COMPLETING THE OPERADIC BUTTERFLY

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Abstract. We complete a certain diagram (the operadic butterfly) of the categories of algebras involving *Com*, *As*, and *Lie* by constructing a type of algebras which have 4 generating operations and 16 relations. The associated operad is self-dual for Koszul duality.

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1. Introduction

The three categories of algebras *Com*, *As*, and *Lie* (for commutative algebras, associative algebras and Lie algebras) are related by two functors

$$Com \hookrightarrow As \xrightarrow{-} Lie \ .$$

There are other types of algebras with similar functors which make up the following "operadic butterfly":
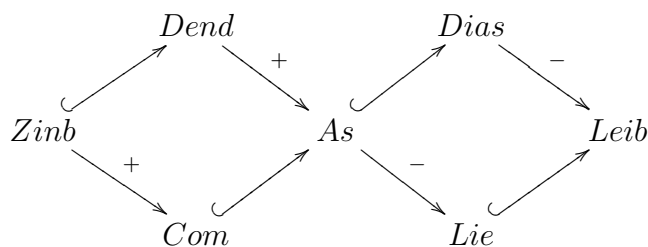
The four other categories of algebras which appear in this diagram are as follows:

Zinb = category of *Zinbiel* algebras. They have one operation $x \cdot y$ (with no symmetry) satisfying

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) + x \cdot (z \cdot y).$$

Dend = category of *dendriform* algebras. They have two operations $x \prec y$ and $x \succ y$ (with no symmetry) satisfying

$$
\begin{cases}
(x \prec y) \prec z &=& x \prec (y \prec z) + x \prec (y \succ z), \\
(x \succ y) \prec z &=& x \succ (y \prec z), \\
(x \prec y) \succ z + (x \succ y) \succ z &=& x \succ (y \succ z).
\end{cases}
$$

$Dias$ = category of *diassociative* algebras (or associative dialgebras). They have two operations $x \dashv y$ and $x \vdash y$ (with no symmetry) satisfying

$$\begin{cases} x \dashv (y \dashv z) = (x \dashv y) \dashv z = x \dashv (y \vdash z), \\ \quad\quad (x \vdash y) \dashv z = x \vdash (y \dashv z), \\ (x \dashv y) \vdash z = x \vdash (y \vdash z) = (x \vdash y) \vdash z. \end{cases}$$

$Leib$ = category of *Leibniz* algebras. They have one operation $[x, y]$ (with no symmetry) satisfying
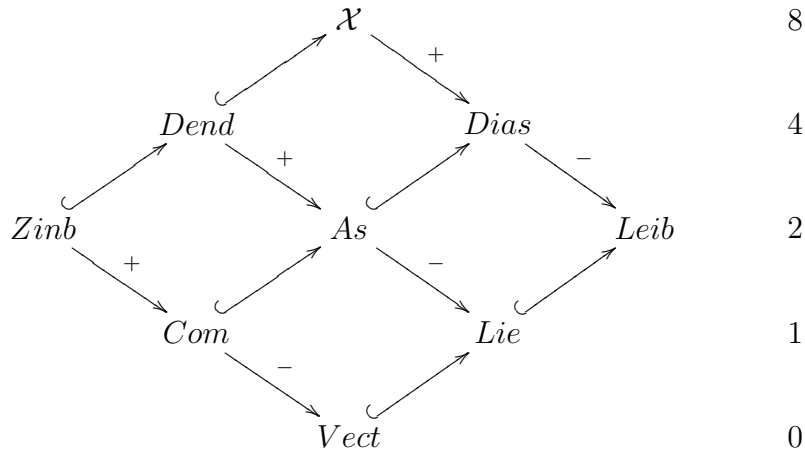
$$[[x, y], z] = [[x, z], y] + [x, [y, z]].$$

An arrow like $\mathcal{A} \hookrightarrow \mathcal{B}$ means that an $\mathcal{A}$-algebra is a $\mathcal{B}$-algebra satisfying some symmetry property. An arrow like $\mathcal{A} \overset{\pm}{\to} \mathcal{B}$ means that any generating operation of a $\mathcal{B}$-algebra is obtained by some addition (or subtraction) of two operations of the $\mathcal{A}$-algebra type (like the Lie bracket from the associative product). We let the reader to write the exact formulas in each case.

