\delta_k \geq 0$ if $k \geq 5$.

3°. If $k=4m+1$, then $\lambda_k=\delta_k=\delta_{k+1}$. By Lemma 4 and 2° we have

$$\dim S_{k/2}(\widetilde{\Gamma}_0(12),\chi) \leq \dim S_{\frac{k+1}{2}}(\widetilde{\Gamma}_0(12),\chi) = \delta_{k+1} = \delta_k. \tag{28}$$

(28), (25) and Lemma 7 imply (16).

4°. If $k=4m+3$, then $\delta_k=\lambda_k=\lambda_{k+1}$. By Lemma 4 and 1° we obtain

$$\dim S_{k/2}(\widetilde{\Gamma}_0(12)) \leq \dim S_{\frac{k+1}{2}}(\widetilde{\Gamma}_0(12)) = \lambda_{k+1} = \lambda_k. \tag{29}$$

From (29), (25) and Lemma 7 follows (16).

2° Directly follows from (18), (20) and (16). □

Remark. From Lemma 9 it follows that $\dim S_{k/2}(\widetilde{\Gamma}_0(12), \chi) = 0$ when $k < 5$. Therefore the basis of the space $S_{k/2}(\widetilde{\Gamma}_0(12), \chi)$ is constructed for any integer k.

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