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ON THE EISENSTEIN SERIES CORRESPONDING TO QUADRATIC FORMS OF CERTAIN TYPE

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Abstract. The Eisenstein series corresponding to quadratic forms of type $(f/2, 4N, \chi)$ (N is a square-free natural number) are constructed using the bases of the spaces of Eisenstein series given in [4].

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We will use the notation and notions from [3] and [4].

In [5] Malyshev constructed a Hardy–Littlewood singular series for any integral positive quadratic form with $f \geq 4$ variables. In [2] Beridze summed up this series. But the use of this result to obtain Fourier coefficients of this series for a given quadratic form needs very long and tedious calculations.

As is known (see [7], p. 160), for any entire modular form $F(\tau)$ there exists a linear combination of Eisenstein series $E(\tau)$ such that $F(\tau) - E(\tau)$ is a cusp form. In this paper, using the results of [4], we find a linear combination of Eisenstein series $E(\tau, Q(x))$ such that $\vartheta(\tau, Q(x)) - E(\tau, Q(x))$ is a cusp form of a certain type.

In what follows, let $f, M \in \mathbb{N}$, $N = p_1 p_2 \cdots p_j$ (p_l is an odd prime number, $l = 1, 2, \dots, j$; $p_{l_1} \neq p_{l_2}$ when $l_1 \neq l_2$). If $Q(X)$ is a quadratic form with integral coefficients and with f variables $m \in \mathbb{N}$, $g \in \mathbb{Z}^f$, then

$$S(Q, m) = \sum_{g \bmod m} \exp\left(\frac{2\pi i}{m} Q(g)\right).$$

Lemma ([7], p. 213; [6]). *Let* $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, $c \neq 0$; $Q(X)$ *be a quadratic form of type* $(f/2, M, \chi)$, A *be the matrix of* $2Q(X)$; $h, g \in \mathbb{Z}^f$. *Then*

$$\vartheta(L\tau, Q(X)) = \frac{(\sqrt{-i(c\tau + d) sgn\, c})^f}{\sqrt{\det A} \cdot |c|^{f/2}} \sum_{\substack{h \bmod M \\ hA \equiv 0 (\bmod M)}} \varphi_h^L \vartheta(\tau; Q(X), 1, h),$$

where

$$\varphi_h^L = \sum_{g \bmod |c|} \exp\left(\frac{2\pi i}{c}\left(aQ(g) + \frac{1}{M} sgn\, c \cdot gAh' + \frac{d}{M^2} Q(h)\right)\right).$$

Proposition 1. *Let* $2 \nmid f$, $f \geq 5$, $Q(X)$ *be a quadratic form of type* $(f/2, 4N, \chi)$ *and*

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$$E(\tau, Q(X)) = E_1(\tau; f/2, 4N, \chi) + \frac{\exp(\pi i f/4)}{\sqrt{\det A}} \Bigg(\sum_{\substack{N_l | N \\ N_l \neq N}} 2^{-f} N_l^{-f/2} \overline{S(Q, 4N_l)}$$

$$\times E_{2,l}(\tau; f/2, 4N, \chi) + \sum_{N_l | N} \left(\frac{-1}{N_l} \right) N_l^{-f/2} \overline{S(Q, N_l)} E_{3,l}(\tau; f/2, 4N, \chi) \Bigg), \quad (1)$$

where $E_1(\tau; f/2, 4N, \chi)$ *and* $E_{r,l}(\tau; f/2, 4N, \chi)$ $(r = 2, 3)$ *are defined in* [4]. *Then*

$$\vartheta(\tau, Q(X)) - E(\tau, Q(X)) \in S_{f/2}(\widetilde{\Gamma}_0(4N), \chi).$$

Proof. It follows from Proposition 2 of [4] and Theorem 2.2, p. 86, of [1] that

$$\vartheta(\tau, Q(X)) - E(\tau, Q(X)) \in M_{f/2}(\widetilde{\Gamma}_0(4N), \chi).$$

Therefore we must only show that $\vartheta(\tau, Q(X))$ and $E(\tau, Q(X))$ assume equal values at cusps. The group $\Gamma_0(4N)$ has 2^{j+1} f-regular cusps (see [4]). Let $\widetilde{L}_1 = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 1 \right)$, $\widetilde{L}_{2,l} = \left(\begin{pmatrix} 1 & 0 \\ -4N_l & 1 \end{pmatrix}, \sqrt{-4N_l\tau + 1} \right)$ $(N_l | N, N_l \neq N)$, $\widetilde{L}_{3,l} = \left(\begin{pmatrix} 1 & 0 \\ -N_l & 1 \end{pmatrix}, -i\varepsilon_{N_l} \sqrt{-N_l\tau + 1} \right)$ $(N_l | N)$; A be the matrix of $2Q(X)$, $h \in \mathbb{Z}^f$. Using the Lemma we get

$$\lim_{\tau \to i\infty} \left(\vartheta(\tau, Q(X))|_{f/2} \widetilde{L}_1 \right) = 1, \quad (2)$$

$$\lim_{\tau \to i\infty} \left(\vartheta(\tau, Q(X))|_{f/2} \widetilde{L}_{2,l} \right) = \lim_{\tau \to i\infty} \Bigg(\frac{1}{\sqrt{\det A}(4N_l)^{f/2}} \cdot \left(\frac{\sqrt{i(-4N_l\tau + 1)}}{\sqrt{-4N_l\tau + 1}} \right)^f$$