Each one of these types of algebras defines a binary quadratic operad. For these operads there is a well-defined notion of Koszul duality theory devised by Ginzburg and Kapranov [3]. Let $\mathcal{P}^!$ be the dual of the operad $\mathcal{P}$ (note that $\mathcal{P}^{!!} = \mathcal{P}$). It turns out that Koszul duality in the operadic butterfly corresponds to symmetry around the vertical axis passing through As:

$$As^! = As, \ Com^! = Lie, \ Zinb^! = Leib, \ Dend^! = Dias .$$

A functor of the form $\hookrightarrow$ is changed into a functor of the form $\overset{\pm}{\leftarrow}$ by duality.

One can slightly enhance the operadic butterfly by putting the category $Vect$ of algebras with no operation (also called *abelian Lie algebras*) in between Com and Lie. The operad $Vect$ is self-dual. So an immediate question comes to mind: can one complete the operadic butterfly by putting some category of algebras $\mathcal{X}$ at the upper place on the middle axis ?

The numbers on the right side of the diagram indicate the dimension of the space of binary operations.

In other words, we would like to find a notion of $\mathcal{X}$-algebra whose operad is binary and quadratic, and satisfies the following properties:

(1) the space of binary operations is 8-dimensional,
(2) the operad is isomorphic to its dual (for Koszul duality),
(3) a dendriform algebra is an $\mathcal{X}$-algebra satisfying some symmetry,
(4) any $\mathcal{X}$-algebra gives, by some symmetrization of the operations, a diassociative algebra,
(5) the functors deduced from the preceding two items make the upper square of the completed operadic butterfly commutative.

The aim of this paper is to answer this question, and the answer is as follows: there are two solutions $\mathcal{X}^+$ and $\mathcal{X}^-$.

An algebra of type $\mathcal{X}^\pm$ has four generating operations denoted by $\searrow$ $\nwarrow$ $\swarrow$ $\nearrow$ and $16 = 5 \times 3 + 1$ relations (we write $(\circ)\bullet$ instead of $(x \circ y) \bullet z$):

$$(\searrow)\searrow = \searrow(\searrow) + \searrow(\swarrow), \quad (\swarrow)\searrow = \swarrow(\searrow), \quad (\searrow)\swarrow + (\swarrow)\swarrow = \swarrow(\swarrow),$$
$$(\searrow)\searrow = \searrow(\searrow) + \searrow(\nearrow), \quad (\swarrow)\searrow = \swarrow(\nearrow), \quad (\searrow)\swarrow + (\swarrow)\swarrow = \swarrow(\searrow),$$
$$(\nearrow)\searrow = \nearrow(\searrow) + \nearrow(\swarrow), \quad (\searrow)\searrow = \searrow(\searrow), \quad (\nearrow)\swarrow + (\searrow)\swarrow = \searrow(\swarrow),$$
$$(\searrow)\nearrow = \nearrow(\nearrow) + \nearrow(\searrow), \quad (\swarrow)\nearrow = \searrow(\nearrow), \quad (\searrow)\searrow + (\swarrow)\searrow = \searrow(\searrow),$$
$$(\nearrow)\nearrow = \nearrow(\nearrow) + \nearrow(\searrow), \quad (\searrow)\nearrow = \searrow(\nearrow), \quad (\nearrow)\searrow + (\searrow)\searrow = \searrow(\searrow),$$

$$(\nearrow)\searrow - (\searrow)\searrow = \pm(\searrow(\swarrow) - \searrow(\searrow)). \tag{16$\pm$}$$

A more conceptual way of describing $\mathcal{X}^\pm$ is as follows. Given two binary quadratic operads $\mathcal{P}$ and $\mathcal{Q}$ with a preferred basis for the space of generating operations, one can construct a new operad, denoted by $\mathcal{P} \blacksquare \mathcal{Q}$, whose set of generating operations is the product of those of $\mathcal{P}$ and $\mathcal{Q}$ and the set of relations is also the product (in a certain sense) of those of $\mathcal{P}$ and $\mathcal{Q}$. This construction has already been used in particular cases in [1], [4] (for $(\mathcal{P} = \mathcal{Q})$ and in [2]. The operad $\mathcal{X}^\pm$ is $Dend \blacksquare Dias$ quotiented by the relation $(16\pm)$. The square product $\blacksquare$ constructed here is a variation, in the monoidal category of quadratic regular operads, of Manin's black product [7].

