

SUMS OF ODD SQUARES

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Abstract. The number of representations of an integer as a sum of n odd squares for $n \equiv 0 \pmod{4}$ is expressed by the number of representations as a sum of n squares of integers and the number of representations by the lattice D_n^+ .

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Let $r_n^{\text{odd}}(m)$ denote the number of representations of a positive integer m as a sum of n odd squares:

$$r_n^{\text{odd}}(m) := \#\{(x_1, \dots, x_n) \in (\mathbb{Z}^{\text{odd}})^n, m = x_1^2 + \dots + x_n^2\},$$

where $\mathbb{Z}^{\text{odd}} := \{g \in \mathbb{Z}, g \equiv 1 \pmod{2}\}$.

Recently, Lass [3] has proved the interesting combinatorial conjectures of Dumont concerning these numbers.

We follow the terminology of Conway–Sloane [1] and define

$$D_n^+ := D_n + \left(\left(\frac{1}{2}, \dots, \frac{1}{2} \right) + D_n \right),$$

where $D_n := \{(x_1, \dots, x_n) \in \mathbb{Z}^n, x_1 + \dots + x_n \equiv 0 \pmod{2}\}$. D_n^+ is a lattice if and only if n is even, in particular, D_8^+ is equal to E_8 .

In this note we prove the following

Proposition. *The identities*

(I) $\frac{1}{2}r_n^{\text{odd}}(8m) = r_{D_n^+}(2m) - r_n(2m)$ for $n \equiv 0 \pmod{8}$,

(II) $\frac{1}{2}r_n^{\text{odd}}(8m+4) = r_{D_n^+}(2m+1)$ for $n \equiv 4 \pmod{8}$

are valid, where $r_n(m) := \#\{(x_1, \dots, x_n) \in \mathbb{Z}^n; m = x_1^2 + \dots + x_n^2\}$ and $r_{D_n^+}(m) := \#\{(x_1, \dots, x_n) \in D_n^+; m = x_1^2 + \dots + x_n^2\}$.

Proof. The theta series of D_n is $\frac{1}{2}((\theta_3(z))^n + (\theta_4(z))^n) = \sum_{m=0}^{\infty} r_n(2m)q^{2m}$ with $q = e^{\pi iz}$ and the theta series of $(\frac{1}{2}, \dots, \frac{1}{2}) + D_n$ is $\frac{1}{2}(\theta_2(z))^n$ (s. [1], p. 118).

Therefore the theta series of D_n^+ is $\frac{1}{2}((\theta_2(z))^n + (\theta_3(z))^n + (\theta_4(z))^n)$ (see [1], p. 120).

Now

$$\theta_2(z) = \sum_{m=-\infty}^{\infty} q^{(m+\frac{1}{2})^2} = \sum_{m=-\infty}^{\infty} q^{\frac{1}{4}(2m+1)^2} = 2 \left(q^{\frac{1}{4}} + q^{\frac{9}{4}} + q^{\frac{25}{4}} + \dots \right).$$

Comparing coefficients gives

$$(I) r_{D_n^+}(2m) = r_n(2m) + \frac{1}{2}r_n^{odd}(8m).$$

Evidently, $r_n^{odd}(8m) \neq 0$ only if $n \equiv 0 \pmod 8$. Furthermore since the theta series of D_n has only nonzero coefficients for even indices, we get by comparing coefficients

$$(II) r_{D_n^+}(2m + 1) = \frac{1}{2}r_n^{odd}(8m + 4).$$

Evidently, $r_n^{odd}(8m + 4) \neq 0$ only if $n \equiv 4 \pmod 8$.

Let us consider in detail the cases $n = 4, 8$ and 24 .

Dimension 4: Since $D_4^+ \cong \mathbb{Z}^4$, we get $r_4^{odd}(8m + 4) = 2r_4(2m + 1)$.

It is well known that

$$r_4(2m + 1) = 8\sigma(2m + 1) = 8 \sum_{d|2m+1} d,$$

thus we get

$$r_4^{odd}(8m + 4) = 16 \sum_{d|2m+1} d = \frac{2}{3}r_4(8m + 4).$$

Dimension 8: Here we get

$$\frac{1}{2}r_8^{odd}(8m) = r_{E_8}(2m) - r_8(2m).$$

Since $r_8(m) = 16\sigma_3^*(m)$, where

$$\sigma_3^*(m) = \begin{cases} \sigma_3(m), & m \text{ odd,} \\ \sigma_3^{even}(m) - \sigma_3^{odd}(m), & m \text{ even} \end{cases}$$

with $\sigma_3^{even}(m) = \sum_{\substack{d|m \\ d \text{ even}}} d^3$, etc. and since $r_{E_8}(2m) = 240\sigma_3(m)$, we get $r_8^{odd}(8m) = 480\sigma_3(m) - 32\sigma_3^*(2m)$.

Specializing for primes $m = p \neq 2$ we get by computation:

$$r_8^{odd}(8p) = 2^8\sigma_3(p).$$

Dimension 24: We have

$$r_{D_{24}^+}(2m) = \frac{65520}{691}\sigma_{11}(m) + \frac{697344}{691}\tau(m),$$

furthermore Ramanujan's formula ([2], p. 155)

$$r_{24}(m) = \frac{16}{691}\sigma_{11}^*(m) + \frac{128}{691}\left((-1)^{m-1}259\tau(m) - 512\tau\left(\frac{m}{2}\right)\right).$$

Thus we get

$$r_{24}^{odd}(8m) = \frac{2}{691}(65520\sigma_{11}(m) - 16\sigma_{11}^*(2m) + 762880\tau(m) + 33152\tau(2m)). \quad \square$$

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