$$\times \sum_{\substack{h \bmod 4N \\ hA \equiv 0 (\bmod 4N)}} \varphi_h^{L_{2,l}} \vartheta(\tau; Q(X), 1, h) \Bigg)$$

$$= \frac{\exp(\pi i f/4)}{\sqrt{\det A}(4N_l)^{f/2}} \sum_{g \bmod 4N_l} \exp\left(-\frac{2\pi i}{4N_l} Q(g) \right)$$

$$= \frac{\exp(\pi i f/4)}{\sqrt{\det A}(4N_l)^{f/2}} \overline{S(Q, 4N_l)}, \quad (3)$$

$$\lim_{\tau \to i\infty} \left(\vartheta(\tau, Q(X))|_{f/2} \widetilde{L}_{3,l} \right) = \lim_{\tau \to i\infty} \Bigg(\frac{1}{\sqrt{\det A} N_l^{f/2}} \cdot \left(\frac{\sqrt{i(-N_l\tau + 1)}}{-i\varepsilon_{N_l} \sqrt{-N_l\tau + 1}} \right)^f$$

$$\times \sum_{\substack{h \bmod 4N \\ hA \equiv 0 (\bmod 4N)}} \varphi_h^{L_{3,l}} \vartheta(\tau; Q(X), 1, h) \Bigg)$$

$$= \frac{i^f \exp(\pi i f/4)}{\sqrt{\det A} N_l^{f/2} \varepsilon_{N_l}^f} \sum_{g \bmod N_l} \exp\left(-\frac{2\pi i}{N_l} Q(g) \right)$$

$$= \frac{i^f \exp(\pi i f/4)}{\sqrt{\det A} N_l^{f/2} \varepsilon_{N_l}^f} \overline{S(Q, N_l)}. \tag{4}$$

It can be readily seen that

$$\lim_{\tau \to i\infty} \left(E_1(\tau; f/2, 4N, \chi)|_{f/2} \widetilde{L}_1 \right) = 1, \tag{5}$$

$$\lim_{\tau \to i\infty} \left(E_1(\tau; f/2, 4N, \chi)|_{f/2} \widetilde{L}_{r,l} \right) = 0, \tag{6}$$

$$\lim_{\tau \to i\infty} \left(E_{r,l}(\tau; f/2, 4N, \chi)|_{f/2} \widetilde{L}_1 \right) = 0 \qquad (r = 2, 3), \tag{7}$$

$$\lim_{\tau \to i\infty} \left(E_{2,l}(\tau; f/2, 4N, \chi)|_{f/2} \widetilde{L}_{2,l} \right) = 1, \tag{8}$$

$$\lim_{\tau \to i\infty} \left(E_{3,l}(\tau; f/2, 4N, \chi)|_{f/2} \widetilde{L}_{3,l} \right) = i^f \left(\frac{-1}{N_l} \right), \tag{9}$$

$$\lim_{\tau \to i\infty} \left(E_{r_1,l_1}(\tau; f/2, 4N, \chi)|_{f/2} \widetilde{L}_{r_2,l_2} \right) = 0 \ \text{ when } \ r_1, r_2 = 2 \ \text{ or } \ 3 \ \text{ and } \ (r_1, l_1) \neq (r_2, l_2). \tag{10}$$

Then the result follows from (1)–(10). □

Proposition 2. *Let $Q(X)$ be a quadratic form of type $(f/2, 4N, \chi)$, $2|f$, $\chi = \phi\psi$, ϕ be a character* mod 4, *ψ be a character* mod N,

$$E(\tau, Q(X)) = E_1(\tau; f/2, 4N, \chi) + \frac{i^{f/2}}{\sqrt{\det A}} \Bigg(\sum_{\substack{N_l | N \\ N_l \neq N}} 2^{-f} N_l^{-f/2} \overline{S(Q, 4N_l)} \times E_{2,l}(\tau; f/2, 4N, \chi) + \sum_{N_l | N} N_l^{-f/2} \overline{S(Q, N_l)} E_{3,l}(\tau; f/2, 4N, \chi) \Bigg)$$

if ϕ is not the principal character mod 4 *and*

$$E(\tau, Q(X)) = E_1(\tau; f/2, 4N, \chi) + \frac{i^{f/2}}{\sqrt{\det A}} \Bigg(\sum_{\substack{N_l | N \\ N_l \neq N}} 2^{-f} N_l^{-f/2} \overline{S(Q, 4N_l)} \times E_{2,l}(\tau; f/2, 4N, \chi) + \sum_{N_l | N} \Big(N_l^{-f/2} \overline{S(Q, N_l)} E_{3,l}(\tau; f/2, 4N, \chi) + (2N_l)^{-f/2} \overline{S(Q, 2N_l)} E_{4,l}(\tau; f/2, 4N, \chi) \Big) \Bigg)$$

if ϕ is the principal character mod 4, *where*

$$E_1(\tau; f/2, 4N, \chi) \quad \text{and} \quad E_{r,l}(\tau; f/2, 4N, \chi) \quad (r = 2, 3, 4)$$

are defined in [4]. *Then*

$$\vartheta(\tau, Q(X)) - E(\tau, Q(X)) \in S_{f/2}(\widetilde{\Gamma}_0(4N), \chi).$$

This Proposition is proved in the same manner as Proposition 1.

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