Content. In the first part we recall the theory of Koszul duality for regular operads and introduce the construction $\mathcal{P} \blacksquare \mathcal{Q}$. In the second part we introduce the operads $Dend$ and $Dias$, and compute $(Dend \blacksquare Dias)^!$. In the third part we show that

$$\mathcal{X}^\pm := Dend \blacksquare Dias / \text{ relation } (16\pm)$$

completes the operadic butterfly.

Convention. All vector spaces are over the field $\mathbb{K}$. The tensor product over $\mathbb{K}$ of the two spaces V and W is denoted by $V \otimes W$.

2. Product of Operads

2.1. **Regular operads.** In this paper we deal with algebras whose structure is defined by generating operations $(x, y) \mapsto x \circ_i y$ (with no symmetry) and relations of the form

$$(r) \qquad \sum_{i,j} \alpha_{ij} (x \circ_i y) \circ_j z = \sum_{i,j} \beta_{ij} x \circ_i (y \circ_j z) ,$$

where α_{ij} and β_{ij} are scalars. In these relations the variables remain in the same order (this is not the case for $Com, Lie, Zinb, Leib$). The associated operad, denoted by $\mathcal{P}$, is *binary* because the generating operations are binary. It is *quadratic* because the relations involve monomials with two operations. It is *regular* because the operations have no symmetry and, in the relations the variables stay in the same order. As a consequence of regularity the free $\mathcal{P}$-algebra over the vector space V is of the form

$$\mathcal{P}(V) = \oplus_{n \geq 1} \big(\mathcal{P}_n \otimes \mathbb{K}[S_n]\big) \otimes_{S_n} V^{\otimes n} = \oplus_{n \geq 1} \mathcal{P}_n \otimes V^{\otimes n},$$

where S_n is the symmetric group. The space $\mathcal{P}_n$ is the space of (non-symmetric) n-ary operations.

Let us denote by $E = \mathcal{P}_2$ the space of (non-symmetric) binary operations. Let $\{\circ_i \mid i \in I\}$ be a basis of E. The space of (non-symmetric) relations R is a subspace of $2E \otimes E$. The first summand corresponds to the parenthesizing $(x \circ_i y) \circ_j z$ and the second summand to the parenthesizing $x \circ_i (y \circ_j z)$. It will be helpful to denote by $(\circ_i)\circ_j$ and $\circ_i(\circ_j)$ the generating elements of $2E \otimes E$. Hence R is the subspace of $2E \otimes E$ spanned by the vectors

$$r = \sum_{i,j} \alpha_{ij} (\circ_i) \circ_j - \sum_{i,j} \beta_{ij} \circ_i (\circ_j) \in 2E \otimes E$$

for each relation (r).

One has $\mathcal{P}_1 = \mathbb{K}$, since, up to multiplication of scalars, there is only one unary operation: the identity. One has $\mathcal{P}_2 = E$ and $\mathcal{P}_3 = (2E \otimes E)/R$.

If an operad $\mathcal{Q}$ is obtained from an operad $\mathcal{P}$ by enlarging the space of relations with some more relations (r), then the space of n-ary operations of $\mathcal{Q}$ is a quotient of the space of n-ary operations of $\mathcal{P}$. By abuse of language we will say that $\mathcal{Q}$ is a quotient of $\mathcal{P}$ by (r) and we write $\mathcal{Q} = \mathcal{P}/(r)$.

The Koszul dual of the operad $\mathcal{P}$, denoted by $\mathcal{P}^!$, is determined by $E^* :=$ $\mathrm{Hom}(E, \mathbb{K})$ and $R^\perp$. Since we have equipped E with a basis, we can take the dual basis for E^* and identify it with E. After this identification $R^\perp$ is described as follows: it is the space orthogonal to R for the inner product $\langle -, - \rangle$ given on $E \otimes E \oplus E \otimes E$ by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

In other words, one has

$$\begin{cases} \langle (\circ_i)\circ_j, (\circ_i)\circ_j \rangle = 1, \\ \langle \circ_i(\circ_j), \circ_i(\circ_j) \rangle = -1, \\ \langle -, - \rangle = 0 \qquad \text{in the other cases.} \end{cases}$$

Since the operations of $\mathcal{P}$ and $\mathcal{P}^!$ do not satisfy, in general, the same relations, it is necessary, sometimes to distinguish between them. So we adopt the notation $\circ^*$ for the latter.

Lemma 2.2. *Let K be an index set and let $(r_k), k \in K$, be the relations defining R. An element*

$$\sum_{i,j} \alpha'_{ij}(\circ^*_i) \circ^*_j - \sum_{i,j} \beta'_{ij} \circ^*_i (\circ^*_j) \in 2E^* \otimes E^*$$

is in $R^{\perp}$ if and only if one has

$$\sum_{i,j} \alpha^{(k)}_{ij} \alpha'_{ij} = \sum_{i,j} \beta^{(k)}_{ij} \beta'_{ij}$$

for all $k \in K$.

Proof. This is an immediate translation of the definition of orthogonality. $\square$

2.3. **The square product of operads.** Let $\mathcal{P}$ be a binary quadratic regular operad defined by binary operations denoted by $\circ_i$, and relations (r) (cf. 2.1). Let $\mathcal{Q}$ be another one with operations $\bullet_k$ and relations (r'). We define the operad $\mathcal{P} \blacksquare \mathcal{Q}$ by the operations $\circ_i \bullet_k$ (product of the two sets of operations), and relations (r, r') given by

$$\sum_{i,j,k,l} \alpha_{ij} \alpha'_{kl}(\circ_i \bullet_k) \circ_j \bullet_l = \sum_{i,j,k,l} \beta_{ij} \beta'_{kl} \circ_i \bullet_k (\circ_j \bullet_l).$$

So, if $\mathcal{P}$ is defined by m relations and $\mathcal{Q}$ by m' relations, then $\mathcal{P} \blacksquare \mathcal{Q}$ is defined by mm' relations.

It immediately follows that the construction $\blacksquare$ is associative, commutative, and its neutral element is the operad As. Indeed, As has only one operation $\cdot$ satisfying $(\cdot)\cdot = \cdot(\cdot)$.

Proposition 2.4. *Let $\mathcal{P}$ and $\mathcal{Q}$ be binary quadratic regular operads. The operad $(\mathcal{P} \blacksquare \mathcal{Q})^!$ is a quotient of the operad $\mathcal{P}^! \blacksquare \mathcal{Q}^!$, so there is a natural forgetful functor of categories of algebras:*

$$(\mathcal{P} \blacksquare \mathcal{Q})^!-\mathrm{alg} \longrightarrow (\mathcal{P}^! \blacksquare \mathcal{Q}^!)-\mathrm{alg}.$$

Proof. The set of generating operations is the same in both cases, only the space of relations is different. For $(\mathcal{P} \blacksquare \mathcal{Q})^!$ it is $T^{\perp}$, where T is the space generated by the relations (r, r') (cf. 2.3). For $\mathcal{P}^! \blacksquare \mathcal{Q}^!$ it is the space S generated by the relations (s, s'), where s is orthogonal to all the relations r, and s' is orthogonal to all the relations r'.

In order to complete the proof it is sufficient to prove that S is included into $T^{\perp}$. Indeed, the expected functor would then simply be the forgetful functor. The space S is included into $T^{\perp}$ if and only if $\langle S, T \rangle = 0$. Let us check this equality for the relations (s, s') and (r, r').

We denote by α, β the structure constants of r as in 2.1 and by γ, δ the structure constants of s (same thing with a prime for r' and s'). By Lemma 2.2, it suffices to prove the equality

$$\sum_{i,j,k,l} \alpha_{ij} \alpha'_{kl} \gamma_{ij} \gamma'_{kl} = \sum_{i,j,k,l} \beta_{ij} \beta'_{kl} \delta_{ij} \delta'_{kl}.$$

The lefthand summand is equal to

$$\Big(\sum_{i,j}\alpha_{ij}\gamma_{ij}\Big)\Big(\sum_{k,l}\alpha'_{kl}\gamma'_{kl}\Big)$$

and similarly for the righthand side. Since the relations r and s (resp. r' and s') are orthogonal, one has

$$\sum_{i,j}\alpha_{ij}\gamma_{ij}=\sum_{i,j}\beta_{ij}\delta_{ij}$$

and similarly with a prime. This proves the equality and so $\langle S,T\rangle = 0$. $\square$

3. Dendriform and Diassociative Algebras

We recall from [5] the definition of dendriform algebra (with a slight change of the notation) and of diassociative algebra.

3.1. **Definition.** A *dendriform* algebra A is a vector space equipped with two operations denoted by $\wedge$ and $\vee$ satisfying the following relations:

$$\begin{cases}
(x\wedge y)\wedge z & = & x\wedge(y\wedge z)+x\wedge(y\vee z), & (i) \\
(x\vee y)\wedge z & = & x\vee(y\wedge z), & (ii) \\
(x\wedge y)\vee z+(x\vee y)\vee z & = & x\vee(y\vee z). & (iii)
\end{cases}$$

3.2. **Definition.** A *diassociative* algebra (or an associative dialgebra) A is a vector space equipped with two operations denoted by $\dashv$ and $\vdash$ satisfying the following relations:

$$\begin{cases}
(x\dashv y)\dashv z & = & x\dashv(y\dashv z), & (1) \\
(x\dashv y)\dashv z & = & x\dashv(y\vdash z), & (2) \\
(x\vdash y)\dashv z & = & x\vdash(y\dashv z), & (3) \\
(x\dashv y)\vdash z & = & x\vdash(y\vdash z), & (4) \\
(x\vdash y)\vdash z & = & x\vdash(y\vdash z). & (5)
\end{cases}$$

It was shown in [5] that the associated operads are dual to each other via the identification $\wedge^* =\dashv$ and $\vee^* =\vdash$. So we have $Dend^! = Dias$ and $Dias^! = Dend$. In fact, the reader can check this immediately from the description of duality for regular operads given in Lemma 2.2.

3.3. **The Operad** $Dend \blacksquare Dias$ **and its dual.** Consider the operad $Dend \blacksquare Dias$. We denote its set of generating operations by

$$\nwarrow := (\wedge,\dashv), \quad \nearrow := (\wedge,\vdash), \quad \searrow := (\vee,\vdash), \quad \swarrow := (\vee,\dashv).$$

Since $Dend$ has 3 relations and since $Dias$ has 5 relations, $Dend \blacksquare Dias$ has 15 relations which read as follows:

$$
\begin{array}{lll}
(\nwarrow)\nwarrow=\nwarrow(\nwarrow)+\nwarrow(\swarrow), & (\swarrow)\nwarrow=\swarrow(\nwarrow), & (\nwarrow)\swarrow+(\swarrow)\swarrow=\swarrow(\swarrow), \\
(\nwarrow)\nwarrow=\nwarrow(\searrow)+\nwarrow(\nearrow), & (\swarrow)\nwarrow=\swarrow(\nearrow), & (\nwarrow)\swarrow+(\swarrow)\swarrow=\swarrow(\searrow), \\
(\nearrow)\nwarrow=\nearrow(\nwarrow)+\nearrow(\swarrow), & (\searrow)\nwarrow=\searrow(\nwarrow), & (\nearrow)\swarrow+(\searrow)\swarrow=\searrow(\swarrow), \\
(\nwarrow)\nearrow=\nearrow(\nearrow)+\nearrow(\nwarrow), & (\swarrow)\nearrow=\searrow(\nwarrow), & (\nwarrow)\searrow+(\swarrow)\searrow=\searrow(\nwarrow), \\
(\nearrow)\nearrow=\nearrow(\nearrow)+\nearrow(\nwarrow), & (\searrow)\nearrow=\searrow(\nwarrow), & (\nearrow)\searrow+(\searrow)\searrow=\searrow(\nwarrow).
\end{array}
$$

The tableau

Proposition 3.4. *The operad of $(Dend \blacksquare Dias)^!$-algebras is isomorphic to the quotient of the operad $Dend \blacksquare Dias$ by the following two relations*

$$\begin{cases} (\nearrow)\searrow -(\nwarrow)\searrow = 0 \ , \\ 0 = \nwarrow(\swarrow) - \nwarrow(\searrow) \ . \end{cases}$$

Proof. The 15 relations of $Dend \blacksquare Dias$ are linearly independent, hence the space of relations is of dimension 15. Its orthogonal space is of dimension $2 \times 4^2 - 15 = 17$. Since $Dend^! = Dias$ and $Dias^! = Dend$, one has $Dend^! \blacksquare Dias^! = Dias \blacksquare Dend \cong Dend \blacksquare Dias$. By Proposition 2.4 $(Dend \blacksquare Dias)^!$ has, at least, the 15 relations of $Dend \blacksquare Dias$ (up to isomorphism) as relations plus two more (linearly independent). Let us show that the two extra operations are the ones indicated.

We first make explicit the isomorphism:

$$\begin{aligned} \nwarrow^* &= (\wedge, \dashv)^* = (\wedge^*, \dashv^*) = (\dashv, \wedge) \cong \nwarrow \ , \\ \nearrow^* &= (\wedge, \vdash)^* = (\wedge^*, \vdash^*) = (\dashv, \vee) \cong \swarrow \ , \\ \searrow^* &= (\vee, \vdash)^* = (\vee^*, \vdash^*) = (\vdash, \vee) \cong \searrow \ , \\ \swarrow^* &= (\vee, \dashv)^* = (\vee^*, \dashv^*) = (\vdash, \wedge) \cong \nearrow \ . \end{aligned}$$

So, in order to write down the 15 relations of $(Dend \blacksquare Dias)^!$ with this basis of generating operations we have to exchange the operations $\nearrow$ and $\swarrow$ in the tableau of the 15 relations. Let us call it the dual tableau. Under this isomorphism the two extra relations read as follows

$$\begin{cases} (\swarrow)\searrow -(\nwarrow)\searrow = 0 \ , \\ 0 = \nwarrow(\nearrow) - \nwarrow(\searrow) \ . \end{cases}$$

Let us check that these two relations are orthogonal with the 15 relations of $Dend \blacksquare Dias$. In 28 cases the verification is immediate because the involved operations are all different. The remaining two cases are

$$\langle\, (\swarrow)\searrow -(\nwarrow)\searrow \, , \, (\nwarrow)\searrow +(\swarrow)\searrow - \searrow(\searrow) \,\rangle = +1 - 1 = 0,$$
$$\langle\, \nwarrow(\nearrow) - \nwarrow(\searrow) \, , \, -(\nwarrow)\nwarrow + \nwarrow(\searrow) + \nwarrow(\nearrow) \,\rangle = -1 + 1 = 0.$$

It is straightforward to check that these 17 relations are linearly independent, therefore we have a complete presentation of $(Dend \blacksquare Dias)^!$. Under the inverse isomorphism of the one described above we get the expected result. $\square$

4. The Missing Operad

In this section we complete the operadic butterfly by constructing the operad $\mathcal{X}$.

Theorem 4.1. *Let $\mathcal{X}^+$ (resp. $\mathcal{X}^-$) be the operad $Dend \blacksquare Dias$ quotiented by the relation $(16+)$ (resp. $(16-)$):*

$$\begin{aligned} (\nearrow)\searrow -(\nwarrow)\searrow &= + \nwarrow(\swarrow) - \nwarrow(\searrow) \ , & (16+) \\ (\nearrow)\searrow -(\nwarrow)\searrow &= - \nwarrow(\swarrow) + \nwarrow(\searrow) \ . & (16-) \end{aligned}$$

The operad $\mathcal{X}^\pm$ completes the operadic butterfly, that is, it satisfies the following properties:

(1) *the space of binary operations is 8-dimensional,*
(2) *the operad is isomorphic to its dual (for Koszul duality),*
(3) *a dendriform algebra is an $\mathcal{X}$-algebra satisfying some symmetry,*
(4) *any $\mathcal{X}$-algebra gives, by some symmetrization of the operations, a diassociative algebra,*
(5) *the functors deduced from the preceding two items make the upper square of the completed operadic butterfly commutative.*

Proof. Let us put $r := (\nearrow) \searrow -(\nwarrow) \searrow$ and $s :=\nwarrow (\swarrow)- \searrow (\searrow)$. We are looking for scalars α and β such that the operad $Dend \ \blacksquare \ Dias/\alpha r + \beta s$ is self-dual. The dual of r (resp. s) is $r^* = (\swarrow) \searrow -(\nwarrow) \searrow$ (resp. $s^* =\nwarrow (\nearrow)- \searrow (\searrow)$). To get the self-duality of the quotient operad we need to have $-\alpha^2 + \beta^2 = 0$. Hence α and β have to be equal up to sign. So the sixteenth relation is either $r = s$ called $(16+)$ or $r = -s$, called $(16-)$. It can be checked immediately that in both cases the $16 \ (= 3 \times 5 + 1))$ relations are linearly independent. Hence we have proved (1) and (2). It turns out that both solutions satisfy also the other requirements as we prove now.

(3) Let $(A, \wedge, \vee)$ be a dendriform algebra. Define $\nwarrow := \wedge =:\nearrow$ and $\swarrow := \vee =:\searrow$. Let us prove that $(A,\nwarrow, \nearrow, \swarrow, \searrow)$ is an $\mathcal{X}^{\pm}$-algebra. The first fifteen relations are fulfilled since relation (n, a) is a consequence of relation (a) for $n = 1, \ldots, 5$ and $a = i, ii, iii$. The relation $(16\pm)$ is also fulfilled since both sides of the equality are 0: the left side because $\nwarrow=\nearrow$ and the right side because $\swarrow=\searrow$.

(4) Let $(A, \nwarrow, \nearrow, \swarrow, \searrow)$ be an $\mathcal{X}^{\pm}$-algebra. Define

$$x \dashv y := x \nwarrow y + x \swarrow y \quad \text{and} \quad x \vdash y := x \nearrow y + x \searrow y.$$

Adding the relations $(n, i), (n, ii)$ and (n, iii) shows that the relation (n) is fulfilled for $n = 1, \ldots, 5$.

(5) Starting with a dendriform algebra $(A, \wedge, \vee)$, we get a diassociative algebra $(A, \dashv, \vdash)$ such that

$$x \dashv y = x \nwarrow y + x \swarrow y = x \wedge y + x \vee y = x \nearrow y + x \searrow y = x \vdash y \ .$$

Therefore the composite $Dend \hookrightarrow \mathcal{X}^{\pm} \xrightarrow{+} Dias$ is equal to the composite $Dend \xrightarrow{+} As \hookrightarrow Dias$ as expected. $\qquad\square$

Remark. In fact, our proof shows that these two solutions are the only quotients of $Dend \ \blacksquare \ Dias$ which satisfy all the required properties.

4.2. **Question.** The operads $Com, Lie, As, Zinb, Leib, Dend, Dias$ are Koszul operads, i.e. the associated Koszul complex is acyclic (cf. [3], [6], [5]). Is also $\mathcal{X}^+$(resp. $\mathcal{X}^-$) a Koszul operad?

Since the operad $\mathcal{X} \ (= \mathcal{X}^+$ or $\mathcal{X}^-)$ is regular and self-dual, the criterion to ensure Koszul duality takes the following form. Let $\mathcal{X}_n$ be the homogeneous part of degree n in the free $\mathcal{X}$-algebra on one generator. Then for each integer $k \geq 1$ there is a finite chain complex

$$0 \to \mathcal{X}_k \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_1 \to \cdots$$
$$\to \bigoplus_{m_1+\cdots+m_n=k} \mathcal{X}_n \otimes \mathcal{X}_{m_1} \otimes \cdots \otimes \mathcal{X}_{m_n} \to \cdots$$
$$\cdots \to \mathcal{X}_1 \otimes \mathcal{X}_k \to 0.$$

The acyclicity of these complexes for $k > 1$ implies Koszulity. It implies that $\mathcal{X}_n$ is of dimension 4^{n-1}.

Added in proofs. The Koszulity of the operad $Dend$ ■ $Dias$ has been recently proved in B. VALLETTE, Manin products, Koszul duality, Loday algebras and Deligne conjecture (2006), ArXiv math.QA/0609002.

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